Unary operations with long pre-periods

C. Ratanaprasert\textsuperscript{a}, K. Denecke\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Silpakorn University, Nakhon Pathom, Thailand
\textsuperscript{b}Institut für Mathematik, Universität Potsdam, Potsdam, Germany

Received 20 March 2005; accepted 18 October 2007
Available online 26 November 2007

Abstract

It is well known that the congruence lattice $\text{Con}\ A$ of an algebra $A$ is uniquely determined by the unary polynomial operations of $A$ (see e.g. [K. Denecke, S.L. Wismath, Universal Algebra and Applications in Theoretical Computer Science, Chapman & Hall, CRC Press, Boca Raton, London, New York, Washington DC, 2002[2]]). Let $A$ be a finite algebra with $|A| = n$. If $\text{Im } f = A$ or $|\text{Im } f| = 1$ for every unary polynomial operation $f$ of $A$, then $A$ is called a permutation algebra. Permutation algebras play an important role in tame congruence theory [D. Hobby, R. McKenzie, The structure of finite algebras, Contemporary Mathematics, vol. 76, Providence, Rhode Island, 1988 [3]]. If $f : A \to A$ is not a permutation then $A \supset \text{Im } f$ and there is a least natural number $\lambda(f)$ with $\text{Im } f^{\lambda(f)} = \text{Im } f^{\lambda(f)+1}$. We consider unary operations with $\lambda(f) = n - 1$ for $n \geq 2$ and $\lambda(f) = n - 2$ for $n \geq 3$ and look for equivalence relations on $A$ which are invariant with respect to such unary operations. As application we show that every finite group which has a unary polynomial operation with one of these properties is simple or has only normal subgroups of index 2.

© 2007 Elsevier B.V. All rights reserved.

MSC: 03B50; 08A30; 08A46

Keywords: Permutation algebra; Unary operation; Pre-period; $LT$-function; $LT_1$-function; Invariant equivalence relation

1. Some facts on unary operations

Let $f : A \to A$ be a unary operation defined on a finite set $A$ with $n := |A|$, for $n \geq 2$. Let $H_A = \{f | f : A \to A\}$ be the set of all unary operations (transformations) defined on $A$ and let $S_A$ be the set of all permutations on $A$. If $\text{id}_A$ is the identity operation on $A$ and $g \in S_A$, then the order $\text{ord}(g)$ of $g$ is the least natural number $m$ with $g^m = \text{id}_A$. Let $f \in H_A$. Then we set $\text{Im } f := \{f(a) | a \in A\}$. Let $\lambda(f)$ be the least natural number $m$ such that $\text{Im } f^m = \text{Im } f^{m+1}$. The number $\lambda(f)$ is called the pre-period of $f$, sometimes also the stabilizer (see e.g. [5]) of $f$. For $f \in H_A$ we put $\omega(f) := \text{ord}(f | \text{Im } f^{\lambda(f)})$ where $f^n$ is defined as $m$-fold composition of $f$ (with $f^0 := \text{id}_A$). Then

Remark 1.1. (i) $0 \leq \lambda(f) \leq |\text{Im } f|$ and $\lambda(f) \leq n - 1$,
(ii) $\lambda(f) = 0 \iff f \in S_A$,
(iii) $\lambda(f) = n - 1$ if and only if there exists an element $d \in A$ such that

$$A = \{d, f(d), f^2(d), \ldots, f^{n-1}(d) = f^n(d)\}.$$

---

\textsuperscript{☆} Research supported by The Thailand Research Fund.
E-mail addresses: ratach@su.ac.th (C. Ratanaprasert), kdenecke@rz.uni-potsdam.de (K. Denecke).

0012-365X/S - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2007.10.027
A unary operation \( f : A \to A \) with \( \lambda(f) = n - 1 \), and \( |A| = n \) was called a \( LT \)-function (long-tailed function) in [1].

Remark 1.1 (iii) characterizes \( LT \)-functions. In the next section we consider unary operations \( f \) with \( \lambda(f) = n - 2 \) for \( n \geq 3 \). The aim of this paper is to characterize equivalence relations which are invariant with respect to unary operations \( f \) with \( \lambda(f) = n - 2 \).

2. \( LT_1 \)-functions

Now we want to characterize unary operations \( f : A \to A \) with \( |A| \geq 3 \) and \( \lambda(f) = n - 2 \).

Definition 2.1. Let \( A \) be a finite set with \( |A| \geq 3 \). Then a unary operation \( f : A \to A \) with \( \lambda(f) = n - 2 \) is said to be a \( LT_1 \)-function.

We will prove some simple properties of \( LT_1 \)-functions:

Lemma 2.2. Let \( f : A \to A \) be a unary operation and assume that \( |A| = n \geq 3 \) and \( \lambda(f) = n - 2 \). Then the following propositions are satisfied:

(i) \( A \supseteq \operatorname{Im} f \supseteq \operatorname{Im} f^2 \supseteq \cdots \supseteq \operatorname{Im} f^{n-2} \),

(ii) \( |\operatorname{Im} f^{n-2}| = 1 \) or \( |\operatorname{Im} f^{n-2}| = 2 \),

(iii) if \( |\operatorname{Im} f^{n-2}| = 2 \), then \( |A| = |\operatorname{Im} f| + 1 \),

(iv) if \( |\operatorname{Im} f^{n-2}| = 1 \), then \( |A| = |\operatorname{Im} f| + 2 \),

(v) \( \operatorname{Im} f^k = |\operatorname{Im} f^{k+1}| + 1 \) for \( k = 1, \ldots, n - 3 \).

Proof. (i) In any cases we have \( A \supseteq \operatorname{Im} f \supseteq \operatorname{Im} f^2 \supseteq \cdots \supseteq \operatorname{Im} f^{n-2} \). Proper inclusions follow from the definition of \( \lambda(f) \) and from \( \lambda(f) = n - 2 \).

(ii) Because of (i) we have \( |\operatorname{Im} f^{n-2}| \leq 2 \) and thus \( |\operatorname{Im} f^{n-2}| = 1 \) or \( |\operatorname{Im} f^{n-2}| = 2 \).

(iii) This is also a consequence of (i).

(iv) By (i) and \( |\operatorname{Im} f^{n-2}| = 1 \) there is a \( k \) with \( 1 \leq k < n - 2 \) such that \( |\operatorname{Im} f^k| = |\operatorname{Im} f^{k+1}| + 2 \). Then there are elements \( a \neq b \) and \( c \neq d \) in \( \operatorname{Im} f^k \) such that \( f(a) = f(b) \) and \( f(c) = f(d) \) in \( \operatorname{Im} f^{k+1} \). Therefore, there are elements \( a, b, c, d \in A \) with \( a \neq b, c \neq d \) such that \( f(a) = f(b) \) and \( f(c) = f(d) \) and this means, \( |A| = |\operatorname{Im} f| + 2 \).

(v) If \( |\operatorname{Im} f^{n-2}| = 2 \), then from (i) we obtain \( |\operatorname{Im} f^k| = |\operatorname{Im} f^{k+1}| + 1 \) for \( k = 0, \ldots, n - 3 \) and if \( |\operatorname{Im} f^{n-2}| = 1 \), then by (iv), \( |A| = |\operatorname{Im} f| + 2 \) and then by (i), \( |\operatorname{Im} f^k| = |\operatorname{Im} f^{k+1}| + 1 \) for \( k = 1, \ldots, n - 3 \).

Our first aim is to characterize unary operations on \( A \) with \( \lambda(f) = n - 2 \). First we will prove some technical lemmas which we need for our characterization theorem. If \( |\operatorname{Im} f^{n-2}| = 1 \), then by Lemma 2.2 (iv), \( |A| = |\operatorname{Im} f| + 2 \). Therefore, there are elements \( a, b, c, d, s, t \in A, a \neq b, c \neq d \) such that \( f(a) = f(b) = s, f(c) = f(d) = t \). Clearly, if \( |A| = 3 \), then \( f \) is constant.

Remark 2.3. If \( |A| \geq 3 \) and \( |A| = |\operatorname{Im} f| + 2 \), then there are different elements \( u, v \in A \) with \( u \neq v \) such that for all \( t' \in A \) we have \( f(t') \notin \{u, v\} \). Moreover, the function \( f|_{A \setminus \{a, b, c, d\}} : A \setminus \{a, b, c, d\} \to A \{s, t, u, v\} \) is a bijection.

Lemma 2.4. Assume that \( f : A \to A \) is a unary operation with \( f(a) = f(b) = s, f(c) = f(d) = t \) and \( |A| = |\operatorname{Im} f| + 2 \). If \( s, t \notin \{a, b, c, d\} \) and \( f(s) \notin \{a, b, c, d\} \) or \( f(t) \notin \{a, b, c, d\} \), then \( |\operatorname{Im} f^k| \geq 2 \) for all \( k \geq 1 \).

Proof. \( s, t \notin \{a, b, c, d\} \) implies \( f(s) \notin \{s, t, u, v\} \) since \( f|_{A \setminus \{a, b, c, d\}} : A \setminus \{a, b, c, d\} \to A \{s, t, u, v\} \) is bijective. Therefore, \( f(s) \neq s \) and the set \( \{s = f^0(s), f(s)\} \) is a two-element subset of \( \operatorname{Im} f \). Inductively, assume that \( \{f^{k-1}(s), f^k(s)\} \) is a two-element subset of \( \operatorname{Im} f^k \). We consider the following cases:

(a) \( f^k(s) \notin \{a, b, c, d\} \) and \( f^{k-1}(s) \notin \{a, b, c, d\} \). Then the injectivity of \( f|_{A \setminus \{a, b, c, d\}} \) implies \( f^k(s) \neq f^{k+1}(s) \).

(b) \( f^k(s) \notin \{a, b, c, d\} \) and \( f^{k-1}(s) \in \{a, b, c, d\} \). Then \( f^k(s) = f(f^{k-1}(s)) \in \{s, t\} \) and \( f^{k+1}(s) = f(f^k(s)) \notin \{s, t\} \), so \( f^{k+1}(s) \neq f^k(s) \).

(c) \( f^k(s) \in \{a, b, c, d\} \). Then \( f^{k+1}(s) \notin \{s, t\} \) where \( s, t \notin \{a, b, c, d\} \) and \( f^k(s) \neq f^{k+1}(s) \).

It follows that \( \{f^k(s), f^{k+1}(s)\} \subseteq \operatorname{Im} f^k \) for all \( k \geq 1 \). This means \( |\operatorname{Im} f^k| \geq 2 \) for all \( k \geq 1 \).
Lemma 2.5. Assume that $\lambda(f) = n - 2$, $|\text{Im} f^{n-2}| = 1$, $f(a) = f(b) = s$ and $f(c) = f(d) = t$. Then,

(i) if $s, t \notin \{a, b, c, d\}$, then $f(s) \not\in \{a, b, c, d\}$ or $f(t) \not\in \{a, b, c, d\}$,

(ii) $s \in \{a, b, c, d\}$ or $t \in \{a, b, c, d\}$,

(iii) if $s \in \{a, b\}$ and $s \neq t$, there exists an $m$ such that $f^m(c) \in \{a, b\}\{s\}$ and $c \in \{a, b\}\{s\}$ \(\cap \{v, u\} \neq \emptyset\) where $u, v \in A$

are the elements which do not occur in $\text{Im} f$(by Remark 2.3),

(iv) if $|A| \geq 4$ and $s = t$, then $\{u, v\} \neq \{b, c\}$ and $\{u, v\} \cap \{b, c\} \neq \emptyset$.

Proof. (i) If $f(s) = s$ and $f(t) = t$ or $f(s) = t$ and $f(t) = s$, then $s, t \in \text{Im} f^k$ for all $k$. This contradicts $|\text{Im} f^{n-1}| = 1$ since $s \neq t$ because of $s, t \notin \{a, b, c, d\}$. Therefore $f(s), f(t) \not\in \{s, t\}$. If $f(s) \in \{a, b\}$ or $f(t) \in \{c, d\}$, then $f^2(s) = s$ or $f^2(t) = t$. This means, $s \in \text{Im} f^{n-2}$ or $t \in \text{Im} f^{n-2}$ and $s \in \text{Im} f^{n-3}$ or $t \in \text{Im} f^{n-3}$ and then $f(s) \neq s \in \text{Im} f^{n-2}$ or $f(t) \neq t \in \text{Im} f^{n-2}$ and $|\text{Im} f^{n-2}| \geq 2$, a contradiction. Thus $f(s) \notin \{a, b\}$ and $f(t) \notin \{c, d\}$. If $f(s) \in \{c, d\}$ and $f(t) \in \{a, b\}$, then $f^2(s) = t$ and $f^2(t) = s$ and then $f(f^2(a)) = f^2(c)$ and $f(f^2(c)) = f^2(a)$. Therefore, $f^2(a)$ and $f^2(c)$ are two elements of $A$ which are mapped to each other and thus they belong to $\text{Im} f^k$ for all $k \geq 1$. Since $f(s) \neq f(t)$, we have $|\text{Im} f^{n-2}| \geq 2$, a contradiction. This means $f(s) \notin \{c, d\}$, or $f(t) \notin \{a, b\}$. Therefore, $f(s) \notin \{a, b, c, d\}$ or $f(t) \notin \{a, b, c, d\}$.

(ii) If $s, t \notin \{a, b, c, d\}$, then by (i) and Lemma 2.4 we get $|\text{Im} f^k| \geq 2$ for all $k \geq 1$ and then $|\text{Im} f^{n-2}| \geq 2$, a contradiction.

(iii) Clearly, $f^{n-2}(c) \in \text{Im} f^{n-2}$ and $f^{n-2}(c) = s$. Let $r$ be the least positive integer such that $f^r(c) = s$. Then $1 < r < n - 2$ and $1 \leq r < n - 2$ and $f^{r-1}(c) \in \{a, b\}$. Indeed, from $f^{r-1}(c) \notin \{a, b\}$ would follow $|\text{Im} f| \leq |A| - 3$, a contradiction. By the choice of $r$ we have $f^{r-1}(c) \in \{a, b\}\{s\}$ with $r - 1 \geq 1$. We choose $m = r - 1$.

Next, suppose that $\{c, d\} \cap \{u, v\} = \emptyset$. Then $\{c, d\} \subseteq \text{Im} f$. Therefore, there are $p, q \in A$ with $f(p) = c$ and $f(q) = d$. Now $a, b, c, d \in \text{Im} f$ with $f(a) = f(b)$ and $f(c) = f(d)$ implies that $|\text{Im} f^2| \leq |\text{Im} f| - 2$, a contradiction.

(iv) Suppose that $\{u, v\} \subseteq \{a, b, c\}$ or $\{u, v\} \cap \{b, c\} = \emptyset$.

(a) $\{u, v\} = \{b, c\}$. Then $A \setminus \{u, v\} = A \setminus \{a, b, c\}$ and $f |A\setminus\{a,b,c\}$ is a permutation on $A \setminus \{a, b, c\}$. Since $|A| = n \geq 4$, we have $|A \setminus \{a, b, c\}| \geq 1$ and $\emptyset = A \setminus \{a, b, c\} \subseteq \text{Im} f^k$ for all $k \geq 1$. Since $a \in \text{Im} f^{n-2}$, we have $|\text{Im} f^{n-2}| \geq 2$, a contradiction.

(b) $\{u, v\} \cap \{b, c\} = \emptyset$ and then $\{b, c\} \subseteq \text{Im} f$. Then there are elements $p, q \in A$ such that $f(p) = b \neq c = f(q)$ which implies that $\{f(a) = a, f(p) = b, f(q) = c\}$ is a subset of $\text{Im} f$. But $f^2(q) = f^2(q) = f^2(a) = a \in f^2$, which shows that $|\text{Im} f^2| \leq |\text{Im} f| - 2 < |\text{Im} f| - 1$, a contradiction. \(\Box\)

From Lemma 2.2 (ii) it follows that we have to consider the two cases $|\text{Im} f^{n-2}| = 1$ and $|\text{Im} f^{n-2}| = 2$. Now we will give a characterization of $LT_1$-functions with $|\text{Im} f^{n-2}| = 1$ which corresponds to the characterization of $LT$-functions given in Remark 1.1.

Theorem 2.6. Let $A$ be a finite set with $|A| = n \geq 3$ and let $f : A \to A$ be an operation. Then $\lambda(f) = n - 2$ and $|\text{Im} f^{n-2}| = 1$ if and only if there are distinct elements $u, v \in A$ such that $A = \{u, v, f(v), \ldots, f^{n-2}v\}$ and such that there is an exponent $k, 0 \leq k \leq n - 2$, with $f(u) = f^{k+1}(v)$ and a number $m$ with $m + k = n - 2$ such that $f^{m+1}(u) = f^m(u)$.

Proof. Assume that $\lambda(f) = n - 2$ and $|\text{Im} f^{n-2}| = 1$. Then by Lemma 2.2 (iv) we get $|A| = |\text{Im} f| + 2$ and there are two elements $u, v$ with $u \neq v$ which do not occur in $\text{Im} f$. Clearly, the restriction $g$ of $f$ on the $(n - 2)$-element set $\text{Im} f$ is a $LT$-function. Then by Remark 1.1 (iii) there is an element $d \in \text{Im} f$ such that $\text{Im} f = \{d, g(d), \ldots, g^{n-3}(d)\} = \{d, f|\text{Im} f(d), \ldots, f^{n-3}|\text{Im} f(d)\}$ and $g^{n-3}(d) = g^{n-2}(d)$. Since $d \in \text{Im} f$, there is an element $q \in A$ such that $d = f(q)$. Since $q \notin \text{Im} f$ we have $q = v$ or $q = u$. Without loss of generality let $q = v$. Further, the second element $u$ which does not belong to $\text{Im} f$ cannot be mapped to $u$ or to $v$. Therefore, it is mapped to one of the elements $d, g(d), \ldots, g^{n-3}(d)$; let us say to $g^k(d) = f^{k+1}(v) = f(u)$. Then $A = \{v, f(v), \ldots, f^k(v)\} \cup \{u, f(u), \ldots, f^m(u)\}$ with $m + k = n - 2$ and because of $|\text{Im} f^{n-2}| = 1$, we have $f^m(u) = f^{m+1}(u)$.

Conversely, we assume that there are different elements $u, v \in A$ such that $A = \{u, v, f(v), \ldots, f^{n-2}v\}$ and such that there is an exponent $k, 0 \leq k \leq n - 2$, with $f(u) = f^{k+1}(v)$ and an integer $m$ with $m + k = n - 2$ and such that $f^{m+1}(u) = f^m(u)$. Clearly, all elements are pairwise distinct and we have $a := f^m(u) \neq f^{m-1}(u) := b$ and $c := f^k(v) \neq u$. Then $f(a) = f^{m+1}(u) = f^m(u) = f(b)$ and $f(c) = f^{k+1}(v) = f(u)$. Hence, $|A| = |\text{Im} f| + 2$. 

Next, we show that $\lambda(f) = n - 2$. Because of $|A| = |\text{Im } f| + 2$, it is enough to show that $A \supset \text{Im } f \supset \text{Im } f^2 \supset \cdots \supset \text{Im } f^{n-2}$.

Since $|A| = |\text{Im } f| + 2$ and $\text{Im } f \supset \{f(v), \ldots, f^{n-2}(v)\}$ we get $\text{Im } f = \{f(v), \ldots, f^{n-2}(v)\}$. Now we consider the cases $m = 1$ and $m > 1$.

If $m = 1$, then $A = \{u, f(u)\} \cup \{v, f(v), \ldots, f^k(v)\}$. If $k = 0$, then $A = \{v, u, f(u)\}$ and $\text{Im } f = \{f(u)\}$, $n = 3$, $\lambda(f) = 1 = 3 - 2$, $|\text{Im } f| = 1$, i.e. $f$ is a $LT_1$-function. If $k > 0$, then $f^k(v) \in \text{Im } f^t$ for all $1 \leq t \leq k$ where $n - 2 = k + 1$; and in this case we have $f(u) \neq f^k(v) \in \text{Im } f^t$, but $f(u) = f^k(v) \notin \text{Im } f^{t+1}$ for all $1 \leq t < k$ which implies $|\text{Im } f^t| \supset |\text{Im } f^{t+1}|$ for all $1 \leq t < k$ and $f$ is a $LT_1$-function.

If $m > 1$, then $f^{k+1}(v) = f(u)$ implies $f^{m-1}(u) = f^{m-2}(f(u)) = f^{m-2}(f^{k+1}(v)) = f^{m-k-1}(v) = f^{-n-3}(v) \in \text{Im } f^{n-3}$. Since $\text{Im } f^{n-3} \supset \text{Im } f^t$ for all $t \leq n - 3$, we have $f^{m-1}(u) \in \text{Im } f^t$ for all $1 \leq t < n - 2$. Now $f^{m-1}(u) \neq f^m(u)$ in $\text{Im } f^t$, whereas $f(f^{m-1}(u)) = f(f^m(u))$ implies that $\text{Im } f^t \supset \text{Im } f^{t+1}$ for all $1 \leq t < n - 2$. This shows $A \supset \text{Im } f \supset \text{Im } f^2 \cdots \supset \text{Im } f^{n-2}$ and together with $|A| = |\text{Im } f| + 2$, this means $|\text{Im } f^{n-2}| = 1$ and $\lambda(f) = n - 2$. This finishes the proof. □

Now we consider the case that $|\text{Im } f^{n-2}| = 2$. Then $|A| = |\text{Im } f| + 1$ and $|\text{Im } f^k| = |\text{Im } f^{k+1}| + 1$ for $k = 1, \ldots, n - 3$. The restriction $f^{n-2}|_{\text{Im } f}$ can be the identity function or another permutation on a two-element set.

**Theorem 2.7.** Let $A$ be a set with $|A| = n \geq 3$ and let $f : A \to A$ be a unary operation. Then $\lambda(f) = n - 2$ and $|\text{Im } f^{n-2}| = 2$ if and only if there are different elements $u, v \in A$ such that either

(i) $A = \{u, v, f(u), \ldots, f^{n-2}(u)\}$ with $v = f(v)$ and $f^{n-1}(u) = f^{-n-2}(u)$, or

(ii) $A = \{u, f(u), f^2(u), \ldots, f^{n-2}(u), f^{n-1}(u)\}$ where $v = f^n(u) = f^{n-2}(u)$.

**Proof.** By Lemma 2.2 (iii) we have $|A| = |\text{Im } f| + 1 > |\text{Im } f|$. Therefore, there are exactly two distinct elements $a, b \in A$ such that $f(a) = f(b) = s$ and there is an element $y \in A$ with $y \neq f(t)$ for all $t \in A$. Then the restriction of $f$ to $A \setminus \{a, b\}$ is a bijective mapping $f|_{A \setminus \{a, b\}} : A \setminus \{a, b\} \to A \{s, y\}$. We consider the two cases $s \in \{a, b\}$ and $s \notin \{a, b\}.

**Case 1:** $s \in \{a, b\}$. Without loss of generality we may assume that $s = a$. Then $f(s) = s \in \text{Im } f^{n-2}$. Now we consider two subcases, $y \in \{a, b\}$ and $y \notin \{a, b\}$.

**Case 1.1:** $y \in \{a, b\}$. Then $\{a, b\} = \{s, y\}$ and $f|_{A \setminus \{a, b\}}$ is a permutation. Since $|A| \geq 3$ and $|\text{Im } f^{n-2}| = 2$, we have $|A| = 3$ because of $s \in \text{Im } f^{n-2}$. Then $A = \{a, y, f(y)\}$ with $f(a) = a$ and $f^2(y) = f(y)$. This corresponds to (i).

**Case 1.2:** $y \notin \{a, b\}$. Because of the bijectivity of $f|_{A \setminus \{a, b\}} : A \setminus \{a, b\} \to A \{s, y\}$, we can choose elements $x_1, x_2, \ldots, x_{q-1}$ from $A \setminus \{a, b, y\}$ and $x_q = y$ such that $f(x_1) = b$ and $f(x_i) = x_{i-1}$ for $1 \leq i \leq q$. So $X = \{x_q, f(x_q), \ldots, f^q(x_q) \} = \{b, f(b), \ldots, f^q(b)\} \subseteq A$. Since $|\text{Im } f^{n-2}| = 2$, we have $A \setminus X \neq \emptyset$ and $f|_{A \setminus X}$ is a permutation. Thus $|A \setminus X| = 1$ and $|A| = q + 3$. It follows that $A = \{c, y, f(y), \ldots, f^{q+1}(y)\}$ with $c \in A \setminus X$, $f(c) = c$ and $f^{q+2}(y) = f^{q+1}(y)$. This corresponds to (i).

**Case 2:** $s \notin \{a, b\}$. Then $s$ is not a fixed point with respect to $f$ since otherwise $|A| > |\text{Im } f| - 2$, a contradiction. We consider the two subcases $f(s) \notin \{a, b\}$ and $f(s) \in \{a, b\}$.

**Case 2.1:** $f(s) \notin \{a, b\}$. Then there is a $k \geq 1$ such that $\{s, f(s), \ldots, f^k(s)\}$ is a subset of $A \setminus \{a, b\}$ consisting of pairwise distinct elements. From the injectivity of $f|_{A \setminus \{a, b\}}$ and $f^{k-1}(s) \neq f^k(s)$ we obtain $f^k(s) \neq f^{k+1}(s)$. If $f^{k+1}(s) = f^t(s)$ for some $1 \leq t < k$, then $1 \leq t - 1 < k$ and $f(f^k(s)) = f(f^{t-1}(s))$ implies $f^{k+1}(s) = f^{t+1}(s)$ which contradicts our assumption. If $f^{k+1}(s) = s$, then by Lemma 2.2 we get another contradiction. This means, $X = \{s, f(s), \ldots, f^k(s)\}$ is an infinite set, a contradiction.

**Case 2.2:** $f(s) \in \{a, b\}$. Assume that $f(s) = a$. Since $f|_{\{a, s\}}$ is a permutation, we obtain $a, s \in \text{Im } f^{n-2}$. So, $f(t) \notin \{a, s\}$ for all $t \notin \{a, s\}$ since $|A| = |\text{Im } f| + 1$. If $y = b$, then $A = \{a, b, s\} = A \{a, s\}$. Hence $f|_{A \setminus \{a, b, s\}}$ is a permutation. There follows that $A = \{a, b, s\} = \{b, f(b), f^2(b)\}$ where $f^3(b) = f(b)$. This corresponds to (ii). The other subcase of 2.2 is that $y \neq b$. Then $b \in A \{s, y\}$. By surjectivity of $f|_{A \setminus \{a, b\}}$ onto $A \{s, y\}$ and because of the finiteness of $A$ we may choose $q - 1$ pairwise distinct elements $x_1, x_2, \ldots, x_{q-1} \in A \{y, s, a, b\}$ which are different from $a$ and from $b$ such that $f(x_i) = f(x_{i-1})$ for $1 \leq i \leq q$ and with $x_0 = b, x_q = y$. Therefore, $X = \{x_q, f(x_q), \ldots, f^q(x_q) = b, f^{q+1}(x_q) = s, f^{q+2}(x_q) = a\} \subseteq A$ and $f|_{A \setminus X}$ is a permutation. Assume that $A \setminus X \neq \emptyset$. Then $f|_{A \setminus X}$ is a permutation and thus $|\text{Im } f^{n-2}| \geq 3$, a contradiction. Thus $A = X$ and $q + 2 = n - 1$, i.e. $q = n - 3$. With $u = x_q$ we have $A = \{u, f(u), \ldots, f^{n-2}(u)\}$ with $f^n(u) = f^{n-2}(u)$ and this corresponds to (ii).
Conversely, let $A$ be a finite set with $|A| \geq 3$ and let $f : A \to A$ be a function satisfying (i) or (ii). Then we have $f^{n-2}(u) \neq f^{n-3}(u)$ (in case (i)) or $f^{n-1}(u) \neq f^{n-2}(u)$ (in case (ii)) but $f(f^{n-2}(u)) = f^{n-1}(u) = f^{n-2}(u) = f(f^{n-3}(u))$ (in case (i)) or $f(f^{n-1}(u)) = f^n(u) = f^{n-2}(u) = f(f^{n-3}(u))$ (in case (ii)). In either cases, we have $A \supset \text{Im } f$.

If $n = 3$, then $A = \{a, b, f(b)\}$ where $f(a) = a$ and $f^2(b) = f(b)$. Thus $\lambda(f) = 1 = 3 - 2$ and $|\text{Im } f| = 2$, i.e. $f$ is a $LT_1$-function. Therefore we may assume that $n \geq 4$. We show that $A \supset \text{Im } f \supset \text{Im } f^2 \supset \cdots \supset \text{Im } f^{n-2} = \text{Im } f^{n-1}$. Since $u \notin \text{Im } f$, we have $A \supset \text{Im } f$. In case (i) the elements $f^{n-3}(u)$ and $f^{n-2}(u)$ are distinct in $\text{Im } f^j$ and have the same image in $\text{Im } f^{j+1}$ for all $1 \leq t \leq n - 3$. Similarly, in case (ii), we have $f^{n-3}(u)$ and $f^{n-1}(u)$ being distinct elements in $\text{Im } f^j$ having the same image in $\text{Im } f^{j+1}$ for all $1 \leq t \leq n - 3$. This shows that $|\text{Im } f^j| \geq |\text{Im } f^{j+1}| + 1$ which implies that $\text{Im } f^j \supset \text{Im } f^{j+1}$ for all $1 \leq t \leq n - 3$. Therefore we have $\lambda(f) = n - 2$. It is left to show that $|\text{Im } f^{n-2}| = 2$. In case (i), $f$ has two different fixed points, $v$ and $f^{n-2}(u)$. Both are elements of $\text{Im } f^{n-2}$. Therefore $|\text{Im } f^{n-2}| = 2$. In case (ii), $f^{n-1}(u)$ and $f^{n-2}(u)$ are different elements from $\text{Im } f^{n-2}$ and therefore, using $\lambda(f) = n - 2$, we have $|\text{Im } f^{n-2}| = 2$. \□

3. Invariant equivalence relations

Let $\theta \subseteq A \times A$ be an equivalence relation on the finite set $A$, with $|A| \geq 2$ and let $f : A \to A$ be an arbitrary unary operation defined on $A$. Then we say, $f$ preserves $\theta$, or $\theta$ is invariant with respect to $f$ if the following is satisfied

$$\forall a, b \in A ((a, b) \in \theta \Rightarrow (f(a), f(b)) \in \theta).$$

Let $\text{Pol}^{(1)} \theta$ be the set of all functions defined on $A$ which preserve $\theta$. Then we ask the following question: which equivalence relations are invariant with respect to $LT$- or $LT_1$-functions? For $LT$-functions the answer is given by the following theorem.

**Theorem 3.1.** Let $A$ be a finite set with $|A| \geq 2$ and let $\theta$ be a non-trivial equivalence relation defined on $A$. Then $f \in \text{Pol}^{(1)} \theta$ is a $LT$-function if and only if there is only one block with respect to $\theta$ which has more than one element.

**Proof.** Since $\lambda(f) = n - 1$, there is an element $d \in A$ such that $A = \{d, f(d), \ldots, f^{n-1}(d)\}$ and $f^{n-1}(d) = f^n(d)$. Since $\theta$ is non-trivial, there is a block $B$ with respect to $\theta$ containing more than one element. Then there exists a least integer $i \geq 0$ and an integer $j$ with $i < j \leq n - 1$ such that $(f^i(d), f^j(d)) \in \theta$. From this we obtain $(f^s(d), f^{n-1}(d)) \in \theta$ for all $s \geq i$. This means, the elements $f^i(d), f^{i+1}(d), \ldots, f^{n-1}(d)$ belong to $B$; and by the choice of $i$, all other elements form singleton blocks.

Conversely, let $\theta$ be a non-trivial equivalence relation on $A$ having only one block with more than one element. We denote the elements of $A$ by $a_0, \ldots, a_{n-1}$ and may assume that $\{a_i, \ldots, a_{n-1}\}$ is the only non-trivial block with respect to $\theta$. Then the operation $f : A \to A$ defined by $f(a_j) = a_{j+1}$ for $0 \leq j < n - 1$ and $f(a_{n-1}) = a_{n-1}$ preserves $\theta$ and is obviously a $LT$-function. \□

We will answer to the same question for $LT_1$-functions, i.e. if $\lambda(f) = n - 2$. Here we have again to distinguish the cases $|\text{Im } f^{n-2}| = 1$ and $|\text{Im } f^{n-2}| = 2$.

**Proposition 3.2.** Let $A$ be a finite set with $|A| = n \geq 3$ and $\theta \subseteq A \times A$ be a non-trivial equivalence relation on $A$. Then there is a unary operation $f$ with $\lambda(f) = n - 2$ and $|\text{Im } f^{n-2}| = 1$ which preserves $\theta$ if and only if either

(i) there exists only one block with respect to $\theta$ with more than one element, or

(ii) there are exactly two blocks with respect to $\theta$ with more than one element and one of them consists of exactly two elements.

**Proof.** Assume that $f : A \to A$ with $\lambda(f) = n - 2$ and $|\text{Im } f^{n-2}| = 1$. Then there are distinct elements $u, v \in A$ and integers $m \geq 1$ and $k \geq 0$ such that $m + k = n - 2$ and $A = \{u, v, f(v), \ldots, f^{n-2}(v)\} = \{u, f(u), \ldots, f^m(u)\} \cup \{v, f(v), \ldots, f^k(v)\}$ where $f^{m+1}(u) = f^m(u) = f^{m+k+1}(v) = f^{m+k}(v) = f(u) = f^{k+1}(v)$. Let $\theta$ be a non-trivial equivalence relation on $A$ which is invariant with respect to $f$ and assume that $X := \{u, v, f(v), \ldots, f^{k+m}(v)\}$ and $Y := \{u, f(u), \ldots, f^m(u)\}$. Then $|X| \geq 2$ and $|Y| \geq 2$ and $f|X$ and $f|Y$ are $LT$-functions. Moreover, $f|X$ preserves
\( \overline{\theta} := \emptyset_{X \times X} \) and \( f|_Y \) preserves \( \overline{\theta} := \emptyset|_Y \). By Theorem 3.1, there is exactly one block with respect to \( \overline{\theta} \) and with respect to \( \overline{\theta} \); respectively, which has more than one element. We consider the following cases:

**Case 1:** The block of \( u \) with respect to \( \overline{\theta} \) consists of only one element. Then \( \theta = \emptyset|_X \cup \{(u, u)\} = \overline{\theta} \cup \{(u, u)\} \).

Hence, there exists only one element of \( A/\theta \) having cardinality greater than one.

**Case 2:** \( (u, f^1(v)) \in \theta \) for some \( 0 \leq t < k \). If \( t = 0 \), then \( (u, v) \in \emptyset \) which implies \( f^{k+1}(v) = f(u) \), \( f(v) \in \emptyset \), so \( \{f(v), \ldots, f^{m+k}(v)\} = X \setminus \{v\} \) is a subset of the block \( C \) of the element \( f(v) \) with respect to \( \overline{\theta} = \emptyset|_X \) and also with respect to \( \theta \); hence \( \emptyset = A \times A \) if \( u \in C \), a contradiction since \( \emptyset \) is non-trivial. This shows \( u \notin C \) and \( B = \{u, v\} \) and \( C = X \setminus \{v\} \) are the only elements of \( A/\theta \) which have more than one element. Since \( k > 0 \), this gives (ii).

If \( t > 0 \), then also \( k > 0 \) and \( f(u) \neq f^k(v) \). Then \( \{f^{t+1}(v), \ldots, f^{k+1}(v) = f(u), \ldots, f^{m+k}(v)\} \) is a subset of the block \( C \) of the element \( f(u) \) with respect to \( \overline{\theta} \) (and also with respect to \( \theta \)) containing \( f^k(v) \) and \( f(u) \); hence \( |C| > 1 \).

If \( u \in C \), then \( C \) is the only block with respect to \( \theta \) having cardinality greater than 1 and if \( u \notin C \), then \( \{u, f^1(u)\} \) and \( C \) are the only blocks with respect to \( \theta \) having cardinalities greater than 1 and \( |\{u, f^1(u)\}| = 2 \).

**Case 3:** \( (u, f^t(v)) \in \emptyset \) for some \( 1 \leq t \leq m \) and \( (u, f^s(v)) \notin \emptyset \) for all \( 0 \leq s < k \). Then \( Y \) is a block with respect to \( \emptyset \) and the block of each \( f^s(v) \) for \( 0 \leq s < k \) is singleton. Therefore, \( Y \) is the only block with respect to \( \theta \) with \( |Y| \geq 2 \).

**Case 4:** \( (u, f^t(v)) \in \emptyset \). If \( (c, d) \notin \emptyset \) for all \( c \neq d \) in \( A \setminus \{u, f^k(v)\} \), then \( \{u, f^k(v)\} \) is the only block with respect to \( \theta \) having more than one element. We consider the case that there are \( c \neq d \) in \( A \setminus \{u, f^k(v)\} \) such that \( (c, d) \notin \emptyset \). If \( c \) or \( d \) belong to \( X \setminus Y \) then \( \{c, d, f^k(v)\} \) is a subset of the only block \( C \) with respect to \( \overline{\theta} \) (hence with respect to \( \theta \)) with \( |C| > 1 \); and so, \( C \cup \{u\} \) is the only block with respect to \( \theta \) which has more than one element. But, if \( c \) and \( d \) both are in \( Y \setminus \{u\} \) then they are in the only block \( C \) with respect to \( \overline{\theta} \) (hence with respect to \( \theta \)) with \( |C| > 1 \); so, in this case, \( C \) and \( \{u, f^k(v)\} \) are the only blocks having more than one element and one of them has cardinality 2.

Conversely, let \( A \) be a set with \( |A| = n \geq 3 \) and let \( \emptyset \subseteq A \times A \) satisfy either case (i) or case (ii). We may assume that \( A = \{a_0, a_1, \ldots, a_{n-1}\} \) and either \( B = \{a_i, a_{i+1}, \ldots, a_{n-1}\} \) for some \( 0 < i < n - 1 \) in case (i) or \( B = \{a_0, a_i\} \) and \( C = \{a_i, a_{i+1}, \ldots, a_{n-1}\} \) for some \( 0 < i < n - 1 \) in case (ii) are the blocks with respect to \( \emptyset \). In either case, we define \( f : A \rightarrow A \) by \( f(a_j) = a_{j+1} \) if \( j \notin \{0, n - 1\} \), \( f(a_0) = a_{i+1} \) and \( f(a_{n-1}) = a_{n-i} \). In both cases \( f \) preserves \( \emptyset \). Further, we have \( f(a_1) = f(a_0) \) and \( f(a_{n-1}) = a_{n-1} = f(a_{n-i}) \) for \( a_i \neq a_0 \) and \( a_{n-1} \neq a_{n-i} \) and there are no other elements \( c \neq d \) in \( A \) such that \( f(c) = f(d) \). Thus \( |\operatorname{Im} f| = |A| - 2 \). Since \( a_{n-1} \neq a_0 \), and \( a_{n-1}, a_{n-2} \in \operatorname{Im} f \) for all \( 1 \leq k < n - 2 \) and \( f(a_{n-1}) = f(a_{n-2}) \), we have \( \operatorname{Im} f^k \supset \operatorname{Im} f^{k+1} \) for all \( 1 \leq k < n - 2 \). Together with \( A = |\operatorname{Im} f| + 2 \) we get \( |\operatorname{Im} f^{n-2}| \leq 1 \).

But, \( a_{n-1} \in \operatorname{Im} f^{n-2} \) implies \( |\operatorname{Im} f^{n-2}| = 1 \) and \( \operatorname{Im} f^{n-2} = \operatorname{Im} f^{n-3} \) and therefore \( \lambda(f) = n - 2 \).

Now we will consider the case \( |\operatorname{Im} f^{n-2}| = 2 \).

**Proposition 3.3.** Let \( A \) be a finite set with \( n \geq 3 \) and let \( \emptyset \subseteq A \times A \) be a non-trivial equivalence relation. Then there is a unary operation \( f : A \rightarrow A \) with \( \lambda(f) = n - 2 \) and \( |\operatorname{Im} f^{n-2}| = 2 \) such that \( \emptyset \) is invariant under \( f \) if and only if either

(i) there is only one block \( B \) with respect to \( \emptyset \) which has more than one element; or

(ii) there are only two blocks \( B \) and \( C \) with respect to \( \emptyset \) which have more than one element and \( |B| - |C| \leq 1 \).

**Proof.** By Theorem 2.7, there are different elements \( u, v \in A \) such that either

(i) \( A = \{v, u, f(u), \ldots, f^{n-2}(u)\} \) with \( v = f(v) \) and \( f^{n-1}(u) = f^{n-2}(u) \), or

(ii) \( A = \{u, f(u), v = f^{n-2}(u), f^{n-1}(u)\} \) where \( v = f^n(u) = f^{n-2}(u) \).

First we consider case (i).

(i) Let \( X := A \setminus \{v\} \). Then \( f|X \) is a LT-function and \( \emptyset|_X \times X \) is invariant with respect to \( f|X \). Therefore, there is only one block \( B' \) with respect to \( \emptyset|_X \times X \) which has more than one element. Let \( B \) be the least block with respect to \( \emptyset \) which contains the block \( B' \). If \( v \notin B \), then \( \emptyset|_X \times X \cup \{(v, v)\} \) and therefore \( B \) is the only block with respect to \( \emptyset \) with more than one element. If \( v \in B \), then \( (v, f^1(u)) \in \emptyset \) for some \( 0 \leq t \leq n - 2 \) and thus \( (v, f^t(u)) \in \emptyset \) for all \( s \) with \( t \leq s \leq n - 2 \) and then also \( (v, f^{n-2}(u)) \in \emptyset \). This means that every block \( B \) with respect to \( \emptyset \) with \( |B| > 1 \) and \( v \in B \) contains also \( f^{n-2}(u) \).

Now \( B = \{v, f^{n-2}(u)\} \) implies that the block of each \( f^t(u) \) for \( 0 \leq t < n - 2 \) is singleton, hence \( B \) is the only block with respect to \( \emptyset \) which has more than one element. But, if \( \{v, f^{n-2}(u)\} \) is a proper subset of \( B \), then \( B \setminus \{v\} \) is the only
block with respect to $\theta|_{X \times X}$ with $|B\setminus\{u\}| > 1$ which implies that the block of each $f^i(u) \notin B$ with respect to $\theta|_{X \times X}$ (and also with respect to $\theta$) is singleton; hence, $B$ is the only block with respect to $\theta$ with $|B| > 1$.

Now we consider the second case.

(ii) From $u = f^n(u) = f^{n-2}(u)$ we obtain $f^{2q}(f^{n-1}(u)) = f^{n-1}(u)$, $f^{2q}(f^{n-2}(u)) = f^{n-2}(u)$, $f^{2q+1}(f^{n-1}(u)) = f^{n-1}(u)$ and $f^{2q+1}(f^{n-2}(u)) = f^{n-2}(u)$ for all $q \geq 1$. Since $\theta$ is non-trivial, there are integers $i, j$ with $0 \leq i < j \leq n-1$ such that $(f^i(u), f^j(u)) \notin \emptyset$. From this we obtain easily $(f^i(u), f^{n-1}(u)) \notin \emptyset$ or $(f^i(u), f^{n-2}(u)) \notin \emptyset$. If $(f^{n-1}(u), f^{n-2}(u)) \notin \emptyset$, there is only one block $B$ with respect to $\theta$ such that $|B| \geq 2$. We assume that $(f^{n-1}(u), f^{n-2}(u)) \notin \emptyset$. Then we get $(f^i(u), f^{n-2}(u)) \notin \emptyset$ or $(f^i(u), f^{n-2}(u)) \notin \emptyset$. Without restriction of the generality we assume that $(f^i(u), f^{n-1}(u)) \notin \emptyset$. Let $f^{n-1}(u) \in B \in A/\theta$ and let $f^{n-2}(u) \in C \in A/\theta$. Then $|B| \geq 2$, $|C| \geq 1$ and $B \cap C = \emptyset$.

Now let $D$ be a block of $\theta$ with more than one element. Then $(f^s(u), f^t(u)) \notin \emptyset$ for some $0 \leq s < t < n-1$. Then from $(f^s(u), f^{n-1}(u)) \notin \emptyset$ or $(f^s(u), f^{n-2}(u)) \notin \emptyset$ there follows that either $D = B$ or $D = C$. In either cases, if $|C| = 1$, there is only one $B \in A/\theta$ such that $|B| \geq 2$. If $|C| \geq 2$, then $B$ and $C$ are the only two elements of $A/\theta$ containing more than one element. For the proof of the last statement in (ii) we have only to consider the case that $(f^{n-1}(u), f^{n-2}(u)) \notin \emptyset$. Let $i$ with $0 \leq i < n - 1$ be the least integer such that $(f^i(u), f^j(u)) \notin \emptyset$ for some $j > i$. If $i$ and $j$ have different parity, it can be easily checked that $(f^{n-1}(u), f^{n-2}(u)) \notin \emptyset$. Therefore $(f^i(u), f^{i+1}(u)) \notin \emptyset$. Let $f^i(u) \in B \in A/\theta$ and $f^{i+1}(u) \in C \in A/\theta$. Then $|B| \geq 2$ since $f^i(u) \notin f^j(u)$ are in $B$. Then we have either $f^{n-1}(u) \in B$ or $f^{n-2}(u) \in B$. In the first case we get $f^{n-2}(u) \in C$ and if $f^s(u), f^t(u) \in B$ or $f^s(u), f^t(u) \in C$, we get $s - t = 2q$ for some $q \geq 1$. By the choice of $i$, we can write $B = \{f^i(u), f^{i+2}(u), \ldots, f^{n-3}(u), f^{n-1}(u)\}$ and $C = \{f^{i+1}(u), f^{i+3}(u), \ldots, f^{n-4}(u), f^{n-2}(u)\}$. So, the function $\alpha : B \to C$, defined by $\alpha(f^i(u)) = f^{i+1}(u)$ if $t \neq n - 1$ and $\alpha(f^{n-1}(u)) = f^{n-2}(u)$ is injective on $B \setminus \{f^{n-1}(u)\}$. Together with $\alpha(f^{n-3}(u)) = f^{n-2}(u) = \alpha(f^{n-1}(u))$ this gives $|B| = |C| + 1$.

In the case $f^{n-2}(u) \in B$ we have $f^{n-1}(u) \in C$ and

$$B = \{f^i(u), f^{i+2}(u), \ldots, f^{n-4}(u), f^{n-2}(u)\},$$

and

$$C = \{f^{i+1}(u), f^{i+3}(u), \ldots, f^{n-3}(u), f^{n-1}(u)\}.$$ 

In this case, $\alpha : B \to C$ defined by $\alpha(f^i(u)) = f^{i+1}(u)$ for $1 \leq t \leq n - 2$ is a bijection and $|B| = |C|$.

Conversely, let $A$ be a set with $|A| = n \geq 3$ and let $\theta \subseteq A \times A$ satisfy either case (i) or case (ii). We may assume that $A = \{a_0, a_1, \ldots, a_{n-1}\}$ and either $B = \{a_i, a_{i+1}, \ldots, a_{n-1}\}$ for some $0 < i < n - 1$ in case (i) or $B = \{a_i, a_{i+2}, a_{i+4}, \ldots, a_{n-1}\}$ and $C = \{a_{i+1}, a_{i+3}, \ldots, a_{n-2}\}$ for some $i$ with $0 < i < n - 1$ such that $i$ and $n - 1$ have the same parity. In case (i) we define $f : A \to A$ by $f(a_i) = a_{i+1}$ if $a_i \notin \{a_{n-2}, a_{n-1}\}$, $f(a_i) = a_i$ if $a_i \in \{a_{n-2}, a_{n-1}\}$. In case (ii), let us define $f : A \to A$ by $f(a_i) = a_{i+1}$ if $a_i \neq a_{n-1}$ and $f(a_{n-1}) = a_{n-2}$. In either cases, it is clear that $\theta$ is invariant with respect to $f$.

Since $f(a_{n-3}) = f(a_{n-2}) = a_{n-2}$ in case (i) and $f(a_{n-3}) = a_{n-2} = f(a_{n-1})$ in case (ii) we have $\text{Im} f \subseteq A$ in both cases.

Since $f|_{\lambda(a_{n-1})}$ is a $LT$-function and $f(a_{n-1}) = a_{n-1}$ in case (i) we have $\lambda(f) = n - 2$ in this case. In case (ii), $a_{n-3}$ and $a_{n-2}$ are in $\text{Im} f^k$ for all $k$ with $1 \leq k < n - 2$ which implies that

$$A \supseteq \text{Im} f \supseteq \cdots \supseteq \text{Im} f^{n-2} \text{ and } \text{Im} f^{n-2} = \{a_{n-1}, a_{n-2}\};$$

hence $\text{Im} f^{n-2} = \text{Im} f^{n-1}$. So $|\text{Im} f^{n-2}| = 2$ and $\lambda(f) = n - 2$. □

4. Applications

First of all we consider affine complete algebras [4] with small cardinalities of its universes.

**Definition 4.1.** An algebra is called **affine complete** if its polynomial operations are precisely the congruence preserving operations [4].

Let $\mathcal{A}$ be a non-simple finite algebra. Then $|A| > 2$. Non-trivial partitions of sets with 3, 4 or 5 elements consist of only one block with more than one element or for $|A| = 4$ of two two-element blocks or for $|A| = 5$ of one-
element and two two-element blocks or of a two-element and a three-element block. Therefore, as a consequence of Theorem 3.1, Propositions 3.2 and 3.3 we have:

**Corollary 4.2.** If \( \mathcal{A} \) is an affine complete non-simple algebra with \(|A| \leq 5\), then \( \mathcal{A} \) has a LT-function or a LT\(_1\)-function among its unary polynomial operations.

**Proof.** Since \( \mathcal{A} \) is not simple and has at most five elements, it must have at least one congruence \( \theta \) of the form described in Theorem 3.1 or in Proposition 3.2 or in Proposition 3.3. Since \( \mathcal{A} \) is affine complete every \( f \in \text{Pol}^{(1)}(\theta) \) is a polynomial operation of \( \mathcal{A} \). Therefore, \( \mathcal{A} \) has a LT-function or a LT\(_1\)-function among its unary polynomial operations.

Congruence relations on groups are determined by normal subgroups and the blocks with respect to a given congruence relation are the cosets of the normal subgroup corresponding to that congruence. Therefore, all blocks have the same cardinality. Universal algebras having this property are called congruence uniform.

**Corollary 4.3.** Every group \( G \) which has a LT-function \( f \) among its unary polynomial operations, is simple.

**Proof.** We may assume that \(|G| > 2\). Assume that the group \( G \) is not simple. Then \( G \) has a congruence \( \theta \) with at least one block containing more than one element. Since \( f \in \text{Pol}^{(1)}(\theta) \), by Theorem 3.1 there is only one block \( B \) with respect to \( \theta \) which has more than one element. Then \( B = G \), a contradiction.

**Corollary 4.4.** Every non-trivial normal subgroup of a group which has a LT\(_1\)-function among its unary polynomial operations, has index 2.

**Proof.** Since all blocks of the congruence corresponding to the given normal subgroup have the same cardinality, by Proposition 3.2 we have exactly two blocks of cardinality two or by Proposition 3.3 we have exactly two blocks with more than one element and both have the same cardinality. In either case the index of the normal subgroup is two.

**References**