Boundary value problem of second order impulsive functional differential equations ✩

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Abstract

This paper discusses a kind of linear boundary value problem for a nonlinear second order impulsive functional differential equations. We establish several existence results by using the lower and upper solutions and monotone iterative techniques. An example is discussed to illustrate the efficiency of the obtained result.
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1. Introduction

The general theory of impulsive differential equations [1–5] is emerging as an important area of investigation since it is much richer and more widely used than the corresponding theory of differential equations. In this paper, we study the boundary value problem for second order impulsive functional differential equation of the following form:

\[-x''(t) = f(t, x(t), x(\theta(t))), \quad t \in J = [0, T], \quad t \neq t_k, \quad k = 1, 2, \ldots, p,
\]

\[\Delta x(t_k) = I_k(x(t_k)), \quad k = 1, 2, \ldots, p,\]

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\[
\Delta x'(t_k) = I^+_k(x(t_k)), \quad k = 1, 2, \ldots, p,
\]
\[
x(0) = x(T) + k_1, \quad x'(0) = \lambda x'(T) + k_2,
\] (1)

where \( f \in C(J \times R^2, R), I_k \in C(R, R), I^*_k \in C(R, R), \theta \in C(J, J). \Delta x(t_k) = x(t^+_k) - x(t^-_k), \) in which \( x(t^+_k), x(t^-_k) \) denote the right and left limits of \( x(t) \) at \( t_k, \) \( \Delta x'(t_k) = x'(t^+_k) - x'(t^-_k), \) in which \( x'(t^+_k), x'(t^-_k) \) denote the right and left limits of \( x'(t) \) at \( t_k, t_k, k = 1, 2, \ldots, p, \) which are fixed points such that \( 0 < t_1 < t_2 < \cdots < t_p < T. \)

The monotone iterative technique, coupled with the method of upper and lower solutions, is widely used in obtaining some existence results for differential or functional differential equations with initial or boundary value problems \([6-17, 20]\). Nevertheless, only a few papers \([6, 7, 10]\) have implemented the technique in second order functional differential equations.

Nonlinear boundary and linear boundary conditions including periodic and anti-periodic boundary conditions value problems for impulsive functional differential equations of first order have been widely studied in recent years \([8-10, 13, 14, 18, 19, 21]\). And in \([7]\), the existence of solutions for a kind of linear boundary condition for second order functional differential equations is considered.

To obtain some existence results for problem (1), we will organize the paper as follows. In Section 2, we will establish some differential inequalities and discuss uniqueness of the solutions for the second order impulsive linear functional differential equations connected with problem (1). The results of this section contain some results of paper \([6]\). In Section 3, using the definition of upper and lower solutions and monotone iterative technique, we will obtain the sufficient conditions for the existence of the solutions, uniqueness solution, and extremal solutions for (1). Finally, we provide an example to verify the required assumptions.

Let \( J_0 = J \setminus \{t_1, t_2, \ldots, t_p\}, \) \( \tau = \max_k \{t_k - t_{k-1} : k = 1, \ldots, p\}, \) here \( t_0 = 0, t_{p+1} = T, \) and \( PC(J) = \{u : J \to R : u \text{ is continuous for any } t \in J_0; u(t^+_k), u(t^-_k) \text{ exist, } k = 1, 2, \ldots, p \text{ and } u(t^-_k) = u(t_k)\}; PC^1(J) = \{u : J \to R : u \text{ is continuously differentiable for any } t \in J_0; u'(t^+_k), u'(t^-_k) \text{ exist, } k = 1, 2, \ldots, p \text{ and } u'(t^-_k) = u'(t_k)\}. \)

\( PC(J) \) and \( PC^1(J) \) are Banach spaces with the norms
\[
\|u\|_{PC(J)} = \sup \{|u(t)| : t \in J\}
\]
and
\[
\|u\|_{PC^1(J)} = \max\{\|u\|_{PC(J)}, \|u'\|_{PC(J)}\}.
\]

Let \( \Omega = PC^1(J) \cap C^2(J_0). \) A function \( x \in \Omega \) is called a solution of (1), if it satisfies (1).

2. Lemmas

Lemma 1. If \( u \in \Omega, \)
\[
-u''(t) \leq -Mu(t) - N \min\{u(\theta(t)), 0\}, \quad t \neq t_k, \ t \in J,
\]
\[
\Delta u(t_k) = L_k u'(t_k), \quad k = 1, 2, \ldots, p,
\]
\[
\Delta u'(t_k) \geq L^*_k u(t_k), \quad k = 1, 2, \ldots, p,
\]
\[
u(0) = u(T), \quad u'(0) \geq \lambda u'(T),
\] (2)

where \( \lambda > 0, M > 0, N \geq 0, L_k \geq 0, L^*_k \geq 0 \) for \( k = 1, 2, \ldots, p, \) and
\[
\left( \sum_{k=1}^{p} L_k + T \right) \left( \sum_{k=1}^{p} L_k^* + (M + N)T \right) \leq \frac{\lambda}{\lambda + 1}
\]  
(3)

hold, then \( u(t) \leq 0 \) on \( J \).

**Proof.** Suppose on the contrary, that \( u(t) > 0 \) for some \( t \in J \). Then there are two cases:

(a) there exists \( \bar{t} \in J \), such that \( u(\bar{t}) > 0 \), and \( u(t) \geq 0 \) for all \( t \in J \);

(b) there exist \( t^*, t_0 \in J \), such that \( u(t^*) > 0 \), \( u(t_0) < 0 \).

In case (a), (2) implies that \( u''(t) \geq 0 \) for \( t \neq t_k \) and \( \Delta u'(t_k) \geq 0 \) \((k = 1, 2, \ldots, p)\), hence \( u'(t) \) is nondecreasing in \( J \). If \( \lambda = 1 \), then \( u'(t) \equiv C \), noting that \( u(\bar{t}) > 0 \), we have \( 0 \equiv u''(t) < 0 \), which is a contradiction. If \( 0 < \lambda < 1 \), then \( u'(0) > 0 \), so \( u'(t) \geq 0 \). Considering that \( \Delta u(t_k) = L_k u'(t_k) \), we have \( u(t) \) is nondecreasing in \( J \). So \( u(t) \equiv C > 0 \), \( 0 \equiv u''(t) \leq -(M + N)C < 0 \) also a contradiction. Similarly we can also have a contradiction for the case \( \lambda > 1 \).

In case (b), let

\[ \inf_{t \in J} u(t) = -\gamma. \]

Then \( \gamma > 0 \), and for some \( i \in \{1, 2, \ldots, p\} \), exists \( t_s \in (t_i, t_{i+1}] \), such that \( u(t_s) = -\gamma \) or \( u(t_i^+) = -\gamma \). And

\[ -u''(t) \leq \gamma (M + N). \]

We only consider \( u(t_s) = -\gamma \), as for the case \( u(t_i^+) = -\gamma \) the proof is similar.

It is obviously there is a contradiction if \( u'(t) > 0 \) for all \( t \in J \). So there exists \( \tilde{t} \in J \) such that \( u'(\tilde{t}) \leq 0 \). Let \( \bar{t} \in (t_k, t_{k+1}] \), \( k \in \{0, 1, 2, \ldots, p\} \). By mean value theorem, we have

\[
u'(t_k) - u'(\tilde{t}) \leq u'(t_k^+) - u'(\bar{t}) + \gamma L_k^* \leq \gamma (M + N) (t_{k+1} - t_k) + \gamma L_k^*,
\]

\[
u'(t_{k-1}) - u'(t_k) \leq \gamma (M + N) (t_k - t_{k-1}) + \gamma L_{k-1}^*,
\]

\[
\vdots
\]

\[
u'(0) - u'(t_1) \leq \gamma (M + N) (t_1 - t_0).
\]

By adding together, we obtain

\[
u'(0) \leq \gamma \left( \sum_{k=1}^{p} L_k^* + (M + N)T \right).
\]

In view of \( u'(0) \geq \lambda u'(T) \), similarly we have

\[
u'(t) \leq u'(T) + \gamma \left( \sum_{k=1}^{p} L_k^* + (M + N)T \right) \leq \frac{\lambda + 1}{\lambda} \gamma \left( \sum_{k=1}^{p} L_k^* + (M + N)T \right).
\]

Now for some \( j \), such that \( j^* \in (t_j, t_{j+1}] \). First assume \( t_s < t^* \), then \( j \geq i \). By mean value theorem, we have

\[
u(t^*) - u(t_j) = u(t^*) - u(t_j^+) + L_j u'(t_j)
\]

\[
\leq \left( L_j + (t_{j+1} - t_j) \right) \left( \frac{\lambda + 1}{\lambda} \gamma \left( \sum_{k=1}^{p} L_k^* + (M + N)T \right) \right),
\]
\[ u(t_j) - u(t_{j-1}) \leq (L_{j-1} + (t_j - t_{j-1}))\left(\frac{\lambda + 1}{\lambda} \gamma \left(\sum_{k=1}^{p} L_k^* + (M + N)T\right)\right) \]

\[ \cdots \]

\[ u(t_{i+1}) - u(t_i) \leq (L_{i+1} + (t_{i+1} - t_{i}))\left(\frac{\lambda + 1}{\lambda} \gamma \left(\sum_{k=1}^{p} L_k^* + (M + N)T\right)\right) \]

(4)

By adding the above inequalities, we obtain

\[ 0 < u(t^*) \leq -\gamma + \frac{\lambda + 1}{\lambda} \gamma \left(\sum_{k=1}^{p} L_k^* + (M + N)T\right) \]

hence

\[ \frac{\lambda + 1}{\lambda} \gamma \left(\sum_{k=1}^{p} L_k^* + (M + N)T\right) > 1, \]

which contradicts (3).

If \( t^* > t^* \), then \( i \geq j \). We can easily get

\[ u(T) - u(t^*) \leq \left(\sum_{k=j+1}^{p} L_k + T - t^*\right)\frac{\lambda + 1}{\lambda} \gamma \left(\sum_{k=1}^{p} L_k^* + (M + N)T\right) \]

(5)

and

\[ u(t^*) - u(0) \leq \left(\sum_{k=1}^{j} L_k + t^* - 0\right)\frac{\lambda + 1}{\lambda} \gamma \left(\sum_{k=1}^{p} L_k^* + (M + N)T\right) \]

(6)

From (5) and (6), we have

\[ \frac{\lambda + 1}{\lambda} \gamma \left(\sum_{k=1}^{p} L_k + T\right)\left(\sum_{k=1}^{p} L_k^* + (M + N)T\right) > 1, \]

which contradicts (3). This completes the proof. \( \square \)

**Remark 1.** It results from Lemma 1 that if \( u \in \Omega \),

\[ -u''(t) \leq -Mu(t) - Nu(\theta(t)), \quad t \not= t_k, \quad t \in J, \]

\[ \Delta u(t_k) = L_ku'(t_k), \quad k = 1, 2, \ldots, p, \]

\[ \Delta u'(t_k) \geq L_k^*u(t_k), \quad k = 1, 2, \ldots, p, \]

\[ u(0) = u(T), \quad u'(0) \geq \lambda u'(T), \]

(7)

where \( \lambda > 0, M > 0, N \geq 0, L_k \geq 0, L_k^* \geq 0 \) for \( k = 1, 2, \ldots, p \), and

\[ \left(\sum_{k=1}^{p} L_k + T\right)\left(\sum_{k=1}^{p} L_k^* + (M + N)T\right) \leq \frac{\lambda}{\lambda + 1} \]

hold, then \( u(t) \leq 0 \) on \( J \).
Remark 2. If we consider the estimate

\[
\left( \sum_{k=1}^{p} L_k + \tau(p + 1) \right) \left( \sum_{k=1}^{p} L_k^* + \tau(p + 1)(M + N) \right) \leq \frac{\lambda}{\lambda + 1},
\]

then the condition (3) of Lemma 1 is true. Furthermore, let \( \lambda = 1 \), then the condition (3) improves that of Lemma 1 in [6] for this kind of inequalities.

Consider the problem

\[-y''(t) = -My(t) - Ny(\theta(t)) + \sigma(t), \quad t \in J, \quad t \neq t_k, \quad k = 1, 2, \ldots, p,\]

\[\Delta y(t_k) = L_k y'(t_k) + I_k(u(t_k)) - L_k u'(t_k), \quad k = 1, 2, \ldots, p,\]

\[\Delta y'(t_k) = L_k^* y(t_k) + I_k^*(u(t_k)) - L_k^* u(t_k), \quad k = 1, 2, \ldots, p,\]

\[y(0) = y(T) + k_1, \quad y'(0) = \lambda y'(T) + k_2,\]

where \( u \in \Omega, \lambda \neq -1. \)

By direct computation we have the following results.

Lemma 2. \( y \in \Omega \) is a solution of (9) if and only if \( y \) is a solution of the impulsive integral equation

\[
y(t) = \omega(t) + \int_{0}^{T} G_1(t, s) \left[ -Ny(\theta(s)) + \sigma(s) \right] ds \\
+ \sum_{0 < t_k < T} G_2(t, t_k) \left( L_k y'(t_k) + I_k(u(t_k)) - L_k u'(t_k) \right) \\
- \sum_{0 < t_k < T} G_1(t, t_k) \left( L_k^* y(t_k) + I_k^*(u(t_k)) - L_k^* u(t_k) \right),
\]

where \( m = \sqrt{M}, C = (1 + \lambda)(1 - \frac{1}{2}e^{mT} - \frac{1}{2}e^{-mT}), \) and

\[
\omega(t) = \frac{1}{2mC} \left( mk_1 e^{-mt} + mk_1 e^{mt} - k_2 e^{-mt} + k_2 e^{mt} \right) \\
- (\lambda mk_1 - k_2) e^{m(T-t)} + (\lambda m + k_2) e^{-m(T-t)},
\]

\[
G_1(t, s) = \frac{1}{2mC} \begin{cases} 
(1 - \frac{1}{2}(1 + \lambda)e^{m(t-s)} - (1 - \frac{1}{2}(1 + \lambda)e^{-m(t-s)})e^{m(t-s)} \\
+ \frac{1}{2}(1 - \lambda)e^{-m(T-t-s)} - \frac{1}{2}(1 - \lambda)e^{m(T-t-s)}, \\
0 \leq s < t \leq T,
\end{cases}
\]

\[
\begin{cases} 
\frac{1}{2}(1 + \lambda)e^{-mT} - \lambda \left( \frac{1}{2}(1 + \lambda)e^{-m(T-t-s)} - \frac{1}{2}(1 + \lambda)e^{m(T-t-s)} \right) \\
+ \frac{1}{2}(1 - \lambda)e^{m(T-t-s)} - \frac{1}{2}(1 - \lambda)e^{m(T-t-s)}, \\
0 \leq t \leq s \leq T,
\end{cases}
\]

\[
G_2(t, t_k) = \frac{1}{2mC} \begin{cases} 
(1 - \frac{1}{2}(1 + \lambda)e^{m(t-t_k)} - (1 - \frac{1}{2}(1 + \lambda)e^{-m(t-t_k)})e^{m(t-t_k)} \\
+ \frac{1}{2}(1 - \lambda)e^{-m(T-t_k-s)} - \frac{1}{2}(1 - \lambda)e^{m(T-t_k-s)}, \\
0 \leq t_k < t \leq T,
\end{cases}
\]

\[
\begin{cases} 
\frac{1}{2}(1 + \lambda)e^{mT} - \lambda \left( \frac{1}{2}(1 + \lambda)e^{m(T-t_k-s)} - \frac{1}{2}(1 + \lambda)e^{-m(T-t_k-s)} \right) \\
+ \frac{1}{2}(1 - \lambda)e^{-m(T-t_k-s)} - \frac{1}{2}(1 - \lambda)e^{-m(T-t_k-s)}, \\
0 \leq t_k < s \leq T,
\end{cases}
\]
\[ G_2(t, s) = \frac{1}{2C} \begin{cases} 
(1 - \frac{1}{2}(1 + \lambda)e^{mT})e^{-m(t-s)} + (1 - \frac{1}{2}(1 + \lambda)e^{-mT})e^{m(t-s)} \\
\frac{1}{2}(1 - \lambda)e^{-m(T-t-s)} + \frac{1}{2}(1 - \lambda)e^{m(T-t-s)}, \\
0 \leq s < t \leq T,
\end{cases} \]

\[ \begin{cases} 
(\frac{1}{2}(1 + \lambda)e^{-mT} - \lambda)e^{-m(t-s)} + (\frac{1}{2}(1 + \lambda)e^{mT} - \lambda)e^{m(t-s)} \\
\frac{1}{2}(1 - \lambda)e^{-m(T-t-s)} + \frac{1}{2}(1 - \lambda)e^{m(T-t-s)}, \\
0 \leq t \leq s \leq T.
\end{cases} \]

Lemma 3. If \( M > 0, N \geq 0, L_k \geq 0, L_k^* \geq 0, \)

\[ g_1 \left( NT + \sum_{k=1}^{p} L_k^* \right) + g_2 \sum_{k=1}^{p} L_k < 1, \]  
(11)

\[ g_2 \left( NT + \sum_{k=1}^{p} L_k^* \right) + Mg_1 \sum_{k=1}^{p} L_k < 1, \]  
(12)

where \( g_1 = \max\{1, \lambda\} \frac{e^{mT} + 1}{m(1 + \lambda)(e^{mT} - 1)}, \) \( g_2 = \max\{1, \lambda\} \frac{1}{1 + \lambda} + \frac{1 - \lambda}{C}, \) \( m = \sqrt{M}, C = (1 + \lambda) \times (1 - \frac{1}{2} e^{mT} - \frac{1}{2} e^{-mT}), \) then Eq. (9) has a unique solution \( y \in \Omega. \)

Proof. By Lemma 2 and Banach fixed point theorem applied to the operator defined by the right-hand side of (10), the proof is apparent. \( \square \)

3. Main results

Functions \( \alpha, \beta \in \Omega \) are called upper and lower solutions of problem (1) if

\[ -\alpha''(t) \geq f\left(t, \alpha(t), \alpha(\theta(t))\right), \quad t \neq t_k, \ t \in J, \]
\[ \Delta \alpha(t_k) = I_k(\alpha(t_k)), \quad k = 1, 2, \ldots, p, \]
\[ \Delta \alpha'(t_k) \leq I_k^*(\alpha(t_k)), \quad k = 1, 2, \ldots, p, \]
\[ \alpha(0) = \alpha(T) + k_1, \quad \alpha'(0) \leq \lambda \alpha'(T) + k_2 \]

and

\[ -\beta''(t) \leq f\left(t, \beta(t), \beta(\theta(t))\right), \quad t \neq t_k, \ t \in J, \]
\[ \Delta \beta(t_k) = I_k(\beta(t_k)), \quad k = 1, 2, \ldots, p, \]
\[ \Delta \beta'(t_k) \geq I_k^*(\beta(t_k)), \quad k = 1, 2, \ldots, p, \]
\[ \beta(0) = \beta(T) + k_1, \quad \beta'(0) \geq \lambda \beta'(T) + k_2. \]

Theorem 1. Let (3), (11), (12) hold and \( \alpha(t), \beta(t) \in \Omega \) be upper and lower solutions of (1) with \( \beta(t) \leq \alpha(t). \) In addition assume that

(A1) The function \( f \in C(J \times R^2, R) \) satisfies

\[ f\left(t, \beta, \beta(\theta)\right) - f\left(t, u, u(\theta)\right) \leq M(u - \beta) + N(u(\theta) - \beta(\theta)) \]

and
\[ f(t, u, u(\theta)) - f(t, \alpha(\theta)) \leq M(\alpha - u) + N(\alpha(\theta) - u(\theta)), \]

\[ \beta(t) \leq u(t) \leq \alpha(t), \quad \beta(\theta(t)) \leq u(\theta(t)) \leq \alpha(\theta(t)), \quad u \in \Omega, \]

where \( M > 0, N \geq 0. \)

\((A_2)\) The function \( I_k \in C(R, R) \) satisfies \( I_k(x(t_k)) = L_kx'(t_k) \) where \( L_k \geq 0 \), and function \( I_k^* \in C(R, R) \) satisfies

\[ I_k^*(\beta(t)) - I_k^*(u(t)) \geq -L_k^*(\alpha(t) - u(t)) \]

and

\[ I_k^*(u(t)) - I_k^*(\alpha(t)) \geq -L_k^*(\alpha(t) - u(t)), \]

wherever \( \beta(t_k) \leq u \leq \alpha(t_k) \), where \( L_k^* > 0 \) and \( k = 1, 2, \ldots, p. \)

Then there exists a solution \( x \) of problem (1), such that \( \beta(t) \leq x \leq \alpha(t) \).

**Proof.** We consider the following problem:

\[-x''(t) = -Mx(t) - Nx(\theta(t)) + \sigma_q(t), \quad t \in J, \quad t \neq t_k, \quad k = 1, 2, \ldots, p, \]

\[ \Delta x(t_k) = L_kx'(t_k) + I_k(q(t_k, x(t_k))), \quad k = 1, 2, \ldots, p, \]

\[ \Delta x'(t_k) = L_k^*x(t_k) + I_k^*(q(t_k, x(t_k))), \quad k = 1, 2, \ldots, p, \]

\[ x(0) = x(T) + k_1, \quad x'(0) = \lambda x'(T) + k_2, \quad (13) \]

where

\[ \sigma_q(t) = f(t, q, q(\theta)) - Mq - Nq(\theta), \]

\[ q(t, x(t)) = \max\{\beta(t), \min(x, \alpha(t))\} = \begin{cases} \beta(t), & x \leq \beta(t), \\ x, & \beta(t) \leq x \leq \alpha(t), \\ \alpha(t), & x > \alpha(t). \end{cases} \]

Then \( x \in \Omega \) such that \( \beta \leq x \leq \alpha, t \in J \) is a solution of (1) if and only if \( x \) is a solution of (13). Next we shall show that problem (13) is solvable and that every solution of (13) satisfies \( \beta \leq x \leq \alpha, t \in J. \)

Assume that \( x \in \Omega \) is a solution of (13), we now show that \( \beta \leq x \leq \alpha, t \in J. \) Set \( v = \beta - x \), from \((A_1), (A_2)\), we have

\[-v''(t) = -(\beta''(t) - x''(t)) \leq f(t, \beta, \beta(\theta)) + Mx + Nx(\theta) - \sigma_q(t) \]

\[ \leq -Mv - Nv(\theta), \quad t \neq t_k, \ t \in J, \]

\[ \Delta v(t_k) = \Delta \beta(t_k) - \Delta x(t_k) = L_kv'(t_k), \quad k = 1, 2, \ldots, p, \]

\[ \Delta v'(t_k) = \Delta \beta'(t_k) - \Delta x'(t_k) \]

\[ \geq I_k^*(\beta(t_k)) - (L_k^*x(t_k) + I_k^*(q(t_k, x(t_k)))) - L_k^*q(t_k, x(t_k))) \]

\[ \geq L_k^*v(t_k), \quad k = 1, 2, \ldots, p, \]

\[ v(0) = \beta(0) - x(0) = v(T), \quad v'(0) = \lambda v'(T). \]

By Remark 1, we get \( v(t) \leq 0, t \in J, \) i.e., \( \beta \leq x \). Similar arguments show that \( x \leq \alpha. \)
Finally, we will show that (13) has at least one solution. By Lemma 2, \( x \in \Omega \) is a solution of (13) if and only if \( y \) is a solution of the impulsive integral equation

\[
x(t) = \omega(t) + \int_0^T G_1(t, s)[-Nx(\theta(s)) + \sigma_q(s)] \, ds \\
+ \sum_{0 < t_k < T} G_2(t, t_k)(L_k x'(t_k) + I_k(q(t_k, x(t_k))) - L_k q'(t_k, x(t_k))) \\
- \sum_{0 < t_k < T} G_1(t, t_k)(L^* k x(t_k) + I^*_k(q(t_k, x(t_k))) - L^*_k q(t_k, x(t_k))).
\]

(14)

Now we define the continuous and compact operator \( F : \Omega \to \Omega \) by the right-hand side of (14). Let \( l > 0 \), such that \( |\alpha(t)| \leq l, |\beta(t)| \leq l \) and take the compact set \( D = \{(t, x, y) \in \mathbb{R}^3; t \in J, \beta(t) \leq x \leq \alpha(t), \beta(\theta(t)) \leq y \leq \alpha(\theta(t))\} \). Since \( f, I_k, I^*_k, \omega(t), \omega'(t) \) are continuous and \( \beta(t) \leq q(t, x(t)) \leq \alpha(t), t \in J \), we can choose constants \( l_1, l_2, l_3, l_4 \) such that

\[
|f(t, x, y)| \leq l_1, |I^*_k(q(t_k, x(t_k))) - L^*_k q(t_k, x(t_k))| \leq l_2, k = 1, 2, \ldots, p, |\omega(t)| \leq l_3, |\omega'(t)| \leq l_4, t \in J.\]

For \( \mu \in (0, 1) \), Let \( x \) be any solution of \( x = \mu Fx \), and so,

\[
\|x\|_{PC} = \mu \|Fx\|_{PC} \\
\leq \sup \left\{ \omega(t) + \int_0^T G_1(t, s)[-Nx(\theta(s)) + \sigma_q(s)] \, ds \\
+ \sum_{0 < t_k < T} G_2(t, t_k)(L_k x'(t_k) + I_k(q(t_k, x(t_k))) - L_k q'(t_k, x(t_k))) \\
- \sum_{0 < t_k < T} G_1(t, t_k)(L^* k x(t_k) + I^*_k(q(t_k, x(t_k))) - L^*_k q(t_k, x(t_k))) \right\} \\
\leq (p + 1)l_2 g_1 + l_3 + g_1 T(l_1 + Ml + Nl) \\
+ \left( g_1 \left( NT + \sum_{k=1}^p L^*_k \right) + g_2 \sum_{k=1}^p L_k \right) \|x\|_{PC},
\]

and

\[
\|x'\|_{PC} = \mu \|Fx'\|_{PC} \leq (p + 1)l_2 g_2 + l_4 + g_2 T(l_1 + Ml + Nl) \\
+ \left( g_2 \left( NT + \sum_{k=1}^p L^*_k \right) + M g_1 \sum_{k=1}^p L_k \right) \|x\|_{PC},
\]

where \( g_1, g_2 \) are the same as in Lemma 3. We have
\[ \|x\|_{PC^1} \leq A, \]
\[ A = \max \left\{ \frac{(p+1)l_2 g_1 + l_3 + g_1 T(l_1 + Ml + Nl)}{1 - c_1}, \frac{(p+1)l_2 g_2 + l_4 + g_2 T(l_1 + Ml + Nl)}{1 - c_2} \right\}, \]
\[ c_1 = g_1 \left( NT + \sum_{k=1}^{p} L_k^* \right) + g_2 \sum_{k=1}^{p} L_k, \quad c_2 = g_2 \left( NT + \sum_{k=1}^{p} L_k^* \right) + Mg \sum_{k=1}^{p} L_k. \]

By Schauder’s fixed point theorem, \( F \) has at least a fixed point \( x \in \Omega \). This completes the proof. \( \square \)

**Theorem 2.** Let (3), (11), (12) hold and \( \alpha(t) \), \( \beta(t) \in \Omega \) be upper and lower solutions of (1) with \( \beta(t) \leq \alpha(t) \). In addition assume that

\( (A_3) \) The function \( f \in C(J \times R^2, R) \) satisfies

\[ f(t, x_1, y_1) - f(t, x_2, y_2) \geq -M(x_1 - x_2) - N(y_1 - y_2), \]

\[ \beta(t) \leq x_2 \leq x_1 \leq \alpha(t), \]

\[ \beta(\theta(t)) \leq y_2 \leq y_1 \leq \alpha(\theta(t)), \quad t \in J, \quad x_1, x_2, y_1, y_2 \in \Omega, \]

where \( M > 0, N \geq 0 \).

\( (A_4) \) The function \( I_k \in C(R, R) \) satisfies \( I_k(x(t_k)) = L_k x'(t_k) \) where \( L_k \geq 0 \), and function \( I_k^* \in C(R, R) \) satisfies

\[ I_k^*(x) - I_k^*(y) \leq L_k^*(x - y), \]

wherever \( \beta(t_k) \leq y \leq x \leq \alpha(t_k) \), where \( L_k^* \geq 0 \) and \( k = 1, 2, \ldots, p \).

Then there exist monotone sequences \( \{\alpha_n(t)\}, \{\beta_n(t)\} \) with \( \alpha_0(t) = \alpha(t), \beta_0(t) = \beta(t) \), which converge in \( \Omega \) to the extremal solutions of (1) in \( [\beta, \alpha] \), \( [\beta, \alpha] = \{ x \in \Omega: \beta(t) \leq x(t) \leq \alpha(t), t \in J \} \).

**Proof.** For any \( u \in [\beta, \alpha] \), consider (9) with

\[ \sigma(t) = f(t, u(t), u(\theta(t))) + Mu(t) + Nu(\theta(t)) \]

by Lemma 3, (11) has a unique solution \( y \in \Omega \). Denote an operator \( A: PC^1(J) \to PC^1(J) \) by \( y = Au \), then the operator \( A \) has the following properties:

(i) \( \beta_0 \leq A\beta_0, A\alpha_0 \leq \alpha_0 \).

Set \( v = \beta_0 - \beta_1, \) where \( \beta_1 = A\beta_0 \).

\[ -v''(t) = -\beta''_1(t) - \left( -\beta''_1(t) \right) \leq f(t, \beta_0, \beta_0(\theta(t))) \]
\[ \leq -M\beta_1 - N\beta_1(\theta(t)) + f(t, \beta_0, \beta_0(\theta(t))) + M\beta_0 + N\beta_0(\theta(t)) \]
\[ = -Mv - Nv(\theta(t)), \quad t \neq t_k, \ t \in J, \]
\[ \Delta v(t_k) = \Delta \beta_0(t_k) - \Delta \beta_1(t_k) \\
= I_k(\beta_0(t_k)) - (L_k \beta_1'(t_k) + I_k(\beta_0(t_k))) - L_k \beta_0'(t_k) \\
= L_k v'(t_k), \quad k = 1, 2, \ldots, p. \]

\[ \Delta v'(t_k) = \Delta \beta_0'(t_k) - \Delta \beta_1'(t_k) \geq I_k^*(\beta_0(t_k)) - (L_k^* \beta_1(t_k) + I_k^*(\beta_0(t_k))) - L_k^* \beta_0(t_k) \\
= L_k^* v(t_k), \quad k = 1, 2, \ldots, p. \]

\[ v(0) = \beta_0(0) - \beta_1(0) = v(T), \]
\[ v'(0) = \beta_0'(0) - \beta_1'(0) \geq \lambda v'(T). \]

By Remark 1, we get \( v(t) \leq 0 \) when \( t \in J \), i.e., \( \beta_0 \leq A \beta_0 \). Similar arguments show that \( A \alpha_0 \leq \alpha_0 \).

(ii) \( A \eta_1 \leq A \eta_2 \), if \( \beta \leq \eta_1 \leq \eta_2 \leq \alpha \).

Let \( u_1 = A \eta_1, u_2 = A \eta_2 \), set \( v = u_1 - u_2 \). Using (A1)–(A3), we get

\[ -v''(t) = -(u_1''(t) - u_2''(t)) \]
\[ = -Mu_1 - Nu_1(\theta(t)) + f(t, \eta_1(\theta(t))) + N \eta_1(\theta(t)) - [-Mu_2 - Nu_2(\theta(t)) + f(t, \eta_2(\theta(t))) + N \eta_2(\theta(t))] \]
\[ \leq -Mu - Nu(\theta(t)), \quad t \neq t_k, \ t \in J, \]

\[ \Delta v(t_k) = \Delta u_1(t_k) - \Delta u_2(t_k) = L_k v'(t_k), \]
\[ \Delta v'(t_k) = \Delta u_1'(t_k) - \Delta u_2'(t_k) \geq L_k^* v(t_k), \]
\[ v(0) = u_1(0) - u_2(0) = v(T), \]
\[ v'(0) \geq \lambda v'(T) \]

from Remark 1, we have \( v(t) \leq 0 \) when \( t \in J \), i.e., \( A \eta_1 \leq A \eta_2 \). From (i) and (ii), we get \( \beta_0 \leq A \beta_0 \leq A \alpha_0 \leq \alpha_0 \), and it is apparent that \( A \beta_0, A \alpha_0 \) are lower and upper solutions of (1), respectively.

Now let \( \alpha_n = A \alpha_{n-1}, \beta_n = A \beta_{n-1}, n = 1, 2, \ldots \). Following (i) and (ii), we have

\[ \beta_0 \leq \beta_1 \leq \cdots \leq \beta_n \leq \cdots \leq \alpha_n \leq \cdots \leq \alpha_1 \leq \alpha_0. \]

Obviously, each \( \alpha_i, \beta_i \ (i = 1, 2, \ldots) \) satisfies

\[
\begin{cases}
-\alpha_n''(t) + M \alpha_n + N \alpha_n(\theta(t)) \\
= f(t, \alpha_{n-1}, \alpha_{n-1}(\theta(t))) + M \alpha_{n-1} + N \alpha_{n-1}(\theta(t)), \quad t \neq t_k, \ t \in J, \\
\Delta \alpha_n(t_k) = L_k \alpha_n'(t_k) + I_k(\alpha_{n-1}(t_k)) - L_k \alpha_{n-1}'(t_k), \quad k = 1, 2, \ldots, p, \\
\Delta \alpha_n'(t_k) = L_k^* \alpha_n(t_k) + I_k^*(\alpha_{n-1}(t_k)) - L_k^* \alpha_{n-1}(t_k), \quad k = 1, 2, \ldots, p, \\
\alpha_n(0) = \alpha_n(T) + k_1, \alpha_n'(0) = \lambda \alpha_n'(T) + k_2
\end{cases}
\]
Theorem 3. Let assumptions of Theorem 2 hold. In addition assume that

\( (A_5) \) The function \( f \in C(\Omega \times \mathbb{R}^2, \mathbb{R}) \) satisfies

\[
\begin{align*}
&f(t, x_1, x_2) - f(t, y_1, y_2) \leq -M_1(x_1 - y_1) - N_1(y_2 - y_2), \\
&\beta(\theta(t)) \leq x_2 \leq x_1 \leq \alpha(t), \quad \beta(\theta(t)) \leq y_2 \leq y_1 \leq \alpha(\theta(t)), \\
&\quad t \in J,
\end{align*}
\]

where \( 0 \leq M_1 < M, 0 \leq N_1 < N \).

\( (A_6) \) The function \( I_k^* \in C(\mathbb{R}, \mathbb{R}) \) satisfies

\[
\begin{align*}
&I_k^*(x) - I_k^*(y) \geq I_k^*(x - y),
\end{align*}
\]

wherever \( \beta(t_k) \leq y \leq x \leq \alpha(t_k) \), where \( 0 \leq I_k^* < L_k^* \) and \( k = 1, 2, \ldots, p \). Then problem (1) has a unique solution in \( \Omega \).

Proof. From Theorem 2, problem (1) has the extremal solutions \( \bar{x}(t), \underline{x}(t) \in \Omega \) which satisfy

\[
\beta(t) \leq \underline{x}(t) \leq \bar{x}(t) \leq \alpha(t).
\]

Setting \( v(t) = \bar{x}(t) - \underline{x}(t) \), then for \( t \in J \), from

\[
\begin{align*}
&-v''(t) = -\left( \bar{x}''(t) - \underline{x}''(t) \right) \leq -M_1 v(t) - N_1 v(\theta(t)), \\
&\Delta v(t_k) = \Delta \bar{x}(t_k) - \Delta \underline{x}(t_k) = L_k v'(t_k), \\
&\Delta v'(t_k) = \Delta \bar{x}'(t_k) - \Delta \underline{x}'(t_k) \geq I_k^* v(t_k), \\
&v(0) = v(T), \quad v'(0) = \lambda v'(T).
\end{align*}
\]
By Lemma 1, we have $v(t) \leq 0$, when $t \in J$, i.e., $\bar{x}(t) \leq x(t)$ on $J$. So $\bar{x}(t) = x(t)$. This completes the proof of the theorem.

**Example 1.** Consider the problem of

$$-x''(t) = -x(t) + \sin(\sqrt{t}) + 1, \quad t \in [0, T], \quad t \neq t_k,$$

$$\Delta x(t_k) = \frac{1}{40} x'(t_k), \quad t = t_k,$$

$$\Delta x'(t_k) = \frac{1}{20} x(t_k), \quad t = t_k,$$

$$x(0) = x(T) - \frac{11}{20}, \quad x'(0) = 2x'(T) - \frac{6}{5},$$

where $T = \frac{1}{2}$, $k = 1$, $t_1 = \frac{T}{2}$. Set

$$\alpha(t) = \begin{cases} t + 2, & t \in \left[0, \frac{T}{2}\right], \\
\frac{11}{10}t + 2, & t \in \left(\frac{T}{2}, T\right], \end{cases} \quad \beta(t) = \begin{cases} t - 10, & t \in \left[0, \frac{T}{2}\right], \\
\frac{11}{10}t - 10, & t \in \left(\frac{T}{2}, T\right], \end{cases}$$

we can easily verify that $\alpha(t)$ is an upper solution, $\beta(t)$ is a lower solution with $\beta(t) \leq \alpha(t)$.

Set $M = 1$, $N = \frac{1}{10}$, $\lambda = 2$, $L_1 = \frac{1}{40}$, $L_1^* = \frac{1}{20}$ the conditions of Theorem 2 are all satisfied. So problem (15) has extremal solutions in the segment $[\beta(t), \alpha(t)]$.

**References**


