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Blending simple polytopes at faces

Fred B. Holt

5520-31st Ave NE; Seattle WA 98105, USA

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Abstract

Blending two simple polytopes together at vertices, at edges, or at other supplementary faces, produces another simple polytope. In pursuing the Hirsch conjectures, vertex-blends produced polytopes which have large diameter and few diametral paths. Here we define and explore blends at higher-dimensional faces. We identify specific blendings which would, given the proper inputs, produce counterexamples to the Hirsch conjecture. We also show that the Hirsch conjecture is sharp for dimension 7.

Keywords: Polytopes; Hirsch conjecture; Paths; Blend; Diameter

1. Introduction

In this paper we define and examine the operation of blending simple polytopes together at supplementary faces. This operation generalizes the vertex-blend which was introduced by Barnette [3] to create nonpolytopal spheres. The vertex-blend was later used by Adler [1] in pursuit of the Hirsch conjecture, and used again recently by Fritzsche, Holt, and Klee [6,10] to establish that the Hirsch conjecture is sharp for bounded polytopes of dimension 8 and higher. Although the blends will be determined by the blending together of certain facets from two polytopes, the impact on the Hirsch conjecture comes from the implied blending together of fast and slow edges.

Once the important concepts have been assembled, we exhibit a construction, a fast–slow blend at simplex faces, which would produce a counterexample to the Hirsch conjecture. However, at this time the construction is an operation with no known input, suggesting an avenue of research and a possible strengthening of the Hirsch conjecture. Other blends at simplex faces demonstrate that the Hirsch conjecture is sharp in dimension 7.

We require the following concepts from the study of polytopes. A (d,n)*-polytope* P is a simple bounded d-dimensional polytope with precisely n facets. A polytope of dimension d is *simple* iff each vertex is incident to precisely d facets. A *facet* is a $(d-1)$-dimensional face of P. An *edge* is a 1-dimensional face, and a *vertex* is a 0-dimensional face. In a simple d-polytope, the simultaneous intersection of k facets is either empty or a $(d-k)$-dimensional face. We use *simplex face* to denote a face of P that is combinatorially a simplex. For background on polytopes, refer to [8,15] and the updated [9].

Let x be a vertex incident to the k-dimensional face F in a simple d-dimensional polytope. Of the d edges incident to x, k connect x to other vertices of the face F. The remaining $d-k$ edges connect x to vertices not incident to F. Relative to the face F, we call these $d-k$ edges *outbound* edges.

For two vertices x and y of a polytope P, the *distance* $\delta_P(x,y)$ is defined as the smallest number of edges of P that can be used to form a path from x to y. The *edge-diameter* $\delta(P)$ of P is the maximum of $\delta_P(x,y)$ over all pairs (x,y) of P's vertices. An undirected edge $[u,v]$ in a polytope P is said to be *slow toward* a vertex x of P iff $\delta_P(u,x)=\delta_P(v,x)$; otherwise, $[u,v]$ is *fast toward* x. $\Delta(d,n)$ denotes the maximum edge-diameter among all (d,n)-polytopes.

E-mail address: holt@math.washington.edu (F.B. Holt).

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As reported by Dantzig [4,5], Hirsch conjectured in 1957 that $\Delta(d,n) \leqslant n-d$ for all $n > d \geqslant 2$. For d-dimensional polytopes with n facets, $n-d$ is the *Hirsch bound*. Although the conjecture originally addressed all polyhedra, unbounded counterexamples were produced in dimension 4 [13], and it has been shown that for each d and n the maximum diameter is attained by a simple polytope [13]. We will therefore consider only simple bounded polytopes, denoted by (d,n)-*polytope*, and we use $\Delta(d,n)$ to refer to the maximum diameter attained by these simple bounded polytopes.

Since the conjecture is still open, we are particularly interested in pairs of vertices whose distance meets the Hirsch bound. For a (d,n)-polytope P, let X and Y be two subsets of vertices. If for every x in X and every y in Y, the distance $\delta_P(x,y) \geqslant j$, then we say that (X,Y) is a *j-pair*. If (X,Y) is an $(n-d)$-pair, we say that (X,Y) is an *H-pair*. We use the notation $(d,n:\ h,k)$ for the set of all triples (P,X,Y) in which P is a (d,n)-polytope, (X,Y) is an H-pair in P, X contains the vertices of an h-dimensional face, and Y contains the vertices of a k-dimensional face.

The Hirsch conjecture originally addressed both unbounded objects (polyhedra) and bounded ones (polytopes). The conjecture holds [11] for $d=3$ and all n, even in the unbounded version. In the bounded case it holds whenever $n-d \leqslant 5$, but the unbounded version fails for $(d,n)=(4,8)$; both of these results appear in [13]. We know two other specific values [7]: $\Delta(4,10)=5$ and $\Delta(5,11)=6$. The bounded Hirsch conjecture is still open for $(4, n>10)$, for $(5, n>11)$ and for all (d,n) with $d \geqslant 6$ and $n-d>5$.

Recent work [6,10] has established that if the Hirsch conjecture is true, then the bound is sharp for all $n > d \geqslant 8$. These results proceed by explicit combinatorial constructions. In particular, an essential construction was the fast–slow blending of two simple polytopes at vertices. We introduce here a generalization of the vertex-blend, by which we can blend together two polytopes at proper faces of any dimension. This generalization has two important consequences. First, face-blending offers new routes to constructing a counterexample to the Hirsch conjecture; for example, a fast–slow blend at a simplex face would produce a counterexample. Second, we construct for any number of facets $n>7$ an H-sharp $(7,n)$-polytope, demonstrating that the Hirsch conjecture is sharp in dimension 7.

2. Embedded polytopes

An embedded d-polytope with n facets and N vertices is completely determined by its facet-supporting hyperplanes $H^{\mathrm{T}}_{n\times(d+1)}$ and also by its vertices $X_{d\times N}$. The relationship between these two representations is

$$H^{\mathrm{T}} V = H^{\mathrm{T}} \begin{bmatrix} \langle 1 \rangle \\ X \end{bmatrix} \leqslant \langle 0 \rangle. \tag{1}$$

Here each column of H contains the normal for a facet of P, and each column of V contains the coordinates of a vertex, prepended by a 1.

Each supporting hyperplane τ_0 is determined by an outward normal h^{T}, unique up to a positive scalar. The hyperplane itself is

$$\tau_0 = \{v \in 1 \times \mathbb{R}^d : h^{\mathrm{T}} v = 0\}.$$

A point v lies *beyond* the hyperplane if $h^{\mathrm{T}} v \geqslant 0$, and if $h^{\mathrm{T}} v \leqslant 0$ then the point v lies *behind* the hyperplane. We denote the closed half-space lying behind the hyperplane as τ_-.

Let F be any proper face of a (d,n)-polytope P. Let τ be a hyperplane such that all the vertices of F lie beyond τ and all other vertices of P lie behind τ. The *truncation* of P at F is the $(d,n+1)$-polytope $\tau P = \tau_F P$ which is the intersection $P \cap \tau_-$. The new facet, denoted $\tau(F)$, is given by $\tau(F) = P \cap \tau_0$.

Since P is a simple polytope, if F is a k-dimensional face of P, then in $\tau_F P$ the facet $\tau(F)$ is combinatorially equivalent to $F \times T^{d-k-1}$. The simplex factor T^{d-k-1} comes from the $d-k$ facets whose supporting hyperplanes define the k-space which contains F.

Combinatorially, a polytope P is completely described by its $n \times N$ facet-vertex incidence matrix M. We can derive this combinatorial description from an embedding as the $\{0,1\}$-matrix $M = \mathrm{IsZero}(H^{\mathrm{T}} V)$.

In constructions it can be difficult to work with embedded polytopes. On the one hand it is easy, given an embedding $H(P)$, to derive $H(P \times I)$ for the prism or $H(\omega P)$ for a wedge. On the other hand, for a truncation $H(\tau P)$ there are many degrees of freedom in choosing the normal for the truncating hyperplane. In a blend $P_1 \bowtie P_2$, the projective transformations and the perturbations of hyperplanes introduce many degrees of freedom. In particular, if the coordinates in $H(P)$ are nice—e.g. integral, rational with small denominator—we can canonically form $H(P \times I)$ and $H(\omega P)$ with nice coordinates. We do not have canonical $H(\tau P)$ or $H(P_1 \bowtie P_2)$, let alone embeddings with nice coordinates. Consequently we often work combinatorially with the incidence matrix M once we have established that the constructions have feasible embeddings.

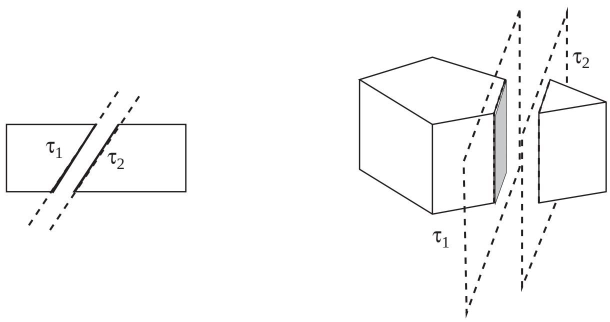

Fig. 1. We extend the notion of supplementary angles to describe two polytopes which fit together seamlessly.

3. Supplementary faces

In this section we develop the concept of *supplementary faces*. We start with a strict definition for embedded polytopes, which expands the familiar concept of supplementary angles. Thereafter we extend and expand this definition until we can use it in the combinatorial setting.

Supplementary at facets: We start with two simple d-polytopes P_1 and P_2 and a facet in each, τ_1 and τ_2, respectively. Each of the τ_i is a $(d-1,m)$-polytope, whose facet-supporting hyperplane in P_i has outward normal $h(\tau_i)$, and whose boundary is given by m facets of P_i with outward normals h_{ij} for $j=1,2,\ldots,m$. The two facets τ_1 and τ_2 are combinatorially equivalent, and this equivalence is reflected in the indexing of the h_{ij}.

Definition 1. The polytopes P_1 and P_2 are *supplementary at the facets* τ_1 and τ_2 if and only if:

(i) τ_1 and τ_2 are congruent $(d-1,m)$-polytopes,
(ii) there is a rigid transformation R which brings

$$h(\tau_1) = -Rh(\tau_2),$$

$$h_{1j} = Rh_{2j}, \quad \text{for } j = 1,\ldots,m.$$

Under this strict definition, the two polytopes P_1 and P_2 are the fragments left from cleaving one polytope P with a hyperplane that misses all vertices of P (refer to Fig. 1). By aligning τ_1 and τ_2 under the rigid transformation R, we can glue P_1 and P_2 back together to form P, and all the vertices of τ_1 and of τ_2 disappear.

Our goal is to blend polytopes together, e.g. obtaining P from P_1 and P_2, with specific combinatorial properties in the resulting polytope. So we broaden our notion of supplementary faces.

For embedded polytopes we say that d-polytopes P_1 and P_2 are *supplementary at facets* τ_1 and τ_2 if there are projective transformations of P_1 and P_2 after which the strict definition 1 applies.

Supplementary faces: We now extend the concept from facets to faces of all dimensions. Let P_1 and P_2 be two embedded polytopes. Let F_1 be a face of P_1 and F_2 be a face of P_2. P_1 and P_2 are *supplementary at the faces* F_1 and F_2 if and only if there are truncations, $\tau P_1 = \tau_{F_1} P_1$ and $\tau P_2 = \tau_{F_2} P_2$, such that τP_1 and τP_2 are supplementary at the introduced facets $\tau(F_1)$ and $\tau(F_2)$.

As noted above, we prefer to work with combinatorial descriptions of polytopes, and we can now provide a definition for combinatorial polytopes. Let P_1 and P_2 be two d-polytopes. Let F_1 be a face of P_1 and F_2 be a face of P_2. The polytopes P_1 and P_2 are *supplementary at the faces* F_1 and F_2 if and only if there exist embeddings such that the embedded P_1 and P_2 are supplementary at F_1 and F_2. The combinatorial type of the facet $\tau(F)$ is known as the *face-figure* of the face F. For the polytopes to be supplementary at these two faces, it is necessary that the face-figures $\tau(F_1)$ and $\tau(F_2)$ have the same combinatorial type.

4. Blending polytopes at supplementary faces

We start with simple polytopes $P_1 \in (d,n_1)$ and $P_2 \in (d,n_2)$. These d-polytopes are supplementary at the faces F_1 and F_2, respectively. Our goal is to blend P_1 and P_2 together at F_1 and F_2, creating another simple d-polytope. This blending of simple polytopes at faces is a generalization of the vertex-blend [1,3,10]. We will identify each facet incident to F_1 in P_1 to a facet incident to F_2 in P_2, so the full notation for this blend has to include the two polytopes, the two faces, and the pairing μ of facets:

$$(P_1, F_1) \bowtie_\mu (P_2, F_2).$$

However, once the faces and pairing are understood we can use the lighter notation $P_1 \bowtie P_2$.

The pairing μ must pair up each facet incident to F_1 with a facet incident to F_2. The correspondence between the face figures of F_1 and F_2 comes with an equivalent correspondence between the hyperplanes which define F_1 and F_2 in the polytopes P_1 and P_2. An *admissible pairing* μ for F_1 and F_2 provides a combinatorial equivalence between the face-figures $\tau(F_1)$ and $\tau(F_2)$.

Unlike the vertex blend, when $k > 0$ the facets incident to F_1 are of two distinct types. Suppose F is a (k,m)-face in a d-polytope P. Then the k-space supporting F is the intersection of $d-k$ facet-supporting hyperplanes of P, and the boundary of F is given by the intersections between this k-space and m additional facet-supporting hyperplanes of P. So the boundary of the new facet $\tau(F)$ consists of $m+d-k$ (truncated) facets of P of two types: $d-k$ *space-defining* facets and m *boundary-defining* facets.

If $\tau(F_1)$ and $\tau(F_2)$ are combinatorially equivalent, then $m_1+d-k_1 = m_2+d-k_2$. For legibility, we suppress the indices when a consistent choice of index makes no difference, e.g. $\tau(F)$ has its boundary defined by $m+d-k$ facet-supporting hyperplanes.

Combinatorially the *blend* of P_1 and P_2 at F_1 and F_2 is a simple polytope constructed by identifying all the facets of P_1 incident to F_1 with those facets of P_2 incident to F_2, under the pairing μ. So while $P_1 \bowtie P_2$ is still d-dimensional, the number of facets is reduced from n_1+n_2 by the $m+d-k$ pairwise identifications of facets.

$$(P_1, F_1) \bowtie_\mu (P_2, F_2) \in (d, n_1+n_2-d-(m-k)). \tag{2}$$

In general, μ could describe any combinatorial equivalence of $\tau(F_1)$ and $\tau(F_2)$. In this general case F_1 and F_2 need not even be of the same dimension, provided $\tau(F_1)$ and $\tau(F_2)$ are combinatorially equivalent.

Once these $m+d-k$ facets are blended together, all lower-dimensional faces incident to them are blended together as well. The $m+d-k$ oriented hyperplanes defining the blended facets form a polyhedron called the *extended waist* of the blend. In general, the combinatorial type of the extended waist will not be well-determined. It will depend instead on the specific embedding. However, at the blend the cross-section of the extended waist will be the face figure $\tau(F)$. Each vertex in $\tau(F_i)$ corresponds to a half-edge in the extended waist of the blend.

4.1. Mechanics of blending

Here we examine how the blend

$$(P_1, F_1) \bowtie_\mu (P_2, F_2)$$

is reflected in the matrices H, V and M.

Denote the $m+d-k$ hyperplanes incident to F_i as $H_i^{\mathrm{T}}(F)$, and permute the rows to respect the pairing μ. Under the strict definition 1, and after applying the rigid transformation R, we have

$$H_1^{\mathrm{T}}(F) = H_2^{\mathrm{T}}(F),$$
$$h(\tau_1) = -h(\tau_2). \tag{3}$$

There is no constraint on supporting hyperplanes $\tilde{H}_i^{\mathrm{T}}$ for facets not incident to F_i. So H_1^{T} and H_2^{T} have the following forms, and $H_{\bowtie}^{\mathrm{T}}$ is given by:

$$H_1^{\mathrm{T}} = \begin{bmatrix} \tilde{H}_1^{\mathrm{T}} \\ H^{\mathrm{T}}(F) \\ h_\tau^{\mathrm{T}} \end{bmatrix}, \quad H_2^{\mathrm{T}} = \begin{bmatrix} \tilde{H}_2^{\mathrm{T}} \\ H^{\mathrm{T}}(F) \\ -h_\tau^{\mathrm{T}} \end{bmatrix} \Rightarrow H_{\bowtie}^{\mathrm{T}} = \begin{bmatrix} \tilde{H}_1^{\mathrm{T}} \\ \tilde{H}_2^{\mathrm{T}} \\ H^{\mathrm{T}}(F) \end{bmatrix}.$$

The truncating hyperplane provides a needed orientation for the blend, and it helps avoid admissible-transformation problems.

To blend two given polytopes, we need to find embeddings which satisfy (3). If we are provided embeddings of the two polytopes, we need to find transformations such that (3) holds for the resulting embeddings. Such transformations may not always be available. For example, consider Fig. 2. If only one of the two cubes was embedded as a regular cube, and the other as a Klee–Minty cube, then there are coincidences of hyperplanes in the regular cube which do not

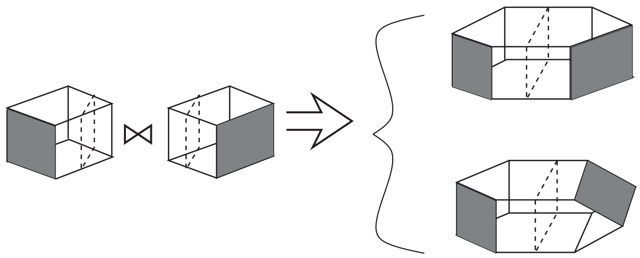

Fig. 2. A blend of two cubes at an edge. Because the face-figure has a symmetry, there are two possible results. The hexagonal cylinder results from pairing boundary-to-boundary and space-to-space.

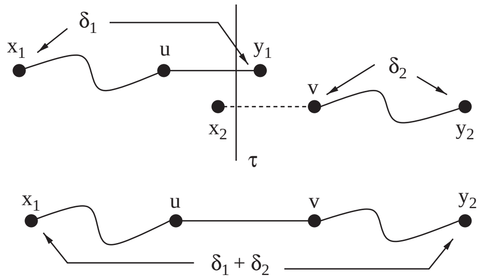

Fig. 3. In the waist of a fast–slow blend, every fast edge is matched with a slow edge. The resulting diameter is the sum of the diameters of the stock polytopes. Here, the fast edge (u, y_1) is matched with the slow edge (x_2, v).

exist in the Klee–Minty cube [2,12]. Coincidences cannot be perturbed apart by projective transformations. So in general, for embedded polytopes we have to consider not only projective transformations but also perturbations of the hyperplanes which preserve the combinatorial type [14].

The set of vertices in the blend is the union of the sets of vertices not incident to the face F_i in each of the polytopes P_i.

$$V_1 = \left[\, \tilde{V}_1 V(F_1) \,\right], \qquad V_2 = \left[\, \tilde{V}_2 V(F_2) \,\right] \Rightarrow V_{\bowtie} = \left[\, \tilde{V}_1 \tilde{V}_2 \,\right].$$

Combinatorially, the blend is a simple manipulation of blocks from the facet-by-vertex incidence matrices $M(P_i)$. Let

$$M(P_1) = \begin{bmatrix} M_1 & \langle 0 \rangle \\ N_1 & M(F_1) \end{bmatrix}, \qquad M(P_1) = \begin{bmatrix} M_2 & \langle 0 \rangle \\ N_2 & M(F_2) \end{bmatrix}.$$

The rows of N_1 and N_2 respect the pairing μ; these $m + d - k$ rows correspond to the facets to be blended together, the facets of P_i incident to F_i. The blocks $M(F_i)$ provide the facet-vertex incidences for the vertices of F_i.

The combinatorial type of the blended polytope is given by

$$M(P_1 \bowtie P_2) = \begin{bmatrix} M_1 & \langle 0 \rangle \\ N_1 & N_2 \\ \langle 0 \rangle & M_2 \end{bmatrix}.$$

4.2. Fast–slow blends

The breakthrough construction in [10] was the fast–slow vertex blend. Identifying pairings μ which blocked every fast edge with a slow edge at the waist elevated vertex-blends from providing a lower bound [1] to generating H-sharp polytopes.

A fast–slow blend at a (k, m)-face with $k \geqslant 1$ would produce a counterexample to the Hirsch conjecture, with the resulting diameter exceeding the Hirsch bound by $m - k$ (Fig. 3).

As shown in [6,10], there are polytopes with an abundance of slow edges available. For $d \equiv 0 \bmod 5$ there are $(d, 2d)$-polytopes with an H-pair, each member of which holds a face of dimension $d - 4$. For example, there is a

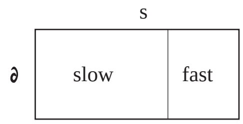

Fig. 4. For simplex faces in simple polytopes, the outbound edges correspond naturally to the cells in a boundary-by-space matrix. In this matrix, the fast edges occur as a block across all boundary indices for some subset of space indices.

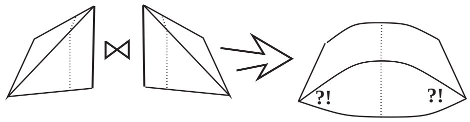

Fig. 5. A boundary-to-boundary blend of two 3-dimensional simplices at an edge is precluded by the unique boundary condition.

$(100,200)$-polytope containing two 96-dimensional faces, such that every vertex from one is at distance 100 from every vertex of the other. To create a fast–slow blend at an edge of one of these 96-dimensional faces, we have only to match each of the four fast edges at each vertex with one of the 95 available slow edges.

4.3. Blending at simplex faces

When F_1 and F_2 are simplex faces of P_1 and P_2, respectively, then the blend $(P,F_1) \bowtie (P,F_2)$ has $d+1$ facets in the extended waist. Having only $d+1$ facets to manipulate for each of P_1 and P_2 offers the prospect of using projective transformations to construct the blend.

For a simplex k-face F, there are $(d-k)(k+1)$ vertices in the face-figure $\tau(F)$. Letting $s=\{1,\ldots,d-k\}$ be an index set for the space-defining facets and $\partial=\{1,\ldots,k+1\}$ for the boundary-defining facets, we have a natural correspondence between the vertices of $\tau(F)$ and the index set $\partial\times s$. Each of the $k+1$ vertices in F can be indexed by the boundary-defining facet *not* incident to it. Of the d edges incident to this vertex, $d-k$ leave the face F and each of these can be indexed by the space-defining facet from which the edge is departing.

Slow edges are provided by a higher-dimensional face in an H-pair, of which F is a proper face. Thus the slow edges appear in the $\partial\times s$ indexing matrix as a block across all rows and across those columns corresponding to facets which are not space-defining for this higher-dimensional face. For a fast–slow blend at simplex faces, we need a pairing μ which matches every entry for a fast edge in one indexing matrix with an entry for a slow edge in the other. This indexing of half-edges reveals that for simplex faces, only a boundary-to-boundary blend could be fast–slow (Fig. 4).

4.4. Barriers

There may be insurmountable barriers to blending two polytopes together at specified faces. One such barrier is a topological one: a pair of boundary facets from P_1 don't intersect in F_1 but have an intersection somewhere else in P_1, and the corresponding pair of boundary facets under μ also don't intersect in F_2 but have an intersection elsewhere in P_2. Simply identifying the facets, we see that in the resulting object the two facets have two connected components in their intersection and so cannot be a polytope, as illustrated in Fig. 5.

Unique boundary condition: Let F_1 and F_2 be combinatorially equivalent (k,m)-faces in P_1 and P_2, respectively. Let μ be an admissible pairing of the facets incident to F_1 with those incident to F_2. The *unique boundary condition* holds for (P_1,F_1) and (P_2,F_2) under μ if and only if

> for every two facets F and G incident to F_1, if F and G intersect in a nonempty face of P_1 not incident to F_1, then $\mu(F)$ and $\mu(G)$ do not intersect in P_2.

Note that if $\mu(F)$ and $\mu(G)$ have intersection incident to F_2, then the admissibility of μ implies that $F\cap G$ is incident to F_1. Thus, while the unique boundary condition is a real barrier for blending at edges or at general faces, blending at triangular or higher-dimensional simplex faces always satisfies the this condition.

There may be other barriers as well. For example, the structure of P_1 and P_2 may constrain the possible embeddings of F_1 and F_2. The difficulties of prescribing the shape of a face are discussed in [14,15].

5. Matching boundaries and spaces

There are two special blends of immediate interest to us. The first arises when F_1 and F_2 are combinatorially equivalent (k,m)-faces. If μ pairs each of the $d-k$ space-defining hyperplanes for F_1 with a space-defining hyperplane for F_2, and it pairs each of the m boundary-defining hyperplanes for F_1 with a boundary-defining hyperplane for F_2, the blend $(P_1,F_1)\bowtie_\mu(P_2,F_2)$ is said to be *boundary-to-boundary* (*and space-to-space*).

As illustrated in Fig. 2, a second case *space-to-boundary* arises when the face figures $\tau(F_1)$ and $\tau(F_2)$ have a symmetry which permits μ to pair each boundary-supporting hyperplane for F_1 with a space-supporting hyperplane for F_2 and vice versa. However, for faces of the same dimension k within simple polytopes, $\tau(F)\equiv F\times T^{d-k-1}$, and so a blend will have to be boundary-to-boundary and space-to-space unless F is combinatorially equivalent to $G\times T^{d-k-1}$ for some $(2k+1-d)$-face G. This is only possible if $2k\geqslant d-1$. Note that this condition applies to the cubes depicted in Fig. 2, making possible an alternative admissible pairing and thus a second possible result.

5.1. *Blends of type boundary-to-boundary and space-to-space*

Let F_1 and F_2 be combinatorially equivalent (k,m)-faces in the simple d-polytopes P_1 and P_2, respectively. We consider a pairing μ for the boundary-to-boundary and space-to-space case.

Denote the hyperplanes for the k-space as $H_i^{\mathrm{T}}(s)$ and those for the boundary as $H_i^{\mathrm{T}}(\delta)$. Under the strict definition 1, and after applying the rigid transformation R, we have

$$
\begin{aligned}
H_1^{\mathrm{T}}(s) &= H_2^{\mathrm{T}}(s),\\
H_1^{\mathrm{T}}(\partial) &= H_2^{\mathrm{T}}(\partial),\\
h(\tau_1) &= -h(\tau_2). \qquad (4)
\end{aligned}
$$

A canonical extended waist for these boundary-to-boundary blends is the cylinder $F\times T^{d-k-1}\times(-\infty,+\infty)$. An open problem is to describe a canonical embedding for boundary-to-boundary blends, even under additional restrictions on the faces (Fig. 6).

5.2. *Blends of space-to-boundary type*

This special case of blending arises when F_1 and F_2 are supplementary faces with $m_1=d-k_2$ and $m_2=d-k_1$. Under this condition there may be admissible pairings μ for which

$$
\begin{aligned}
H_1^{\mathrm{T}}(s) &= H_2^{\mathrm{T}}(\partial),\\
H_1^{\mathrm{T}}(\partial) &= H_2^{\mathrm{T}}(s),\\
h(\tau_1) &= -h(\tau_2). \qquad (5)
\end{aligned}
$$

Again, there is no constraint on supporting hyperplanes $\tilde{H}_i^{\mathrm{T}}$ for facets not incident to F_i.

Since $\tau(F)$ is combinatorially equivalent to $F\times T^{d-k-1}$, with the m boundary hyperplanes corresponding to the factor F, a space-to-boundary blend must have

$$F_1\cong T^{d-k_2-1}\quad\text{and}\quad F_2\cong T^{d-k_1-1}.$$

The face-figure $\tau(F)\cong T^{d-k_1-1}\times T^{d-k_2-1}$, and so $d=k_1+k_2+1$.

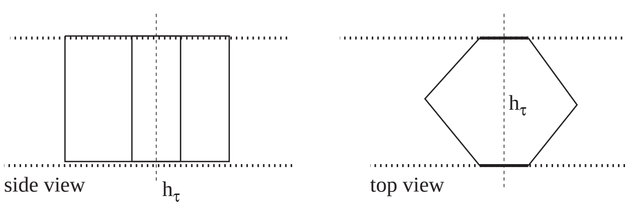

Fig. 6. A boundary-to-boundary blend of two 3-dimensional cubes at an edge produces a hexagonal cylinder. Here the blend is illustrated inside the square $(T^1\times T^1)$ cylinder that is the extended waist in this blend.

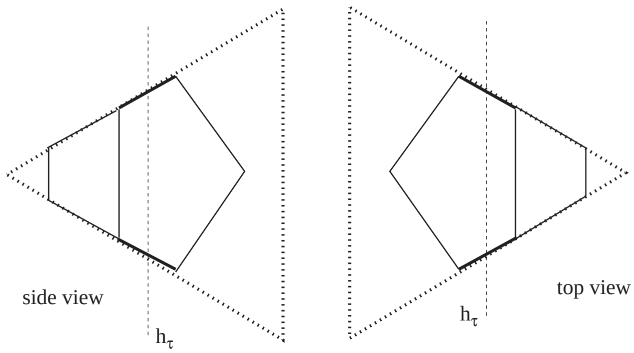

Fig. 7. A space-to-boundary blend of two 3-dimensional cubes at an edge. Here the blend is illustrated inside the simplex that is the extended waist in this blend.

The identification of fast edges in Fig. 4 indicates that no fast–slow space-to-boundary blend is possible. In a space-to-boundary blend, we would be aligning the matrix from one stock polytope with the transpose of the matrix for the other stock polytope. Thus, there would always be a block of fast edges blended with fast edges, lowering the diameter by one. If the stock polytopes were H-sharp, then the resulting polytope would also be H-sharp (Fig. 7).

5.3. Projective transformations suffice

When a polytope is embedded as described in Section 2, a projective transformation is given by a $(d+1)\times(d+1)$ matrix T.

$$H^{\mathrm{T}} \to H^{\mathrm{T}}T$$

$$V \to T^{-1}V/\sim$$

The equivalence relation is the usual one for homogeneous coordinates: $u\sim v$ iff $u=cv$ for some nonzero constant c. Since we work with $1\times\mathbb{R}^d$, we choose c to restore the first coordinate in each column to 1 if possible. If the first entry in a column is 0, then the vertex corresponding to that column has been mapped to an ideal point.

Lemma 2. *For $1\leqslant k_1<d$, let F_1 be a simplex k_1-face in a simple d-polytope P_1.* For $k_2=d-k_1-1$, *let F_2 be a simplex k_2-face in a simple d-polytope P_2. Then projective transformations suffice to create the space-to-boundary blend*

$$(P_1,F_1)\bowtie_\mu(P_2,F_2)$$

for any admissible pairing μ.

Proof. For the extended waist we use a standard simplex T^d, given by the hyperplanes

$$H^{\mathrm{T}}(T^d)=\begin{bmatrix}-1 & \langle 1\rangle\\ \langle 0\rangle & -I\end{bmatrix}.$$

Let the first $d-k_1$ rows of $H^{\mathrm{T}}(T^d)$ be the space-defining facets of $\tilde{F}_1$. Let the remaining $d-k_2$ rows be the space-defining facets of $\tilde{F}_2$. Let τ be the hyperplane bisecting every edge between $\tilde{F}_1$ and $\tilde{F}_2$. There exist projective transformations which map P_i into the simplex T^d such that F_i is mapped to $\tilde{F}_i$. All pairings μ which match the space-defining hyperplanes of F_1 to those of $\tilde{F}_1$ are possible. Let $\tilde{P}_i$ be the image of P_i in T^d.

It suffices for us to show that there exists a projective transformation of $\tilde{P}_1$ which leaves T^d fixed but makes τ a truncating hyperplane for $\tilde{F}_1$. Write $H^{\mathrm{T}}(T^d)$ in block form as follows:

$$H^{\mathrm{T}}(T^d)=\begin{bmatrix}-1 & \langle 1\rangle & \langle 1\rangle\\ \langle 0\rangle & -I & \langle 0\rangle\\ \langle 0\rangle & \langle 0\rangle & -I\end{bmatrix}.$$

The top blocks are a single row, the middle blocks have $d-k_1-1$ rows, and the lower blocks have k_1+1 rows; the top and middle blocks are the space-defining hyperplanes for $\tilde{F}_1$.

For large positive Λ and small positive λ, a transformation of the form

$$T = \begin{bmatrix} \lambda & \langle 0 \rangle & \langle \Lambda - \lambda \rangle \\ \langle 0 \rangle & \lambda I & \langle 0 \rangle \\ \langle 0 \rangle & \langle 0 \rangle & \Lambda I \end{bmatrix}$$

keeps T^d fixed but pushes all points interior to T^d away from the face $\tilde{F}_1$. Let $h^{\mathrm{T}} = [\varepsilon - 1 \langle 0 \rangle \, \langle 1 \rangle]$, with $\varepsilon > 0$ small enough to truncate $\tilde{F}_1$ in $\tilde{P}_1$. Then if $\Lambda/\lambda = \frac{1-\varepsilon}{\varepsilon}$, $h^{\mathrm{T}}T$ coincides with τ, and τ is a truncating hyperplane for $\tilde{F}_1$ in the resulting image of $\tilde{P}_1$. $\square$

6. Implications for the Hirsch conjecture

Blending polytopes together at supplementary faces enables us to demonstrate that more pairs (d, n) are H-sharp, and it offers at least two routes to constructing a counterexample to the Hirsch conjecture.

6.1. More H-sharp polytopes

Theorem 3. *All* $(7, n)$ *are H-sharp.*

Proof. We use copies of $P = \omega^3 Q_4$. From [6] we know that $(P, X, Y) \in (7, 12 : 3, 3)$. Following the proof of Lemma 2, we use a standard 7-simplex T^7 for the extended waist. In T^7 we identify a disjoint pair of tetrahedra $\tilde{X}$ and $\tilde{Y}$, and we pick a hyperplane τ separating this pair.

Take two embeddings of P, P_1 and P_2. To create the blend $(P_1, Y_1) \bowtie (P_2, X_2)$ we use admissible projective transformations. First we map the $d+1$ hyperplanes of P_1 onto T^7 such that Y_1 is mapped onto $\tilde{Y}$. Similarly we map P_2 into T^7 such that X_2 is mapped onto $\tilde{X}$. As necessary we contract the image of P_1 toward $\tilde{X}$ and the image of P_2 toward $\tilde{Y}$ so that τ truncates Y_1 in P_1 and X_2 in P_2. The result B is H-sharp, $(B, X_1, Y_2) \in (7, 16 : 3, 3)$.

We iterate this construction, producing a sequence of H-sharp polytopes, $(B^k, X, Y) \in (7, 12 + 4k : 3, 3)$. Truncations in the faces held by the H-pair provide H-sharp polytopes for the pairs $(7, n)$ in between these blended examples. $\square$

6.2. A fast–slow blend would produce a counterexample

A fast–slow blend of two polytopes at simplex faces would produce a counterexample to the Hirsch conjecture. In dimension 9 we have sufficiently many slow edges to consider blending at an edge; however, the unique boundary condition complicates this construction. In dimension 10 we can consider creating a fast–slow blend at a triangular face. For simplex faces, only the boundary-to-boundary blends could be fast–slow. Thus, the Hirsch conjecture implies that there is always some barrier to a fast–slow (boundary-to-boundary) blend for every $(P, X, Y) \in (d, n : h, k)$.

6.3. Other blends could produce counterexamples

If the faces to be blended are not simplices, then we are identifying even more facets in the blend, and a possibly weaker condition than fast–slow blending suffices to produce a counterexample. Let F_i be a (k_i, m_i)-face in P_i, such that $\tau(F_1) \equiv \tau(F_2)$. Let x be a vertex in P_1 which is far from the vertices of F_1, and let y be a vertex in P_2 which is far from the vertices of F_2. For every vertex u that is an edge-neighbor to F_1 in P_1, denote by $\mu(u)$ the vertex in P_2 such that $[u, \mu(u)]$ will be an edge in the waist of the blend. The blend $P_1 \bowtie P_2$ will be a counterexample to the Hirsch conjecture if for every such u,

$$\delta_{P_1}(x, u) + \delta_{P_2}(\mu(u), y) \geqslant (n_1 - d) + (n_2 - d) - (m + d - k).$$

Both of the above constructions are well-defined combinatorially. To produce a counterexample, each construction requires suitable stock polytopes and a description of an embedding, as Lemma 2 provides for the space-to-boundary blend.

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