Convergence of Global and Bounded Solutions of the Wave Equation with Linear Dissipation and Analytic Nonlinearity

Mohamed Ali Jendoubi

Université Pierre et Marie Curie, Laboratoire d’Analyse Numérique,
Tour 55-65, 5ème étage, 4 pl. Jussieu, 75252 Paris cedex 05, France
E-mail: jendoubi@ann.jussieu.fr

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We prove convergence of global, bounded, and smooth solutions of the wave equation with linear dissipation and analytic nonlinearity. A generalization and examples of applications are given at the end of the paper.

1. INTRODUCTION

In this paper we study the asymptotic behavior, as \( t \to \infty \), of global and bounded solutions of the second-order evolution problem

\[
\begin{align*}
\partial_t u + \partial_x u &= \Delta u + f(x, u) \quad &\text{in } \mathbb{R}^+ \times \Omega \\
\partial_t u &= 0 \quad &\text{on } \mathbb{R}^+ \times \partial \Omega \\
\partial_x u \big|_{t=0} &= u_0(x) \quad &\text{in } \Omega \\
\partial_t u \big|_{t=0} &= u_1(x) \quad &\text{in } \Omega
\end{align*}
\]

(1.1)

where \( \Omega \) is a bounded smooth domain of \( \mathbb{R}^N \), \( \alpha > 0 \) and

\( f: \Omega \times \mathbb{R} \to \mathbb{R} \)

\((x, s) \mapsto f(x, s)\)

is a smooth function.

The parabolic equation

\[
\begin{align*}
\partial_t u - \Delta u &= f(u) \quad &\text{in } \mathbb{R}^+ \times \Omega \\
\partial_t u &= 0 \quad &\text{on } \mathbb{R}^+ \times \partial \Omega
\end{align*}
\]

(1.2)

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has been studied earlier in the literature. When \( N = 1 \), H. Matano [8] and T. J. Zelenyak [12] have independently proved convergence of global and bounded solutions of (1.2) to some equilibrium point solution of the associated stationary equation. For \( N > 1 \), and by using the maximum principle, P. L. Lions [7] has proved that “most” bounded solutions of (1.2) are convergent; L. Simon [10] has proved convergence if the nonlinearity is real analytic. J. K. Hale and G. Raugel [2] (see also [9], and [1] for a generalization) used a perturbation method to obtain the convergence result for (1.2) under the assumption that the domain is sufficiently close to an arc. This last result is also true for Eq. (1.1). A. Haraux and P. Polacik [5] have proved convergence of all non-negative bounded solutions of (1.1) and (1.2) when \( \Omega \) is a ball and \( f \) is independent of \( x \). In general, it is unknown whether convergence occurs for (1.1).

It is well-known that every bounded precompact semi-orbit \((u, u_t)\) of (1.1) is such that

\[
\lim_{t \to \infty} \left\{ \|u_t\| + \text{dist}_{H^1(\Omega)}(u(t, \cdot), \mathcal{S}) \right\} = 0 \tag{1.3}
\]

where we denote by \( \|x\| \) the norm of \( x \) in \( L^2(\Omega) \) and

\[
\mathcal{S} = \{ \varphi \in H^2(\Omega) \cap H^1(\Omega) : -\Delta \varphi = f(x, \varphi) \text{ in } \Omega \}.
\]

In [4], we studied global bounded solutions of the following second-order gradient-like ordinary differential system:

\[
U_{tt} + U_t = F(U), \tag{1.4}
\]

where \( F: \mathbb{R}^N \to \mathbb{R} \) is analytic, and by introducing a new Liapunov function we established convergence to an equilibrium of any such solution. After this result, it becomes natural to study the convergence problem for (1.1) when \( f \) is analytic. The method will require additional regularity assumptions on \( u \) itself. More precisely, in this paper we assume that

\[
f \text{ is analytic in } s, \tag{1.5}
\]

\[
f(x, s), \frac{\partial f}{\partial s} (x, s) \text{ and } \frac{\partial^2 f}{\partial s^2} (x, s) \text{ are bounded in } \Omega \times (-\beta, \beta) \forall \beta > 0, \tag{1.6}
\]

and we prove the following result.

**Theorem 1.1.** Under the hypotheses (1.5), (1.6), let \( u \) be a solution of (1.1) and assume that there exists \( p \geq 2 \) such that

\[
\bigcup_{t \geq 1} \{ u(t, \cdot), u_t(t, \cdot) \} \text{ is precompact in } W^{2, p}(\Omega) \times W^{1, p}(\Omega) \tag{1.7}
\]
with \( p > N/2 \) if \( N \leq 6 \), and \( p > N \) if \( N > 6 \). Then there exists \( \varphi \in \mathcal{S} \) such that
\[
\lim_{t \to +\infty} \left\{ \| u_t \| + \| u(t, \cdot) - \varphi(\cdot) \|_{W^{2,p}(\Omega)} \right\} = 0.
\]

**Remark 1.2.** In the Theorem, we can change \(-A\) by a general second-order self-adjoint, elliptic operator, and we do not have to assume that the coefficients are analytic.

Below we denote by \( \| u \|_{2,p} \) the norm of \( u \) in \( W^{2,p}(\Omega) \), and \( C_1, C_2, ... \) denote positive constants.

The paper is organized as follows: In Section 2 we prove Theorem 1.1 and we give some examples of application. In Section 3 we give a generalization of our main result. In each section some remarks are presented.

2. PROOF OF THEOREM 1.1 AND EXAMPLES OF APPLICATION

2.1. **Proof of Theorem 1.1**

For the proof of Theorem 1.1, we have to use the following result from [10].

**Proposition 2.1 (Theorem of Simon).** Under hypotheses (1.5), (1.6), let \( \varphi \in \mathcal{S} \), then there exist \( \theta \in (0, 1/2) \) and \( \sigma > 0 \) such that \( \forall u \in W^{2,p}(\Omega) \cap H^1_0(\Omega) \),
\[
\| u - \varphi \|_{2,p} < \sigma
\]
\[
\| Au + f(x, u) \| \geq |E(u) - E(\varphi)|^{1-\theta}
\]
(2.1)
where \( E(u) = \int_{\Omega} \frac{1}{2} |\nabla u|^2 - f(x, u) \, dx \) and \( F(x, u) = \int_0^u f(x, s) \, ds \).

For the proof of this Proposition, we refer to [10, Theorem 3, p. 537].

**Proof of Theorem 1.1.** In all what follows, \( \alpha = 1 \), the case \( \alpha \neq 1 \) can be treated by the same way. First, we have (cf. [3, Theorem 2.4.3, p. 32])
\[
\| u \| \to 0 \quad \text{when} \quad t \to \infty.
\]
(2.2)

Now let \((u_0, u_1) \in H^1_0(\Omega) \times L^1(\Omega)\) such that \((u_0, u_1)\) satisfies (1.7), and let us define the \( \omega \)-limit set of \((u_0, u_1)\) by:
\[
\omega(u_0, u_1) = \{ (\varphi, 0); \varphi \in W^{2,p}(\Omega) \cap H^1_0(\Omega),
\]
\[
3t_n \to +\infty, \lim_{n \to +\infty} \| u(t_n, \cdot) - \varphi \|_{2,p} = 0 \}.
\]
(2.3)

We know that \( \omega(u_0, u_1) \) is a nonempty compact, connected set. As a consequence of (2.2) we also have (and we refer to [3] for a simple proof) \( \omega(u_0, u_1) \subset \mathcal{S} \).
Let \((\varphi, 0) \in \mathcal{C}(u_0, u_1)\), up to the change of variable \(u = \varphi + v\), we can assume that \(\varphi = 0\) and then \(f(x, 0) = 0\ \forall x \in \Omega\).

Now let \(\varepsilon\) be a positive real number, and we define for all \(t \geq 1\)

\[
H(t) = \frac{1}{2} \int_{\Omega} |u_t|^2 \, dx + \epsilon \int_{\Omega} [Au + f(x, u)] \, u_t \, dx + \epsilon \int_{\Omega} |\nabla u_t|^2 \, dx
+ \epsilon \int_{\Omega} [Au + f(x, u)]^2 \, dx - \epsilon \int_{\Omega} f'(x, u)|u_t|^2 \, dx.
\]

We note that \(H\) makes sense as a consequence of (1.7).

We have for all \(t \geq 1\):

\[
H'(t) = -\int_{\Omega} |u_t|^2 + \epsilon \int_{\Omega} [Au + f(x, u)]^2 \, dx + \epsilon \int_{\Omega} [Au + f(x, u)] \, u_t \, dx
- \epsilon \int_{\Omega} [Au + f'(x, u)]u_t \, dx - 2\varepsilon \int_{\Omega} Au_t u_{tt} \, dx
+ 2\varepsilon \int_{\Omega} [Au + f'(x, u)]u_t |u_t + u_t| \, dx
- 2\varepsilon \int_{\Omega} f'(x, u) u_t u_{tt} \, dx - \epsilon \int_{\Omega} f'(x, u) |u_t|^2 \, dx.
\]

Then for all \(t \geq 1\), using the Cauchy–Schwarz in the term \(\int_{\Omega} [Au + f(x, u)] \, u_t \, dx\) we find

\[
H'(t) \leq \int_{\Omega} [-\frac{1}{2} + \epsilon f'(x, u) - \epsilon f'(x, u) |u_t|] \, u_t \, dx - \epsilon \int_{\Omega} |\nabla u_t|^2 \, dx
- \epsilon \int_{\Omega} |\nabla u|^2 \, dx.
\]

This calculation is formal, but can be rigorously justified by using hypothesis (1.7), the fact that \(W^{1, r}(\Omega) \hookrightarrow L^q(\Omega)\) and a density argument.

We note that by hypothesis (1.7), \(u \in W^{2, r}(\Omega) \hookrightarrow L^q(\Omega)\) and then \(f'(x, u)\) and \(f''(x, u)\) remain bounded.

Now, when \(N \leq 6\) we have \(H^1_0(\Omega) \hookrightarrow L^q(\Omega)\) with continuous imbedding and then we obtain

\[
-\epsilon \int_{\Omega} f'(x, u) u_t \, dx - \epsilon \int_{\Omega} |\nabla u_t|^2 \, dx \leq \epsilon C_1 \|u_t\|_{L^q(\Omega)}^2 - \epsilon \int_{\Omega} |\nabla u|^2 \, dx
\]

\[
\leq \epsilon C_2 \|u_t\|_{L^q(\Omega)}^2 - \epsilon \int_{\Omega} |\nabla u|^2 \, dx.
\]
By combining hypotheses (1.7) and (2.2), there exists $T > 0$ such that for all $t \geq T$

$$-\varepsilon \int_Q f^*(x, u) u_t^i \, dx - \varepsilon \int_Q |\nabla u_t|^2 \, dx \leq -\frac{\varepsilon}{2} \|\nabla u_t\|^2.$$  

In the case $N > 6$, we have $p > N$ and then $u_t \in W^{1,p} \hookrightarrow L^\infty$. So

$$-\varepsilon \int_Q f^*(x, u) u_t^i \, dx \leq \varepsilon C_3 \int_Q u_t^i \, dx.$$  

In both cases, by choosing $\varepsilon$ small enough there exists $C_4 > 0$ such that we have for all $t \geq T$

$$-H'(t) \geq C_4 \\{ \|u_t\|^2 + \|Au + f(x, u)\|^2 + \|\nabla u_t\|^2 \} \geq \frac{C_4}{3} \\{ \|u_t\|^2 + \|Au + f(x, u)\|^2 + \|\nabla u_t\|^2 \}.$$  

Then $H$ is nonincreasing on $[T, \infty)$, since $0 \leq \omega(u_0, u_1)$ then $H(t) \geq 0$, $\forall t \geq T$ and $H(t) \to 0$ when $t \to \infty$.

On the other hand, let $\theta$ be as in Proposition 2.1, then we obtain for all $t \geq 0$

$$-\frac{d}{dt}[H(t)] = -\theta H(t) [H(t)]^{\theta-1}. \quad (2.5)$$  

By using Holder’s inequality, we get

$$[H(t)]^{1-\theta} \leq C_5 \\{ \|u_t\|^{2(1-\theta)} + \|E(u)\|^{(1-\theta)} + \|\nabla u_t\|^{2(1-\theta)} + \|Au + f(x, u)\|^{2(1-\theta)} \}.$$  

Thanks to Young’s inequality we obtain

$$\|Au + f(x, u)\|^{(1-\theta)} \|u_t\|^{(1-\theta)} \leq \|Au + f(x, u)\| + \|u_t\|^{(1-\theta)\theta}.$$  

Then (2.6) becomes

$$[H(t)]^{1-\theta} \leq C_5 \\{ \|u_t\|^{2(1-\theta)} + \|E(u)\|^{(1-\theta)} + \|\nabla u_t\|^{2(1-\theta)} \}.$$  

By using (1.7), (2.2), the fact that $(1-\theta)/\theta \geq 1$ and $0 \leq \omega(u_0, u_1)$ we obtain for all $t \geq T$

$$[H(t)]^{1-\theta} \leq C_6 \\{ \|u_t\| + |E(u)|^{1-(1-\theta)} + \|\nabla u_t\| + \|Au + f(x, u)\| \}. \quad (2.7)$$
Since $0 \in o(U_0, \sigma)$, there exists a sequence $t_n \to \infty$ such that $u(t_n, \cdot) \to 0$ when $n \to \infty$. Then for all $\delta > 0$ ($\delta \ll \sigma$) there exists $N = N(\delta) > 0$ such that $t_N > T$ and

$$||u(t_N, \cdot)|| < \frac{\delta}{2}, \quad ||u(t_N, \cdot)||_{2, p} < \frac{\delta}{2}, \quad \frac{6C_6}{\partial C_4} [H(t_N)]^\alpha < \frac{\delta}{2} \quad (2.8)$$

Let

$$\ell = \text{Sup}\{ t \geq t_N/||u(s, \cdot)||_{2, p} < \sigma, \forall s \in [t_N, t] \}. \quad (2.9)$$

So by using (2.1), (2.7) we have for all $t \in (t_N, \ell)$

$$[H(t)]^{1-\alpha} \leq 2C_6(||u_t|| + ||\Delta u + f(x, u)|| + ||\nabla u||). \quad (2.10)$$

By combining (2.4), (2.5), and (2.10) we obtain for all $t \in (t_N, \ell)$:

$$-\frac{d}{dt} [H(t)]^\alpha \geq \frac{\theta C_4}{6C_6} (||u_t|| + ||\Delta u + f(x, u)|| + ||\nabla u||). \quad (2.11)$$

By integrating (2.11) over $(t_N, \ell)$ we get

$$\int_{t_N}^\ell ||u_t|| \, dt \leq \frac{6C_6}{\partial C_4} [H(t_N)]^\alpha. \quad (2.12)$$

Now assuming $\ell < \infty$ we can write

$$||u(\ell, \cdot)|| \leq \int_{t_N}^\ell ||u_t|| \, dt + ||u(t_N, \cdot)||.$$ 

Then by (2.8), (2.12) we have

$$||u(\ell, \cdot)|| < \delta.$$ 

It suffices now to use (1.7) and to choose $\delta$ small enough in (2.8) to obtain $||u(\ell, \cdot)||_{2, p} < \sigma$, which contradicts (2.9) if $\ell < \infty$. Therefore $\ell = \infty$.

So (2.12) becomes

$$\int_{t_N}^{\infty} ||u_t|| \, dt \leq \frac{6C_6}{\partial C_4} [H(t_N)]^\alpha < \infty. \quad (2.14)$$

(2.14) implies the convergence of $u$ in $L^2(\Omega)$, by (1.7) $u$ converges in $W^{2, p}(\Omega)$. Theorem 1.1 is completely proved.
2.2. Examples

Here we give examples of nonlinearities for which we can apply our result.

**Example 1.** The Sine–Gordon equation

\[
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \Delta u = \beta \sin u & \quad \text{in } \mathbb{R}^+ \times \Omega \\
\frac{\partial u}{\partial t}(., t) = 0 & \quad \text{on } \mathbb{R}^+ \times \partial \Omega \\
u(0, .) = u_0(., .) & \quad \text{in } \Omega
\end{aligned}
\]

(2.15)

where \( \alpha > 0, \beta \in \mathbb{R} \).

**Example 2.** The Relativistic quantum mechanics equation.

\[
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \Delta u + Cu^p = \lambda u & \quad \text{in } \mathbb{R}^+ \times \Omega \\
u(., t) = 0 & \quad \text{on } \mathbb{R}^+ \times \partial \Omega \\
u(0, .) = u_0(., .) & \quad \text{in } \Omega \\
u_t(0, .) = u_t(., .) & \quad \text{in } \Omega
\end{aligned}
\]

(2.16)

where \( p \) is an odd integer, \( C > 0 \) and \( \lambda > \lambda_1 \).

The case \( \lambda < \lambda_1 \) is not interesting since \( \mathcal{S} = \{0\} \).

**Remark 2.2.** When \( N = p = 3 \) and \( \lambda = 0 \), (2.16) is the equation of Relativistic quantum mechanics. In this case, convergence of solution can be obtained from the fact that \( \mathcal{S} = \{0\} \).

**Remark 2.3.** If \( \mathcal{S} \) is discrete, then by the connectedness of \( \omega(u_0, u_1) \), we know that \( \omega(u_0, u_1) \) is a singleton and the convergence follows. In Example 2, when \( C = 1, p = 3, \) and \( \Omega \) is the unit ball of \( \mathbb{R}^2 \), Haraux [3] has proved that \( \mathcal{S} \) is a continuum of solutions.

3. GENERALIZATION AND EXAMPLES

Let \( d \) be an integer, \( H := L^2(\Omega, \mathbb{R}^d) \) the usual Hilbert space and we consider a real Hilbert space \( V \subset L^2(\Omega, \mathbb{R}^d) \), such that \( V \) is dense in \( L^2(\Omega, \mathbb{R}^d) \) and the imbedding of \( V \) in \( L^2(\Omega, \mathbb{R}^d) \) is compact. We identify \( L^2(\Omega, \mathbb{R}^d) \) with its dual and we denote by \( V^* \) the dual of \( V \). The inner product and
the norm in $\mathbb{R}^d$ are denoted by $\langle \cdot, \cdot \rangle$, $|\cdot|$, respectively. The norm in $V$ and $L^2(\Omega, \mathbb{R}^d)$ are denoted by $\| \cdot \|_V$, $\| \cdot \|$, respectively.

Let, on the other hand, $a(u, v)$ be a bilinear continuous form on $V$ which is symmetric and coercive, which means:

$$\exists C > 0, \quad a(u, u) \geq C \|u\|_V^2 \quad \forall u \in V.$$  

With this form we associate the linear operator $A$ from $V$ into $V'$ defined by

$$\int_{\Omega} (Au, v) \, dx = a(u, v), \quad \forall u, v \in V,$$

$A$ is an algebraic and topological isomorphism from $V$ into $V'$ and it can also be considered as a self-adjoint unbounded operator in $L^2(\Omega, \mathbb{R}^d)$ with domain $D(A) \subset V$.

$$D(A) = \{ v \in V, Av \in L^2(\Omega, \mathbb{R}^d) \}.$$

The space $D(A)$ is equipped with the graph norm $\|Au\| + \|u\|$ or more simply with the norm $\|Au\|$ which is equivalent. The spectral theory of these operators allows us to define the powers $A^s$ of $A$ for $s \in \mathbb{R}$. For every $s > 0$, $A^s$ is an unbounded self-adjoint operator in $L^2(\Omega, \mathbb{R}^d)$ with a dense domain $D(A^s) \subset V$. The operator $A^s$ is strictly positive and injective. The space $D(A^s)$ is endowed with the scalar product and the norm

$$\langle u, v \rangle_{D(A^s)} = \int_{\Omega} (A^s u, A^s v) \, dx,$$

$$\|u\|_{D(A^s)} = \left( \langle u, u \rangle_{D(A^s)} \right)^{1/2},$$

which makes it a Hilbert space and $A^s$ is an isomorphism from $D(A^s)$ onto $L^2(\Omega, \mathbb{R}^d)$. For $s = 1$, we recover $D(A)$ and for $s = \frac{1}{2}$, $D(A^{1/2}) = V$.

For all $q \geq 1$ we denote by $X_q$ the space $A^{-1}(L^q(\Omega, \mathbb{R}^d))$, endowed with the graph norm for which it becomes a Banach space.

Furthermore we assume that there exist $p \geq 2$ such that:

(H1) $A^{-1}(L^p(\Omega, \mathbb{R}^d)) \hookrightarrow L^{\infty}(\Omega, \mathbb{R}^d)$ with continuous injection.

(H2) For all $a \in L^\infty(\Omega, \mathbb{R}^d)$, $h \in L^\infty(\Omega, \mathbb{R}^d)$, if $u \in D(A)$ is a solution of $Au + a(x)u = h$, then $u \in L^p(\Omega, \mathbb{R}^d)$.

Let $B : H \rightarrow H$ be a bounded linear operator satisfying the coerciveness condition

$$\forall x \in H \quad B(x, x) \geq \alpha \|x\|^2$$

for some $\alpha > 0$. 

Now let
\[ F: \Omega \times \mathbb{R}^d \to \mathbb{R} \]
\[(x, s) \mapsto F(x, s)\]
which verify that.

\[(H3) \ F \text{ is analytic with respect to } s \text{ "uniformly" in } x \in \Omega, \nabla F(.,.) \text{ and } \nabla^2 F(.,.) \text{ are bounded on } \Omega \times (-\beta, \beta)^d, \forall \beta \in \mathbb{R}^+ .\]

We consider the evolution problem
\[
\begin{cases}
  u_t + Bu_t + Au = f(x, u) \\
  u(0, .) = u_0( . ) \\
  u_t(0, .) = u_1( . )
\end{cases}
\] (3.1)
and the stationary problem
\[
Au = f(x, u) \] (3.2)
where \(f: X \to L^p(\Omega, \mathbb{R}^d)\) with \(f(x, u(x)) = \nabla F(x, u(x)).\) (Here \(\nabla = (\partial/\partial s_1, ..., \partial/\partial s_d).\))

We set
\[
E(u) = \frac{1}{2} \int_{\Omega} (Au, u) \, dx - \int_{\Omega} F(x, u) \, dx
\]
and \(\mathcal{S} = \{ \psi \in D(A)/A\psi = f(x, \psi) \}.\) We denote by \(X\) the space \(A^{-1}(L^p(\Omega, \mathbb{R}^d))\) and by \(\|u\|_X, \|u\|_p\) the norms of \(u\) in \(X, L^p(\Omega, \mathbb{R}^d),\) respectively.

Under these hypotheses, we have the following result:

**Theorem 3.1.** Let \(u\) be a solution of (3.1) and assume that
\[
\bigcup_{t > 1} \{ u(t, .), u_t(t, .) \} \text{ is precompact in } X \times (V \cap L^\infty(\Omega, \mathbb{R}^d)).
\]
Then there exists \(\psi \in D(A)\) solution of (3.2) such that
\[
\lim_{t \to \infty} \{ \|u(t, .)\| + \|u(t, .) - \psi( . )\|_X \} = 0.
\]

The proof of this result is the same as Theorem 1.1, but we have to use the following proposition instead of Proposition 2.1.
PROPOSITION 3.2. Let \( \varphi \in D(A) \) be a solution of (3.2), then there exist \( \theta \in (0, \frac{1}{2}) \) and \( \sigma > 0 \) such that \( \forall u \in X \),
\[
\|Au + f(x, u)\| \geq |E(u) - E(\varphi)|^{1-\theta}.
\]

We refer to [6] for the proof of this proposition.

EXAMPLE 3.3. We consider a system of Sine-Gordon equations occurring in the Josephson junctions (see R. Temam [11, p.217] and the reference therein). The unknown function \( u = (u_1, u_2) \) is a vector. It satisfies \((k \geq 0)\)
\[
\begin{align*}
\frac{\partial^2 u_1}{\partial t^2} + \frac{\partial u_1}{\partial t} - Au_1 + \sin u_1 + k(u_1 - u_2) &= 0 \\
\frac{\partial^2 u_2}{\partial t^2} + \frac{\partial u_2}{\partial t} - Au_2 + \sin u_2 + k(u_2 - u_1) &= 0
\end{align*}
\]
\[
u_{i}(t, x) = 0 \quad \text{on} \quad \mathbb{R}^+ \times \partial \Omega \quad i = 1, 2.
\]
In this example \( d = 2 \), \( V = (H^1_0(\Omega))^2 \), \( D(A) = (H^1_0(\Omega) \cap H^2(\Omega))^2 \), \( A(\varphi_1, \varphi_2) = (-A\varphi_1, -A\varphi_2) \), \( F(\varphi_1, \varphi_2) = \cos \varphi_1 + \cos \varphi_2 - (k/2)(\varphi_1 - \varphi_2)^2 \) and by the elliptic regularity results we can take \( X = W^{2,p}(\Omega, \mathbb{R}^2) \cap H^1_0(\Omega, \mathbb{R}^2) \) with \( p > (N/2) \) and \( p \geq 2 \).

EXAMPLE 3.4.
\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + A' u &= f(u) \quad \text{in} \quad \mathbb{R}^+ \times \Omega \\
u &= Au = 0 \quad \text{on} \quad \mathbb{R}^+ \times \partial \Omega \\
u(0, .) &= u_0( .) \quad \text{in} \quad \Omega \\
u_t(0, .) &= u_1( .) \quad \text{in} \quad \Omega
\end{align*}
\]
In this case \( f \) is a real analytic function, \( d = 1 \), \( H = L^2(\Omega) \), \( A = A' \), \( D(A) = H^4 \cap H^2_0 \), and \( X = W^{4,p}(\Omega) \cap H^2_0(\Omega) \) with \( p > N/4 \) and \( p \geq 2 \).

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