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On w-compatible mappings and common coupled coincidence point in cone metric spaces

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ABSTRACT

Recently, Abbas et al. [M. Abbas, M.A. Khan, S. Radenović, Common coupled fixed point theorems in cone metric spaces for w-compatible mapping, Applied Mathematics and Computation (2010) doi:10.1016/j.amc.2010.05.042] introduced the concept of w-compatible mappings and obtained results on coupled coincidence point for nonlinear contractive mappings in a cone metric space. In the present paper, we introduce the concept of a common coupled coincidence point of the mappings $F, G : X \times X \to X$ and $f : X \to X$ and we prove some theorems for nonlinear contractive mappings in a cone metric space with a cone having nonempty interior. Our results generalize several well known comparable results in the literature.

1. Introduction

Huang and Zhang [1] introduced the concept of a cone metric space, replacing the set of real numbers by an ordered Banach space. They proved some fixed point theorems of contractive type mappings over cone metric spaces. Later, many authors generalized their fixed point theorems in different type. For a survey of coincidence point theory over cone metric spaces, we refer the reader (as examples) to [1–19]. Bhaskar and Lakshmikantham [20] introduced the concept of a coupled coincidence point of a mapping F from $X \times X$ into X and a mapping g from X into X and studied fixed point theorems in partially ordered metric spaces. In [21], Lakshmikantham and Ćirić studied some coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. For more details, on coupled fixed point results, we refer the reader as example to [22–27]. Recently, Abbas et al. [2], defined the concept of w-compatible mappings and obtained coupled coincidence point theorems for nonlinear contractive mappings in a cone metric space with a cone having nonempty interior. The aim of this paper is to introduce the concept of common coupled coincidence point and obtain some common coupled coincidence results for nonlinear contractive mappings in a cone metric space with a cone having nonempty interior.

2. Basic concepts

In the present paper, E stands for a real Banach space. Let P be a subset of E with $\mathrm{int}(P) \neq \emptyset$, where $\mathrm{int}(P)$ denotes the interior of P. Then P is called a cone if the following conditions are satisfied:

1. P is closed and $P \neq \{\theta\}$, where θ is referred as the "zero" element of the Banach space E.
2. $a, b \in \mathbf{R}^+$, $x, y \in P$ implies $ax + by \in P$.
3. $x \in P \cap -P$ implies $x = \theta$.

For a cone P, define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \mathrm{int}P$. It can be easily shown that $\lambda\mathrm{int}(P) \subseteq \mathrm{int}(P)$ for all positive scalar λ.

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Definition 2.1 (*[1]*). Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

1. $\theta < d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Definition 2.2 (*[1]*). Let (X, d) be a cone metric space. Let (x_n) be a sequence in X and $x \in X$. If for every $c \in E$ with $\theta \ll c$, there is an $N \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n \geq N$, then (x_n) is said to be convergent and (x_n) converges to x and x is the limit of (x_n). We denote this by $\lim_{n\to+\infty} x_n = x$ or $x_n \to x$ as $n \to +\infty$. If for every $c \in E$ with $\theta \ll c$ there is an $N \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$, then (x_n) is called a Cauchy sequence in X. The space (X, d) is called a complete cone metric space if every Cauchy sequence is convergent.

The cone P in a real Banach space E is called normal if there is a number $k > 0$ such that for all $x, y \in E$,

$$\theta \leq x \leq y \quad \text{implies } \|x\| \leq k\|y\|.$$

Rezapour and Hamlbarani [14] proved that there are no normal cones with normal constant $k < 1$ and that for each $h > 1$ there are cones with normal constant $K > h$. Also, omitting the assumption of normality, they obtain generalizations of some results of [1].

Let (X, d) be a cone metric space with cone P not necessary to be normal. Then the following properties are useful in our subsequent arguments.

1. If $a \leq ha$ and $h \in [0, 1)$, then $a = \theta$.
2. If $\theta \leq u \ll c$ for each $\theta \ll c$, then $u = \theta$.
3. If $u \leq v$ and $v \ll w$, then $u \ll w$.

Definition 2.3 (*[21]*). An element $(x, y) \in X \times X$ is called a common coupled fixed point of mappings $F : X \times X \to X$ and $f : X \to X$ if $F(x, y) = fx = x$ and $F(y, x) = fy = y$.

Definition 2.4 (*[21]*). An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$F(x, y) = gx \quad \text{and} \quad F(y, x) = gy.$$

Definition 2.5 (*[2]*). Let X be a nonempty set. Then we say that the mappings $F : X \times X \to X$ and $g : X \to X$ are w-compatible if $gF(x, y) = F(gx, gy)$ whenever $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

3. Main results

In order to proceed in our work and achieve our results, we introduce the following definition.

Definition 3.1. Let X be a nonempty set. Then the point (x, y) in $X \times X$ is called a common coupled coincidence point of the mappings $F, G : X \times X \to X$ and $f : X \to X$ if

$$F(x, y) = G(x, y) = fx$$

and

$$F(y, x) = G(y, x) = fy.$$

The point (fx, fy) is called a common coupled point of coincidence.

Theorem 3.1. *Let (X, d) be a cone metric space with a cone P having nonempty interior. Let $F, G : X \times X \to X$ and $f : X \to X$ be functions such that*

$$\begin{aligned} d(F(x, y), G(u, v)) \leq\ & a_1 d(fx, fu) + a_2 d(fy, fv) + a_3(d(F(x, y), fx) + d(G(u, v), fu)) \\ & + a_4(d(F(x, y), fu) + d(G(u, v), fx)) \end{aligned}$$

holds for all $x, y, u, v \in X$. Assume that F, G and f satisfy the following conditions:

1. *$F(X \times X) \subseteq f(X)$,*
2. *$G(X \times X) \subseteq f(X)$, and*
3. *$f(X)$ is a complete subspace of X.*

If a_1, a_2, a_3 and a_4 are nonnegative real numbers with $a_1 + a_2 + 2a_3 + 2a_4 < 1$, then F, G and f have a common coupled coincidence point.

Proof. Let x_0, y_0 be two arbitrary elements in X. Since $F(X \times X) \subseteq f(X)$, we can choose $x_1, y_1 \in X$ such that $fx_1 = F(x_0, y_0)$ and $fy_1 = F(y_0, x_0)$. Again since $G(X \times X) \subseteq f(X)$, we can choose $x_2, y_2 \in X$ such that $fx_2 = G(x_1, y_1)$ and $fy_2 = G(y_1, x_1)$. Continuing the same process, we can construct two sequences (x_n) and (y_n) in X such that

$$\begin{aligned} fx_{2n+1} &= F(x_{2n}, y_{2n}), \\ fy_{2n+1} &= F(y_{2n}, x_{2n}), \\ fx_{2n+2} &= G(x_{2n+1}, y_{2n+1}), \end{aligned}$$

and

$$fy_{2n+2} = G(x_{2n+1}, y_{2n+1}).$$

Let $n \in \mathbb{N} \cup \{0\}$. Then we have

$$\begin{aligned} d(fx_{2n+1}, fx_{2n+2}) &= d(F(x_{2n}, y_{2n}), G(x_{2n+1}, y_{2n+1})) \\ &\leq a_1 d(fx_{2n}, fx_{2n+1}) + a_2 d(fy_{2n}, fy_{2n+1}) + a_3(d(fx_{2n+1}, fx_{2n}) + d(fx_{2n+2}, fx_{2n+1})) \\ &\quad + a_4(d(fx_{2n+1}, fx_{2n+1}) + d(fx_{2n+2}, fx_{2n})). \end{aligned}$$

Since

$$d(fx_{2n+2}, fx_{2n}) \leq d(fx_{2n+2}, fx_{2n+1}) + d(fx_{2n+1}, fx_{2n}),$$

we have

$$(1 - a_3 - a_4) d(fx_{2n+1}, fx_{2n+2}) \leq (a_1 + a_3 + a_4) d(fx_{2n}, fx_{2n+1}) + a_2 d(fy_{2n}, fy_{2n+1}). \tag{1}$$

Similarly, we get

$$(1 - a_3 - a_4) d(fy_{2n+1}, fy_{2n+2}) \leq (a_1 + a_3 + a_4) d(fy_{2n}, fy_{2n+1}) + a_2 d(fx_{2n}, fx_{2n+1}). \tag{2}$$

Let

$$r = \frac{a_1 + a_2 + a_3 + a_4}{1 - a_3 - a_4}.$$

By adding inequality (1) to inequality (2), we get

$$d(fx_{2n+1}, fx_{2n+2}) + d(fy_{2n+1}, fy_{2n+2}) \leq r(d(fx_{2n}, fx_{2n+1}) + d(fy_{2n}, fy_{2n+1})). \tag{3}$$

On the other hand, for $n \in \mathbb{N} \cup \{0\}$ we have

$$\begin{aligned} d(fx_{2n+1}, fx_{2n}) &= d(F(x_{2n}, y_{2n}), G(x_{2n-1}, y_{2n-1})) \\ &\leq a_1 d(fx_{2n}, fx_{2n-1}) + a_2 d(fy_{2n}, fy_{2n-1}) + a_3(d(fx_{2n+1}, fx_{2n}) + d(fx_{2n}, fx_{2n-1})) \\ &\quad + a_4(d(fx_{2n+1}, fx_{2n-1}) + d(fx_{2n}, fx_{2n})). \end{aligned}$$

Since

$$d(fx_{2n+1}, fx_{2n-1}) \leq d(fx_{2n+1}, fx_{2n}) + d(fx_{2n}, fx_{2n-1}),$$

we get

$$(1 - a_3 - a_4) d(fx_{2n}, fx_{2n+1}) \leq (a_1 + a_3 + a_4) d(fx_{2n}, fx_{2n-1}) + a_2 d(fy_{2n}, fy_{2n-1}). \tag{4}$$

From similar arguments as above, we get

$$(1 - a_3 - a_4) d(fy_{2n}, fy_{2n+1}) \leq (a_1 + a_3 + a_4) d(fy_{2n}, fy_{2n-1}) + a_2 d(fx_{2n}, fx_{2n-1}). \tag{5}$$

Adding inequality (4) to inequality (5), we get

$$d(fx_{2n}, fx_{2n+1}) + d(fy_{2n}, fy_{2n+1}) \leq r(d(fx_{2n}, fx_{2n-1}) + d(fy_{2n}, fy_{2n-1})). \tag{6}$$

From inequalities (3) and (6), we have

$$\begin{aligned} d(fx_{2n+1}, fx_{2n+2}) + d(fy_{2n+1}, fy_{2n+2}) &\leq r(d(fx_{2n}, fx_{2n+1}) + d(fy_{2n}, fy_{2n+1})) \\ &\leq r^2(d(fx_{2n}, fx_{2n-1}) + d(fy_{2n}, fy_{2n-1})) \\ &\ \ \vdots \\ &\leq r^{2n+1}(d(fx_0, fx_1) + d(fy_0, fy_1)). \end{aligned}$$

Let

$$(w_n)_{n=0}^{\infty} = (fx_0, fx_1, fx_2, \ldots)$$

and

$$(z_n)_{n=0}^{\infty} = (fy_0, fy_1, fy_2, \ldots).$$

Then for $n \in \mathbb{N}$, we have

$$d(w_n, w_{n+1}) + d(z_n, z_{n+1}) \leq r^n(d(w_0, w_1) + d(z_0, z_1)). \tag{7}$$

Case 1: $d(w_0, w_1) + d(z_0, z_1) = \theta$. This case yields that $w_0 = w_1$ and $z_0 = z_1$. By inequality (7), we get that $w_0 = w_n$ and $z_0 = z_n$ for each $n \in \mathbb{N}$. Hence

$$fx_0 = fx_1 = F(x_0, y_0)$$

and

$$fy_0 = fy_1 = F(y_0, x_0).$$

Now, we will show that $G(x_0, y_0) = fx_0$ and $G(y_0, x_0) = fy_0$. For that, we have

$$\begin{aligned} d(fx_0, G(x_0, y_0)) &= d(F(x_0, y_0), G(x_0, y_0)) \\ &\leq a_1 d(fx_0, fx_0) + a_2 d(fy_0, fy_0) + a_3(d(fx_0, fx_0) + d(G(x_0, y_0), fx_0)) \\ &\quad + a_4(d(fx_0, fx_0) + d(G(x_0, y_0), fx_0)). \end{aligned}$$

Hence

$$d(fx_0, G(x_0, y_0)) \leq (a_3 + a_4) d(fx_0, G(x_0, y_0)).$$

Since $a_3 + a_4 < 1$, we get $d(fx_0, G(x_0, y_0)) = \theta$, and hence $fx_0 = G(x_0, y_0)$. Similarly, we may show that $fy_0 = G(y_0, x_0)$. Therefore, we get that the point (x_0, y_0) is a common coupled coincidence point of F, G and f.

Case 2: $d(w_0, w_1) + d(z_0, z_1) \neq \theta$. For $m > n$ we get

$$d(w_n, w_m) \leq d(w_n, w_{n+1}) + \cdots + d(w_{m-1}, w_m)$$

and

$$d(z_n, z_m) \leq d(z_n, z_{n+1}) + \cdots + d(z_{m-1}, z_m).$$

By inequality (7) and the fact that $r < 1$, we have

$$d(w_n, w_m) + d(z_n, z_m) \leq \frac{r^n}{1 - r}(d(w_0, w_1) + d(z_0, z_1)) \to \theta \quad \text{as } n \to +\infty.$$

Thus for $c \gg \theta$, we can find $k \in \mathbb{N}$ such that

$$\frac{r^n}{1 - r}(d(w_0, w_1) + d(z_0, z_1)) \ll c$$

for all $n \geq k$. Hence

$$d(w_n, w_m) + d(z_n, z_m) \ll c$$

for all $n \geq k$. Since

$$d(w_n, w_m) \leq d(w_n, w_m) + d(z_n, z_m)$$

and

$$d(z_n, z_m) \leq d(w_n, w_m) + d(z_n, z_m),$$

we conclude that (w_n) and (z_n) are Cauchy sequences in $f(X)$. Since $f(X)$ is complete, we find x, y in X such that $w_n = f(x_n) \to f(x)$ and $z_n = f(y_n) \to f(y)$ as $n \to +\infty$. Thus, we have $f(x_{2n+1}) \to f(x), f(x_{2n}) \to f(x), f(y_{2n+1}) \to f(y)$, and $f(y_{2n}) \to f(y)$. Now, we will prove that $F(x, y) = G(x, y) = f(x)$ and $F(y, x) = G(y, x) = fx$. For that

$$d(F(x, y), fx) \leq d(F(x, y), f(x_{2n+2})) + d(f(x_{2n+2}), fx). \tag{8}$$

But

$$\begin{aligned} d(F(x, y), f(x_{2n+2})) &= d(F(x, y), G(x_{2n+1}, y_{2n+1})) \leq a_1 d(fx, fx_{2n+1}) + a_2 d(fy, fy_{2n+1}) \\ &\quad + a_3(d(F(x, y), fx) + d(fx_{2n+2}, fx_{2n+1})) + a_4(d(F(x, y), fx_{2n+1}) + d(fx_{2n+2}, fx)). \end{aligned}$$

Since

$$d(fx_{2n+2}, fx_{2n+1}) \leq d(fx_{2n+2}, fx) + d(fx, fx_{2n+1})$$

and

$$d(F(x,y), fx_{2n+1}) \leq d(F(x,y), fx) + d(fx, fx_{2n+1}),$$

inequality (8), becomes

$$d(F(x,y), fx) \leq \frac{1+a_3+a_4}{1-a_3-a_4} d(fx_{2n+2}, fx) + \frac{a_1+a_3+a_4}{1-a_3-a_4} d(fx, fx_{2n+1}) + \frac{a_2}{1-a_3-a_4} d(fy, fy_{2n+1}).$$

Since $fx_{2n+1} \to fx, fy_{2n+1} \to fy$, and $fx_{2n+2} \to fx$ as $n \to +\infty$, then for $c \gg \theta$ there is $N_0 \in \mathbb{N}$ such that

$$d(fx_{2n+2}, fx) \ll \frac{1-a_3-a_4}{3(1+a_3+a_4)} c, \qquad d(fx_{2n+1}, fx) \ll \frac{1-a_3-a_4}{3(a_1+a_3+a_4)} c$$

and

$$d(fy_{2n+2}, fy) \ll \frac{1-a_3-a_4}{3a_2} c$$

hold for all $n \geq N_0$. So, $d(F(x,y), fx) \ll c$ and hence $F(x,y) = fx$. By similar arguments as above and the aid of the following inequality:

$$d(fx, G(x,y)) \leq d(fx, fx_{2n+1}) + d(fx_{2n+1}, G(x,y)) = d(fx, fx_{2n+1}) + d(F(x_{2n}, y_{2n}), G(x,y)),$$

we get $G(x,y) = fx$. Hence $F(x,y) = G(x,y) = fx$. By the same method, we may show that $F(y,x) = G(y,x) = fy$. Therefore (x,y) is a common coupled coincidence point of the mappings F, G and f. □

Theorem 3.2. *In addition to the hypotheses of Theorem 3.1, suppose that the mappings F, f are w-compatible and the mappings G, f are w-compatible; then F, G and f have a unique common coupled fixed point of the form (u,u) for some $u \in X$.*

Proof. By Theorem 3.1, the maps F, G and f have a common coupled coincidence point, say (x,y), that is, $F(x,y) = G(x,y) = fx$ and $F(y,x) = G(y,x) = fy$. Now, we will show that $fx = fy$. For that

$$\begin{aligned} d(fx, fy) &= d(F(x,y), G(y,x)) \leq a_1 d(fx, fy) + a_2 d(fy, fx) + a_3(d(fx, fx) + d(fy, fy)) \\ &\quad + a_4(d(fx, fy) + d(fy, fx)) = (a_1 + a_2 + 2a_4) d(fx, fy). \end{aligned}$$

Since $a_1 + a_2 + 2a_4 < 1$, we have $d(fx, fy) = \theta$. Hence $fx = fy$. Let (x^*, y^*) be another common coupled coincidence point of F, G and f. Then $fx^* = fy^*$. Now, we will show that $fx = fx^*$ and $fy = fy^*$. For that, from

$$\begin{aligned} d(fx, fy^*) &= d(F(x,y), G(y^*, x^*)) \leq a_1 d(fx, fy^*) + a_2 d(fy, fx^*) + a_3(d(fx, fx) + d(fy^*, fy^*)) \\ &\quad + a_4(d(fx, fy^*) + d(fy^*, fx)) = (a_1 + 2a_4) d(fx, fy^*) + a_2 d(fy, fx^*) \end{aligned}$$

and

$$\begin{aligned} d(fx^*, fy) &= d(F(x^*, y^*), G(y,x)) \leq a_1 d(fx^*, fy) + a_2 d(fy^*, fx) + a_3(d(fx^*, fx^*) + d(fy, fy)) \\ &\quad + a_4(d(fx^*, fy) + d(fy, fx^*)) = (a_1 + 2a_4) d(fx^*, fy) + a_2 d(fy^*, fx), \end{aligned}$$

we get

$$d(fx, fy^*) + d(fx^*, fy) \leq (a_1 + a_2 + 2a_4)(d(fx, fy^*) + d(fx^*, fy)).$$

Since $a_1 + a_2 + 2a_4 < 1$, we get $fx = fy^*$ and $fx^* = fy$, and hence $fx = fx^*$ and $fy = fy^*$. Therefore the common coupled point of coincidence is unique. Let $u = fx$. Then using the fact that F, f are w-compatible and G, f are w-compatible, we have

$$fu = f(fx) = f(F(x,x)) = F(fx, fx) = F(u,u),$$

and

$$fu = f(fx) = f(G(x,x)) = G(fx, fx) = G(u,u).$$

Thus (fu, fu) is also a common coupled point of coincidence. By uniqueness we get that $fu = fx$. Hence $fu = u$. □

Corollary 3.1 ([2, Theorem 2.4])**.** *Let X be a cone metric space with a cone P having non-empty interior, $F : X \times X \to X$ and $f : X \to X$ be mappings satisfying:*

$$\begin{aligned} d(F(x,y), F(u,v)) &\leq a_1 d(fx, fu) + a_2 d(fy, fv) + a_3 d(F(x,y), fx) + a_4 d(F(u,v), fu) \\ &\quad + a_5 d(F(x,y), fu) + a_6 d(F(u,v), fx) \end{aligned}$$

for all $x, y, u, v \in X$, where a_1, a_2, a_3, a_4, a_5 and a_6 are nonnegative real numbers such that $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 < 1$. If $F(X \times X) \subseteq f(X)$ and $f(X)$ is a complete subset of X, then F and f have a coupled coincidence point in X.

Proof. From

$$d(F(x,y),F(u,v)) \le a_1 d(fx,fu) + a_2 d(fy,fv) + a_3 d(F(x,y),fx) + a_4 d(F(u,v),fu) + a_5 d(F(x,y),fu) + a_6 d(F(u,v),fx)$$

and

$$d(F(u,v),F(x,y)) \le a_1 d(fu,fx) + a_2 d(fv,fy) + a_3 d(F(u,v),fu) + a_4 d(F(x,y),fx) + a_5 d(F(u,v),fx) + a_6 d(F(x,y),fu),$$

we get

$$2d(F(x,y),F(u,v)) \le 2a_1 d(fx,fu) + 2a_2 d(fy,fv) + (a_3+a_4)(d(F(x,y),fx) + d(F(u,v),fu)) + (a_5+a_6)(d(F(x,y),fu) + d(F(u,v),fx)).$$

Hence

$$d(F(x,y),F(u,v)) \le a_1 d(fx,fu) + a_2 d(fy,fv) + \frac{a_3+a_4}{2}(d(F(x,y),fx) + d(F(u,v),fu)) + \frac{a_5+a_6}{2}(d(F(x,y),fu) + d(F(u,v),fx)).$$

Define $G : X \times X \to X$, by $G(x,y) = F(x,y)$. Then we get

$$d(F(x,y),G(u,v)) \le a_1 d(fx,fu) + a_2 d(fy,fv) + \frac{a_3+a_4}{2}(d(F(x,y),fx) + d(G(u,v),fu)) + \frac{a_5+a_6}{2}(d(F(x,y),fu) + d(G(u,v),fx)).$$

Since

$$a_1 + a_2 + 2\left(\frac{a_3+a_4}{2}\right) + 2\left(\frac{a_5+a_6}{2}\right) < 1,$$

then by Theorem 3.1, we conclude that F and f have a coupled coincidence point of X. □

Corollary 3.2 *([2, Theorem 2.5]). In addition to the hypotheses of Corollary 3.1, suppose that the mappings F, f are w-compatible and the mappings G, f are w-compatible; then F, G and f have a unique common coupled fixed point of the form (u,u) for some $u \in X$.*

Proof. Follows from Theorem 3.2 and Corollary 3.1. □

Remark. As results of Corollaries 3.1 and 3.2, Theorems 2.2, 2.5, 2.6, and Corollaries 2.3, 2.7, 2.8 of [15] become special cases of our results.

Corollary 3.3. *Let (X,d) be a complete cone metric space with a cone P having nonempty interior. Let $F, G : X \times X \to X$ be two maps such that*

$$d(F(x,y),G(u,v)) \le a_1 d(x,u) + a_2 d(y,v) + a_3(d(F(x,y),x) + d(G(u,v),u)) + a_4(d(F(x,y),u) + d(G(u,v),x))$$

holds for all $x, y, u, v \in X$. If a_1, a_2, a_3 and a_4 are nonnegative real numbers with $a_1 + a_2 + 2a_3 + 2a_4 < 1$, then F and G have a unique common coupled coincidence fixed point.

Proof. Defining $f : X \to X$ by $f(x) = x$, then F, G and f satisfy the hypotheses of Theorems 3.1 and 3.2. □

Now, we introduce an example to support our results.

Example 3.1. Let $X = [0,1]$, $E = \mathbb{R}^2$. Let $P = \{(a,b) : a \ge 0, b \ge 0\}$ be the cone with $d(x,y) = (|x-y|, |x-y|)$. Then (X,d) is a complete cone metric space with normal cone P (see [1]). Define $F, G : X \times X \to X$ by $F(x,y) = \frac{1}{8}x^2$, $T(x,y) = 0$ and $f : X \to X$ by $fx = x^2$. Then F, T and f have a unique common coincidence point in $X \times X$.

Proof. For $x, y, u, v \in X$, we have

$$d(F(x,y),G(u,v)) = d\left(\frac{1}{8}x^2, 0\right) = \left(\frac{1}{8}x^2, \frac{1}{8}x^2\right) = \frac{1}{8}\left(x^2, x^2\right) = \frac{1}{8}d(0,x^2) = \frac{1}{8}d(G(u,v),fx) \le \frac{1}{8}(d(F(x,y),fu) + d(G(u,v),fx)).$$

Take $a_1 = a_2 = a_3 = 0$ and $a_4 = \frac{1}{8}$. Then by Theorem 3.1, F, G and f have a common coupled coincidence point. Here $(0, 0)$ is a common coupled coincidence point of F, G and f. Moreover, F and f are w-compatible. To show this, suppose that $fx = F(x, y)$ and $fy = F(y, x)$ for some $(x, y) \in X \times X$. Then $x^2 = \frac{1}{8}x^2$ and $y^2 = \frac{1}{8}y^2$. Hence $x = y = 0$. So, $fF(x, y) = fF(0, 0) = f0 = 0$ and $F(fx, fy) = F(f0, f0) = F(0, 0) = 0$. Therefore, $fF(x, y) = F(fx, fy)$. Hence F and f are w-compatible. By similar way, we may show that G and f are w-compatible. So, by Theorem 3.2 we deduce that $(0, 0)$ is the unique common coupled fixed point of F, G and f. $\square$

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