Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 312 (2007) 142–151
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Hilbert series of algebras associated to directed graphs

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Received 9 January 2006

Available online 29 November 2006

Communicated by Efim Zelmanov

Abstract

We compute the Hilbert series of some algebras associated to directed graphs and related to factorizations of noncommutative polynomials.

Keywords: Hilbert series; Directed graphs; Quadratic algebras

1. Introduction

In [2] we introduced a new class of algebras $A(\Gamma)$ associated to layered directed graphs Γ. These algebras arose as generalizations of the algebras Q_n (which are related to factorizations of noncommutative polynomials, see [1,3,6]), but the new class of algebras seems to be interesting by itself.

Various results have been proven for algebras $A(\Gamma)$. In [2] we constructed a linear basis in $A(\Gamma)$. In [4] we showed that algebras $A(\Gamma)$ are defined by quadratic relations for a large class of directed graphs and proved that in this case they are Koszul algebras. It follows immediately that the dual algebras to $A(\Gamma)$ are also Koszul and that their Hilbert series are related.

In this paper we continue to study algebras $A(\Gamma)$. In Section 2 we recall the definition of the algebra $A(\Gamma)$ and the construction of a basis for $A(\Gamma)$ given in [2]. In Section 3 we prove the main result of the paper, an expression for the Hilbert series, $H(A(\Gamma), t)$ of the algebra $A(\Gamma)$

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2006.06.048

corresponding to a layered graph Γ with a unique element $*$ of level 0. In stating this we denote the level of v by $|v|$ and write $v > w$ to indicate that v and w are vertices of the directed graph Γ and that there is a directed path from v to w. Then we have:

$$H\bigl(A(\Gamma), t\bigr) = \frac{1-t}{1+\sum_{v_1>v_2>\cdots>v_l \geqslant *}(-1)^l t^{|v_1|-|v_l|+1}}.$$

The proof uses matrices $\zeta(t)$ and $\zeta(t)^{-1}$ which generalize the zeta function and the Möbius function for partially ordered sets.

In Section 4 we specialize our results to the case of the Hasse graph of the lattice of subsets of a finite set, giving a derivation of the Hilbert series for the algebras Q_n that is shorter and more conceptual than that in [1]. In Section 5 we treat the case of the Hasse graph of the lattice of subspaces of a finite-dimensional vector space over a finite field. Finally, in Section 6, we define the complete layered graph $\mathbf{C}[m_n, m_{n-1}, \dots, m_1, m_0]$ and compute the Hilbert series of $A(\mathbf{C}[m_n, m_{n-1}, \dots, m_1, 1])$.

2. The algebra $A(\Gamma)$

We begin by recalling the definition of the algebra $A(\Gamma)$. Let $\Gamma = (V, E)$ be a **directed graph**. That is, V is a set (of vertices), E is a set (of edges), and $\mathbf{t}: E \to V$ and $\mathbf{h}: E \to V$ are functions. ($\mathbf{t}(e)$ is the *tail* of e and $\mathbf{h}(e)$ is the *head* of e.)

We say that Γ is **layered** if $V = \bigcup_{i=0}^n V_i$, $E = \bigcup_{i=1}^n E_i$, $\mathbf{t}: E_i \to V_i$, $\mathbf{h}: E_i \to V_{i-1}$. If $v \in V_i$ we will write $|v| = i$.

We will assume throughout the remainder of the paper that $\Gamma = (V, E)$ is a layered graph with $V = \bigcup_{i=0}^n V_i$, that $V_0 = \{*\}$, and that, for every $v \in V_+ = \bigcup_{i=1}^n V_i$, $\{e \in E \mid \mathbf{t}(e) = v\} \neq \emptyset$. For each $v \in V_+$ fix, arbitrarily, some $e_v \in E$ with $\mathbf{t}(e_v) = v$.

If $v, w \in V$, a **path** from v to w is a sequence of edges $\pi = \{e_1, e_2, \dots, e_m\}$ with $\mathbf{t}(e_1) = v$, $\mathbf{h}(e_m) = w$ and $\mathbf{t}(e_{i+1}) = \mathbf{h}(e_i)$ for $1 \leqslant i < m$. We write $v = \mathbf{t}(\pi)$, $w = \mathbf{h}(\pi)$. We also write $v > w$ if there is a path from v to w. Define $P_\pi(\tau) = (\tau - e_1)(\tau - e_2)\cdots(\tau - e_m) \in T(E)[\tau]$ and write

$$P_\pi(\tau) = \sum_{j=0}^m e(\pi, j)\tau^{m-j}.$$

Let π_v denote the path $\{e_1, \dots, e_{|v|}\}$ from v to $*$ with $e_1 = e_v$, $e_{i+1} = e_{\mathbf{h}(e_i)}$ for $1 \leqslant i < |v|$, and $\mathbf{h}(e_{|v|}) = *$.

Recall that R is the ideal of $T(E)$ generated by

$$\bigl\{e(\pi_1, k) - e(\pi_2, k) \bigm| \mathbf{t}(\pi_1) = \mathbf{t}(\pi_2),\ \mathbf{h}(\pi_1) = \mathbf{h}(\pi_2),\ 1 \leqslant k \leqslant l(\pi_1)\bigr\}.$$

The algebra $A(\Gamma)$ is the quotient $T(E)/R$.

For $v \in V_+$ and $1 \leqslant k \leqslant |v|$ we define $\hat{e}(v, k)$ to be the image in $A(\Gamma)$ of the product $e_1 \cdots e_k$ in $T(E)$ where $\pi_v = \{e_1, \dots, e_{|v|}\}$.

If $(v, k), (u, l) \in V \times \mathbf{N}$ we say (v, k) **covers** (u, l) if $v > u$ and $k = |v| - |u|$. In this case we write $(v, k) > (u, l)$. (In [2] we used different terminology and notation: if $(v, l) > (u, l)$ we said (v, l) can be composed with (u, l) and wrote $(v, l) \models (u, l)$.)

The following theorem is proved in [2, Corollary 4.5].

Theorem 1. *Let $\Gamma = (V, E)$ be a layered graph, $V = \bigcup_{i=0}^{n} V_i$, and $V_0 = \{*\}$ where $*$ is the unique minimal vertex of Γ. Then*

$$\bigl\{\hat{e}(v_1, k_1) \cdots \hat{e}(v_l, k_l) \bigm| l \geqslant 0, v_1, \ldots, v_l \in V_+,\ 1 \leqslant k_i \leqslant |v_i|,\ (v_i, k_i) \not> (v_{i+1}, k_{i+1})\bigr\}$$

is a basis for $A(\Gamma)$.

3. The Hilbert series of $A(\Gamma)$

Let $h(t)$ denote the Hilbert series $H(A(\Gamma), t)$, where Γ is a layered graph with unique minimal element $*$ of level 0. If $X \subseteq A(\Gamma)$ is a set of homogeneous elements (so $X = \bigcup_{i=0}^{\infty} X_i$ where $X_i = X \cap A(\Gamma)_i$), denote the "graded cardinality" $\sum_{i=0}^{\infty} |X_i| t^i$ of X by $\|X\|$. Let B denote the basis for $A(\Gamma)$ described in Theorem 1 and, for $v \in V_+$, let $B_v = \{\hat{e}(v_1, k_1) \cdots \hat{e}(v_l, k_l) \in B \mid v_1 = v\}$. Then $B = \{1\} \cup \bigcup_{v \in V_+} B_v$. Let $h_v(t)$ denote the graded dimension of the subspace of $A(\Gamma)$ spanned by B_v. Since B is linearly independent, we have $\|B\| = h(t)$ and $\|B_v\| = h_v(t)$. Then

$$\|B\| = h(t) = 1 + \sum_{v \in V_+} h_v(t).$$

Let $C_v = \bigcup_{k=1}^{|v|} \hat{e}(v, k) B$. Then

$$\|C_v\| = \bigl(t + \cdots + t^{|v|}\bigr) h(t) = t\left(\frac{t^{|v|} - 1}{t - 1}\right) h(t).$$

Now $C_v \supseteq B_v$. Let D_v denote the compliment of B_v in C_v. Then

$$D_v = \bigl\{\hat{e}(v, k)\hat{e}(v_1, k_1) \cdots \hat{e}(v_l, k_l) \bigm| 1 \leqslant k \leqslant |v|,$$
$$(v, k) > (v_1, k_1), \hat{e}(v_1, k_1) \cdots \hat{e}(v_l, k_l) \in B\bigr\}$$

and so

$$D_v = \bigcup_{v > v_1 > *} \hat{e}\bigl(v, |v| - |v_1|\bigr) B_{v_1}.$$

Then $\|D_v\| = \sum_{v > v_1 > *} t^{|v| - |v_1|} h_{v_1}(t)$ and so

$$h_v(t) = \|B_v\| = \|C_v\| - \|D_v\| = t\left(\frac{t^{|v|} - 1}{t - 1}\right) h(t) - \sum_{v > w > *} t^{|v| - |w|} h_w(t).$$

This equation may be written in matrix form. Arrange the elements of V in nonincreasing order and index the elements of vectors and matrices by this ordered set. Let $\mathbf{h}(t)$ denote the column vector with entry $h_v(t)$ in the v-position (where we set $h_*(t) = 1$), let $\mathbf{u}$ denote the vector with $t^{|v|}$ in the v-position, $\mathbf{e}_*$ denote the vector with δ_{*v} in the v-position, let $\mathbf{1}$ denote the column vector all of whose entries are 1, and let $\zeta(t)$ denote the matrix with entries $\zeta_{v,w}(t)$ for $v, w \in V$ where $\zeta_{v,w}(t) = t^{|v|-|w|}$ if $v \geqslant w$ and 0 otherwise. Note that

$$\zeta(t)\mathbf{e}_* = \mathbf{u}.$$

Then we have

$$\zeta(t)\big(\mathbf{h}(t) - \mathbf{e}_*\big) = \frac{t}{t-1}(\mathbf{u} - \mathbf{1})h(t)$$

and so

$$\mathbf{h}(t) - \mathbf{e}_* = \frac{t}{t-1}\big(\mathbf{u} - \zeta(t)^{-1}\mathbf{1}\big)h(t).$$

Then

$$\mathbf{1}^T\big(\mathbf{h}(t) - \mathbf{e}_*\big) = \frac{t}{t-1}\big(\mathbf{1}^T\mathbf{u} - \mathbf{1}^T\zeta(t)^{-1}\mathbf{1}\big)h(t)$$

or

$$h(t) - 1 = \frac{t}{t-1}\big(1 - \mathbf{1}^T\zeta(t)^{-1}\mathbf{1}\big)h(t).$$

Consequently, we have

Lemma 2.

$$\frac{1-t}{h(t)} = 1 - t\mathbf{1}^T\zeta(t)^{-1}\mathbf{1}.$$

Now $N(t) = \zeta(t) - I$ is a strictly upper triangular matrix and so $\zeta(t)$ is invertible. In fact, $\zeta(t)^{-1} = I - N(t) + N(t)^2 - \cdots$ and so the (v, w)-entry of $\zeta(t)^{-1}$ is

$$\sum_{v=v_1>\cdots>v_l=w\geqslant *} (-1)^{l+1}t^{|v|-|w|}.$$

Combining this remark with Lemma 2 we obtain the following result.

Theorem 3. *Let Γ be a layered graph with unique minimal element $*$ of level 0 and $h(t)$ denote the Hilbert series of $A(\Gamma)$. Then*

$$\frac{1-t}{h(t)} = 1 + \sum_{v_1>v_2>\cdots>v_l\geqslant *} (-1)^l t^{|v_1|-|v_l|+1}.$$

We remark that the matrices $\zeta(1)$ and $\zeta(1)^{-1}$ are well known as the zeta-matrix and the Möbius-matrix of V (cf. [5]).

In the remaining sections of this paper we will use Theorem 3 to compute the Hilbert series of the algebras $A(\Gamma)$ associated with certain layered graphs.

4. The Hilbert series of the algebra associated with the Hasse graph of the lattice of subsets of $\{1, \dots, n\}$

Let Γ_n denote the Hasse graph of the lattice of all subsets of $\{1, \dots, n\}$. Thus the vertices of Γ_n are subsets of $\{1, \dots, n\}$, the order relation $>$ is set inclusion $\supset$, the level $|v|$ of a set v is its cardinality, and the unique minimal vertex $*$ is the empty set $\emptyset$. Then the algebra $A(\Gamma_n)$ is the algebra Q_n defined in [3]. In this section we will prove the following theorem (from [1]). The present proof is much shorter and more conceptual than that in [1].

Theorem 4.

$$H(Q_n, t) = \frac{1-t}{1-t(2-t)^n}.$$

Our computations depend on the following lemma and corollary.

Lemma 5. *Let w be a finite set. Then*

$$\sum_{w \supset w_2 \supset \cdots \supset w_l = \emptyset} (-1)^l = (-1)^{|w|+1}.$$

Proof. If $|w| = 1$, both sides are $+1$. Assume the result holds for all sets of cardinality $< |w|$. Then

$$\sum_{w \supset w_2 \supset \cdots \supset w_l = \emptyset} (-1)^l = \sum_{w \supset w_2 \supseteq \emptyset} \sum_{w_2 \supset \cdots \supset w_l = \emptyset} (-1)^l$$

and, by the induction assumption, this is equal to

$$\sum_{w \supset w_2 \supseteq \emptyset} (-1)^{|w_2|}.$$

Since

$$\sum_{w \supset w_2 \supseteq \emptyset} (-1)^{|w_2|} = \sum_{w \supseteq w_2 \supseteq \emptyset} (-1)^{|w_2|} - (-1)^{|w|} = 0 + (-1)^{|w|+1} = (-1)^{|w|+1}$$

the proof is complete. □

Corollary 6. *Let $v \supseteq w$ be finite sets. Then*

$$\sum_{v = v_1 \supset v_2 \supset \cdots \supset v_l = w} (-1)^l = (-1)^{|v|-|w|+1}.$$

Proof. Let w' denote the complement of w in v. Sets u satisfying $v \subseteq u \subseteq w$ are in one-to-one correspondence with subsets of w' via the map $u \mapsto u \cap w'$. Thus

$$\sum_{v = v_1 \supset v_2 \supset \cdots \supset v_l = w} (-1)^l = \sum_{w' = v'_1 \supset \cdots \supset v'_l = \emptyset} (-1)^l.$$

By the lemma, this is $(-1)^{|w'|+1}$, giving the result. $\square$

To prove the theorem we observe that

$$\sum_{\substack{v_1 \supset v_2 \supset \cdots \supset v_l \supseteq v_l \supseteq \emptyset \\ l \geqslant 1}} (-1)^l t^{|v_1|-|v_l|+1} = \sum_{\{1,\ldots,n\} \supseteq v_1 \supseteq \emptyset} t^{|v_1|-|v_l|+1} \sum_{v_1 \supset \cdots \supset v_l \supseteq \emptyset} (-1)^l.$$

By Corollary 6, this is

$$\sum_{\{1,\ldots,n\} \supseteq v_1 \supseteq v_l \supseteq \emptyset} t^{|v_1|-|v_l|+1} (-1)^{|v_1|-|v_l|+1}.$$

Let u denote the compliment of v_l in v_1 and u' denote the complement of u in $\{1,\ldots,n\}$. Then the coefficient of t^{k+1} in the above expression is the number of ways of choosing a k-element subset $u \subseteq \{1,\ldots,n\}$ times the number of ways of choosing a subset $v \subseteq u'$. This is $\binom{n}{k} 2^{n-k}$. Thus

$$\sum_{\substack{v_1 \supset v_2 \supset \cdots \supset v_l \supseteq \emptyset \\ l \geqslant 1}} (-1)^l t^{|v_1|-|v_l|+1} = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} (-t)^{k+1} = -t(2-t)^k.$$

In view of Theorem 3, this completes the proof of the theorem.

5. The Hilbert series of algebras associated with the Hasse graph of the lattice of subspaces of a finite-dimensional vector space over a finite field

We will denote by $\mathbf{L}(\mathbf{n},\mathbf{q})$ the Hasse graph of the lattice of subspaces of an n-dimensional space over the field $\mathbf{F_q}$ of q elements. Thus the vertices of $\mathbf{L}(\mathbf{n},\mathbf{q})$ are subspaces of $\mathbf{F}_\mathbf{q}^n$, the order relation $>$ is inclusion of subspaces $\supset$, the level $|U|$ of a subspace U is its dimension, and the unique minimal vertex $*$ is the zero subspace (0).

Theorem 7.

$$\frac{1-t}{H(A(\mathbf{L}(\mathbf{n},\mathbf{q})),t)} = 1 - t \sum_{m=0}^{n} \binom{n}{m}_q (1-t)(1-tq)\cdots\left(1-tq^{n-m-1}\right).$$

Our proof depends on the following lemma and corollary.

Lemma 8. *Let U be a finite-dimensional vector space over $\mathbf{F_q}$. Then*

$$\sum_{\substack{U = U_1 \supset U_2 \supset \cdots \supset U_l = (0) \\ l \geqslant 1}} (-1)^l = (-1)^{|U|+1} q^{\binom{|U|}{2}}.$$

Proof. If $|U| = 0$, the sum occurring in the lemma has a single term corresponding to $l = 1$, $U = U_1 = (0)$. Then both sides of the expression in the lemma are equal to -1. Now let U be a finite-dimensional vector space and assume the result holds for all spaces of dimension less than $|U|$. Then

$$\sum_{\substack{U=U_1\supset U_2\supset\cdots\supset U_l=(0)\\ l\geqslant 1}}(-1)^l=\sum_{U=U_1\supset U_2}\ \sum_{\substack{U_2\supset\cdots\supset U_l=(0)\\ l\geqslant 1}}(-1)^l.$$

By the induction assumption, this is equal to

$$\sum_{U\supset U_2}(-1)^{|U_2|}q^{\binom{|U_2|}{2}}.$$

It is well known that the number of m-dimensional subspaces of the space U is given by the q-binomial coefficient $\binom{|U|}{m}_q$.

Hence

$$\sum_{\substack{U=U_1\supset U_2\supset\cdots\supset U_l=(0)\\ l\geqslant 1}}(-1)^l=\sum_{|U_2|=0}^{|U|-1}\binom{|U|}{|U_2|}_q(-1)^{|U_2|}q^{\binom{|U_2|}{2}}.$$

Recall the q-binomial theorem

$$\prod_{i=0}^{m-1}\left(1+xq^i\right)=\sum_{j=0}^{m}\binom{m}{j}_q q^{\binom{j}{2}}x^j.$$

Set $x=-1$. Then the $i=0$ factor in the product is 0 and so we have

$$\sum_{j=0}^{m-1}\binom{m}{j}_q(-1)^j q^{\binom{j}{2}}=(-1)^{m+1}q^{\binom{m}{2}}.$$

Thus

$$\sum_{\substack{U=U_1\supset U_2\supset\cdots\supset U_l=(0)\\ l\geqslant 1}}(-1)^l=(-1)^{|U|+1}q^{\binom{|U|}{2}}$$

as required. □

Corollary 9. *Let $V\supseteq W$ be subspaces of $\mathbf{F_q}$. Then*

$$\sum_{V=V_1\supset V_2\supset\cdots\supseteq V_l=W}(-1)^l=(-1)^{|V/W|+1}q^{\binom{|V/W|}{2}}.$$

Proof. Since subspaces Y, $V\supseteq Y\supseteq W$, are in one-to-one correspondence with subspaces of V/W via the map $Y\mapsto Y/W$, this is immediate from the lemma. □

To prove the theorem, we observe that

$$\sum_{\substack{V_1\supset V_2\supset\cdots\supset V_l\supseteq(0)\\ l\geqslant 1}} (-1)^l t^{|V_1/V_l|+1} = \sum_{\mathbf{F}_\mathbf{q}^n\supseteq V_1\supseteq V_l\supseteq(0)} t^{|V_1/V_l|+1} \sum_{\substack{V_1\supset V_2\supset\cdots\supset V_l\supseteq(0)\\ l\geqslant 1}} (-1)^l.$$

By Corollary 9, this is equal to

$$\sum_{\mathbf{F}_\mathbf{q}^n\supseteq V_1\supseteq V_l\supseteq(0)} t^{|V_1/V_l|+1}(-1)^{|V_1/V_l|+1} q^{\binom{|V_1/V_l|}{2}}.$$

Set $|v_l| = m$ and $|V_1/V_l| = k$. Then the number of possible V_l is $\binom{n}{m}_q$ and, for fixed V_l, the number of possible V_1 is the number of k-dimensional subspaces of $\mathbf{F}_\mathbf{q}^n/V_l$ which is $\binom{n-m}{k}_q$. Thus

$$\sum_{\substack{V_1\supset V_2\supset\cdots\supset V_l\supseteq(0)\\ l\geqslant 1}} (-1)^l t^{|V_1/V_l|+1} = \sum_{\substack{0<k,m\\ k+mn}} \leqslant \binom{n}{m}_q \binom{n-m}{k}_q (-t)^{k+1} q^{\binom{k}{2}}$$

$$= (-t)\sum_{m=0}^{n} \binom{n}{m}_q \sum_{k=0}^{n-m} \binom{n-m}{k}_q (-t)^k q^{\binom{k}{2}}.$$

Setting $x=-t$ in the q-binomial theorem shows that

$$\sum_{k=0}^{n-m} \binom{n-m}{k}_q (-t)^k q^{\binom{k}{2}} = \prod_{i=0}^{n-m-1} \left(1-tq^i\right).$$

Therefore

$$\sum_{\substack{V_1\supset V_2\supset\cdots\supset V_l\supseteq(0)\\ l\geqslant 1}} (-1)^l t^{|V_1/V_l|+1} = (-t)\sum_{m=0}^{n} \binom{n}{m}_q \prod_{i=0}^{n-m-1} \left(1-tq^i\right).$$

In view of Theorem 3, the theorem is proved.

Note that setting $q=1$ in the expression in Theorem 7 gives $1-t(2-t)^n$. By Theorem 4, this is $\frac{1-t}{H(Q_n,t)}$.

Recall (cf. [7]) that if A is a quadratic algebra it has a dual quadratic algebra, denoted $A^!$ and that if A is a Koszul algebra the Hilbert series of A and $A^!$ are related by

$$H(A,t)H\left(A^!,-t\right)=1.$$

Since by [4] $A(\mathbf{L}(\mathbf{n},\mathbf{q}))$ is a Koszul algebra, we have the following

Corollary 10.

$$H\left(A\left(\mathbf{L}(\mathbf{n},\mathbf{q})\right)^!,t\right) = 1 + \sum_{m=0}^{n-1} \binom{n}{m}_q (1+tq)\cdots\left(1+tq^{n-m-1}\right).$$

6. The Hilbert series of algebras associated with complete layered graphs

We say that a layered graph $\Gamma = (V, E)$ with $V = \bigcup_{i=0}^{n} V_i$ is **complete** if for every i, $1 \leqslant i \leqslant n$, and every $v \in V_1$, $w \in V_{i-1}$, there is a unique edge e with $\mathbf{t}(e) = v$, $\mathbf{h}(e) = w$. A complete layered graph is determined (up to isomorphism) by the cardinalities of the V_i. We denote the complete layered graph with $V = \bigcup_{i=0}^{n} V_i, |V_i| = m_i$ for $0 \leqslant i \leqslant n$, by $\mathbf{C}[m_n, m_{n-1}, \ldots, m_1, m_0]$. Note that the graph $\mathbf{C}[m_n, m_{n-1}, \ldots, m_1, 1]$ has a unique minimal vertex of level 0 and so Theorem 3 applies to $A(\mathbf{C}[m_n, m_{n-1}, \ldots, m_1, 1])$. We will show:

Theorem 11.

$$\frac{1-t}{H(A(\mathbf{C}[m_n, m_{n-1}, \ldots, m_1, 1]), t)} = 1 - \sum_{k=0}^{n} \sum_{a=k}^{n} (-1)^k m_a (m_{a-1} - 1)(m_{a-2} - 1) \cdots (m_{a-k+1} - 1) m_{a-k} t^{k+1}.$$

Proof. We first compute

$$\sum_{\substack{v_1 > v_2 > \cdots > v_l \geqslant \emptyset \\ l \geqslant 1}} (-1)^l t^{|v_1| - |v_l| + 1}.$$

The coefficient of t^{k+1} in the sum is

$$\sum_{\substack{v_1 > v_2 > \cdots > v_l \geqslant * \\ l \geqslant 1,\ |v_1| - |v_l| = k}} (-1)^l = \sum_{|v_1| = k}^{n} \sum_{\substack{v_1 > \cdots > v_l \\ |v_1| - |v_l| = k}} (-1)^l.$$

Note that the number of chains $v_1 > \cdots > v_l$ with $|v_i| = a_i$ for $1 \leqslant i \leqslant l$ is $m_{a_1} m_{a_2} \cdots m_{a_l}$. Then, writing $k = |v_1| - |v_l|$ and $a_1 = a$ we have

$$\begin{aligned}
\sum_{\substack{v_1 > v_2 > \cdots > v_l \geqslant * \\ l \geqslant 1}} (-1)^l t^{|v_1| - |v_l| + 1} &= \sum_{k=0}^{n} \Bigg(\sum_{\substack{v_1 > \cdots > v_l \geqslant * \\ l \geqslant 1,\ |v_1| - |v_l| = k}} (-1)^l \Bigg) t^{k+1} \\
&= \sum_{k=0}^{n} \Bigg(\sum_{\substack{a_1 > \cdots > a_{l-1} > a_1 - k \geqslant 0 \\ l \geqslant 1}} (-1)^l m_{a_1} m_{a_2} \cdots m_{a_{l-1}} m_{a_1 - k} \Bigg) t^{k+1} \\
&= \sum_{k=0}^{n} \Bigg(\sum_{a=k}^{n} m_a (1 - m_{a-1}) \cdots (1 - m_{a-k+1}) m_{a-k} \Bigg) t^{k+1}.
\end{aligned}$$

The theorem now follows from Theorem 3. □

This result applies, in particular, to the case $m_0 = m_1 = \cdots = m_n = 1$. The resulting algebra $A(\mathbf{C}[1, \dots, 1])$ has n generators and no relations. Theorem 11 shows that

$$\frac{1-t}{H(A(\mathbf{C}[1, \dots, 1]), t)} = 1 - \sum_{a=0}^{n} t + \sum_{a=1}^{n} t^2 = (1-t)(1-nt).$$

Thus $H(A(\mathbf{C}[1, \dots, 1]), t) = \frac{1}{1-nt}$ and we have recovered the well-known expression for the Hilbert series of the free associative algebra on n generators.

Since by [4] the algebras associated to complete directed graphs are Koszul algebras, we have the following corollary.

Corollary 12.

$$\begin{aligned} &H\bigl(A\bigl(\mathbf{C}[m_n, m_{n-1}, \dots, m_1, 1]\bigr)^!, t\bigr) \\ &\quad = 1 + \sum_{k=1}^{n} \sum_{a=k}^{n} m_a (m_{a-1} - 1)(m_{a-2} - 1) \cdots (m_{a-k+1} - 1) t^k. \end{aligned}$$

Acknowledgments

During preparation of this paper Vladimir Retakh was partially supported by NSA. We thank the referee for useful comments.

References

[1] I. Gelfand, S. Gelfand, V. Retakh, S. Serconek, R. Wilson, Hilbert series of quadratic algebras associated with decompositions of noncommutative polynomials, J. Algebra 254 (2002) 279–299.
[2] I. Gelfand, V. Retakh, S. Serconek, R. Wilson, On a class of algebras associated to directed graphs, Selecta Math. (N.S.) 11 (2005) 281–295.
[3] I. Gelfand, V. Retakh, R. Wilson, Quadratic-linear algebras associated with decompositions of noncommutative polynomials and differential polynomials, Selecta Math. (N.S.) 7 (2001) 493–523.
[4] V. Retakh, S. Serconek, R. Wilson, On a class of Koszul algebras associated to directed graphs, J. Algebra 304 (2) (2006) 1114–1129.
[5] G.-C. Rota, On the foundations of combinatorial theory, I. Theory of Möbius functions, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 2 (1964) 340–368.
[6] S. Serconek, R. Wilson, Quadratic algebras associated with decompositions of noncommutative polynomials are Koszul algebras, J. Algebra 278 (2004) 473–493.
[7] V.A. Ufnarovskij, Combinatorial and asymptotic methods in algebra, in: A.I. Kostrikin, I.R. Shafarevich (Eds.), Algebra, vol. VI, Springer-Verlag, New York, 1995, pp. 1–196.