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Algebras of operations in K-theory

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Abstract

We describe explicitly the algebras of degree zero operations in connective and periodic p-local complex K-theory. Operations are written uniquely in terms of certain infinite linear combinations of Adams operations, and we give formulas for the product and coproduct structure maps. It is shown that these rings of operations are not Noetherian. Versions of the results are provided for the Adams summand and for real K-theory.

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1. Introduction

The complex K-theory of a space or spectrum may be usefully endowed with operations. The most well-known of these are the Adams operations Ψ^j arising out of the geometry of vector bundles. As originally constructed by Adams, these are *unstable* operations. A stable operation is given by a sequence of maps, commuting with the Bott periodicity isomorphism. For an Adams operation Ψ^j to be *stable* requires j to be a unit in the coefficient ring one is working over. Integrally, we only have Ψ^1 and Ψ^{-1}, corresponding to the identity and complex conjugation. Following work of Adams et al. [3] in which the structure of the dual object, the algebra of cooperations, was determined, Adams and Clarke [2] showed

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that there are uncountably many integral stable operations. It is remarkable that, after more than forty years of topological K-theory, no-one knows explicitly any integral stable operations, apart from linear combinations of the identity and complex conjugation.

In this paper, we give a new description of K-theory operations in the p-local setting. This is considerably closer to the integral situation and more subtle than the rather better understood case of p-adic coefficients; see [7,14]. In p-local connective K-theory, the degree zero operations $k^0_{(p)}(k_{(p)})$ form a bicommutative bialgebra, which we denote by $k^0(k)_{(p)}$. Note, however that this is not the p-localisation of the bialgebra $k^0(k)$ of integral operations, but is isomorphic to the *completed* tensor product $k^0(k)\widehat{\otimes}\mathbb{Z}_{(p)}$. In the periodic theory, the corresponding object $K^0(K)_{(p)} = K^0_{(p)}(K_{(p)}) = K^0(K)\widehat{\otimes}\mathbb{Z}_{(p)}$ is a bicommutative Hopf algebra, as it also possesses an antipode. We provide explicit descriptions for both of these algebras of operations, together with formulas for all their structure maps. These results build on our recent paper [8], in which we gave new additive bases for the ring of cooperations in p-local K-theory. Such an understanding of the degree zero operations is sufficient to determine the whole of $K^*(K)_{(p)}$ and the torsion-free part of $k^*(k)_{(p)}$; see [13].

Since we have a stable Adams operation Ψ^α for each $\alpha \in \mathbb{Z}^\times_{(p)}$, and since these multiply according to the formula $\Psi^\alpha\Psi^\beta = \Psi^{\alpha\beta}$, the group ring $\mathbb{Z}_{(p)}[\mathbb{Z}^\times_{(p)}]$ is a subring of the ring of operations. Our results express operations in terms of certain infinite sums involving Adams operations and thus describe the ring of operations as a completion of the group ring. This idea is certainly not new; Johnson [11] also has basis elements for p-local operations of this form. However, our description has considerable advantages in the form of explicit formulas, which in the connective case are particularly nice. We note that Madsen et al. [14] have also considered operations defined as infinite sums of Adams operations, but for them the p-adic context is essential.

Our results will be used in a later paper to give a re-working of Bousfield's study of the $K_{(p)}$-local category; see [6].

We now outline the structure of the paper.

Sections 2 and 3 are concerned with the case of operations in connective K-theory. In Section 2 we describe the bialgebra $k^0(k)_{(p)}$ in an explicit form which enables us to describe the structure maps. We also give formulas for the action on the coefficient ring and on the Hopf bundle over $\mathbb{C}P^\infty$. In Section 3 we show that, as a ring, $k^0(k)_{(p)}$ is not Noetherian, and we characterise its units. We also indicate how it can be considered as a completion of a polynomial ring.

In Section 4 we consider the idempotents in connective K-theory which were introduced by Adams, and we show how the results of Sections 2 and 3 extend to the algebra of operations on the Adams summand. We prove in Section 5 that the ring of operations on the p-adic Adams summand is a power series ring.

In Section 6 we show how the results of Sections 2–4 generalise to periodic K-theory, and in Section 7 we discuss the relation between the connective and periodic cases.

In Sections 8 and 9 we work over the prime 2. We outline how our results from the preceding sections need to be adapted, and we consider operations in *KO*-theory.

Finally, in an appendix we give some general relations among polynomials which underpin a number of the formulas given in the preceding sections.

Unless otherwise stated, p is assumed to be an odd prime. Having chosen p, we fix q to be an integer which is primitive modulo p^2, and hence primitive modulo any power of p.

2. Degree zero operations in connective K-theory

For each non-negative integer n, we define $\theta_n(X) \in \mathbb{Z}[X]$ by

$$\theta_n(X) = \prod_{i=0}^{n-1} (X - q^i),$$

where q, as stated in the Introduction, is primitive modulo p^2. The notation derives from [9]. Generalisations of these polynomials are considered later in this paper.

The Gaussian polynomials in the variable q (or q-binomial coefficients) may be defined as

$$\begin{bmatrix} n \\ j \end{bmatrix} = \frac{\theta_j(q^n)}{\theta_j(q^j)}.$$

Definition 2.1. Define elements $\varphi_n \in k^0(k)_{(p)}$, for $n \geqslant 0$, by

$$\varphi_n = \theta_n(\Psi^q),$$

where Ψ^q is the Adams operation.

Thus, for example, $\varphi_0 = 1$, $\varphi_1 = \Psi^q - 1$ and $\varphi_2 = (\Psi^q - 1)(\Psi^q - q)$.

Theorem 2.2. *The elements of $k^0(k)_{(p)}$ can be expressed uniquely as infinite sums*

$$\sum_{n \geqslant 0} a_n \varphi_n,$$

where $a_n \in \mathbb{Z}_{(p)}$.

Proof. The bialgebra $k^0(k)_{(p)} = K^0(k)_{(p)}$ is the $\mathbb{Z}_{(p)}$-dual of the bialgebra $K_0(k)_{(p)}$; see [7,11]. In Proposition 3 of [8] we gave a basis for $K_0(k)_{(p)}$, consisting of the polynomials $f_n(w) = \theta_n(w)/\theta_n(q^n)$, for $n \geqslant 0$. The theorem follows from the fact (which is implicit in [9]) that the φ_n are dual to this basis. To see this, recall that a coalgebra admits an action of its dual which, in the case of $K_0(k)_{(p)}$, is determined by $\Psi^r \cdot f(w) = f(rw)$. By a simple induction on m, this implies that

$$\varphi_m \cdot f_n(w) = q^{m(m-n)} w^m f_{n-m}(w),$$

where $f_i(w)$ is understood as 0 if $i < 0$, so that $\varphi_m \cdot f_n(w) = 0$ if $n < m$. The Kronecker pairing can be recovered from this action by evaluating at $w = 1$, hence

$$\langle \varphi_m, f_n(w) \rangle = q^{m(m-n)} f_{n-m}(1) = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

Remark 2.3. Operations in K-theory are determined by their action on coefficients [10]. It is therefore instructive to see how an infinite sum $\sum_{n \geqslant 0} a_n \varphi_n$ acts. Since Ψ^q acts on $\pi_{2i}(k_{(p)})$ as multiplication by q^i, we see that $\sum_{n\geqslant 0} a_n\varphi_n$ acts on the coefficient group $\pi_{2i}(k_{(p)})$ as multiplication by

$$\sum_{n=0}^{i} a_n\theta_n(q^i).$$

The sum is finite since $\theta_n(q^i)=0$ for $n>i$.

In particular, the augmentation $\varepsilon : k^0(k)_{(p)} \to \mathbb{Z}_{(p)}$ given by the action on $\pi_0(k_{(p)})$, satisfies $\varepsilon(\sum_{n\geqslant 0} a_n\varphi_n)=a_0$.

It is easy to see by induction that $\theta_n(X)=\sum_{j=0}^{n}(-1)^{n-j}q^{\binom{n-j}{2}}\begin{bmatrix} n \\ j \end{bmatrix}X^j$; see [4, (3.3.6)] and also [8, Proposition 8]. Hence we can express each φ_n explicitly as a linear combination of Adams operations.

Proposition 2.4. *For all $n\geqslant 0$,*

$$\varphi_n=\sum_{j=0}^{n}(-1)^{n-j}q^{\binom{n-j}{2}}\begin{bmatrix} n \\ j \end{bmatrix}\Psi^{q^j}.$$

Conversely, our proof of Theorem 2.2 shows how to express all the stable Adams operations in terms of the φ_n.

Proposition 2.5. *If $j\in\mathbb{Z}_{(p)}^{\times}$,*

$$\Psi^j=\sum_{n\geqslant 0}\frac{\theta_n(j)}{\theta_n(q^n)}\varphi_n.$$

In particular, for $i\in\mathbb{Z}$,

$$\Psi^{q^i}=\sum_{n\geqslant 0}\begin{bmatrix} i \\ n \end{bmatrix}\varphi_n.$$

Note that this is a finite sum for $i\geqslant 0$.

Additive operations in K-theory are determined by their action on the Hopf bundle over $\mathbb{C}P^\infty$; see [5]. Writing

$$k^0(\mathbb{C}P^\infty)_{(p)}=K^0(\mathbb{C}P^\infty)_{(p)}=\mathbb{Z}_{(p)}[[t]],$$

where $1+t$ is the Hopf bundle, we have the following formula for the action of $k^0(k)_{(p)}$ on the Hopf bundle.

Proposition 2.6. *For all $n\geqslant 0$,*

$$\varphi_n(1+t)=\sum_{i\geqslant n}\left(\sum_{j=0}^{n}(-1)^{n-j}q^{\binom{n-j}{2}}\binom{q^j}{i}\begin{bmatrix} n \\ j \end{bmatrix}\right)t^i.$$

Proof. Since Ψ^r acts on line bundles by raising them to the rth power, Proposition 2.4 shows that

$$\varphi_n(1+t) = \sum_{j=0}^{n} (-1)^{n-j} q^{\binom{n-j}{2}} \begin{bmatrix} n \\ j \end{bmatrix} (1+t)^{q^j}.$$

The formula now follows by using the binomial expansion and reversing the order of summation. That the coefficient of t^i in $\varphi_n(1+t)$ is zero for $i<n$ can be proved by a simple induction, using the identity $\varphi_{n+1} = (\Psi^q - q^n)\varphi_n$. $\square$

The product structure on $k^0(k)_{(p)}$ is determined by the following formula.

Proposition 2.7.

$$\varphi_r\varphi_s = \sum_{i=0}^{\min(r,s)} \frac{\theta_i(q^r)\theta_i(q^s)}{\theta_i(q^i)} \varphi_{r+s-i} = \sum_{j=\max(r,s)}^{r+s} \frac{\theta_{r+s-j}(q^r)\theta_{r+s-j}(q^s)}{\theta_{r+s-j}(q^{r+s-j})} \varphi_j.$$

Proof. This result is essentially a fact about the polynomials $\theta_n(X)$. Since the algebra of these polynomials underlies many of our results, we have gathered together the relevant facts, in appropriate generality, in Appendix A. In particular, we need to use here Proposition A.4 with $m=0$, $X=\Psi^q$ and $\mathbf{c}=\mathbf{q}=(1,q,q^2,\ldots)$.

To show that the coefficient $A_{r,j-s}(\mathbf{q},\mathbf{q}[s])$ given by that proposition is equal to $\theta_{r+s-j}(q^r)\theta_{r+s-j}(q^s)/\theta_{r+s-j}(q^{r+s-j})$, it is only necessary to verify that this expression satisfies the recurrence (A.5), i.e., taking $i=r+s-j$ that

$$\frac{\theta_{i+1}(q^{r+1})\theta_{i+1}(q^s)}{\theta_{i+1}(q^{i+1})} = (q^{r+s-i} - q^r)\frac{\theta_i(q^r)\theta_i(q^s)}{\theta_i(q^i)} + \frac{\theta_{i+1}(q^r)\theta_{i+1}(q^s)}{\theta_{i+1}(q^{i+1})}.$$

Using the identities

$$\theta_{i+1}(q^s) = \theta_i(q^s)(q^s - q^i) \quad \text{and} \quad \theta_{i+1}(q^{r+1}) = \theta_i(q^r)(q^{r+i+1} - q^i),$$

this is easy to check. $\square$

Proposition 2.7 clarifies how it is that infinite sums can be multiplied without producing infinite coefficients:

$$\left(\sum_{r\geqslant 0} a_r\varphi_r\right)\left(\sum_{s\geqslant 0} b_s\varphi_s\right) = \sum_{j\geqslant 0}\left(\sum_{\substack{r,s\leqslant j\\ r+s\geqslant j}} a_r b_s \frac{\theta_{r+s-j}(q^r)\theta_{r+s-j}(q^s)}{\theta_{r+s-j}(q^{r+s-j})}\right)\varphi_j,$$

where the important point is that the inner summations are finite.

The multiplicative structure of $k^0(k)_{(p)}$ is very intricate; we study it in more detail in Section 3, in which we will use the following generalisation of Proposition 2.7.

Proposition 2.8. *For all* $r, s \geqslant m$,

$$\varphi_r\varphi_s = \varphi_m \sum_{j=\max(r,s)}^{r+s-m} c_{r,s}^{m,j}\varphi_j,$$

where the coefficients are given by

$$c_{r,s}^{m,j} = \frac{\theta_{r+s-j}(q^r)\theta_{r+s-j}(q^s)}{q^{(r+s-j-m)m}\theta_{r+s-j-m}(q^{r+s-j-m})\theta_m(q^r)\theta_m(q^s)}.$$

Proof. Interchanging the role of r and s if necessary, we may assume that $s \geqslant r$. Then Proposition A.4 provides the given expansion for $\varphi_r\varphi_s$ with $c_{r,s}^{m,j} = A_{r-m,j-s}(\mathbf{q}[m], \mathbf{q}[s])$. The formula for $c_{r,s}^{m,j}$ holds since this expression satisfies the recurrence (A.5). □

The multiplication formula for the $f_n(w)$ (Proposition 7 of [8]) leads by duality to the following result.

Proposition 2.9. *The coproduct satisfies*

$$\Delta\varphi_n = \sum_{\substack{r,s\geqslant 0\\ r+s\leqslant n}} \frac{\theta_{r+s}(q^n)}{\theta_r(q^r)\theta_s(q^s)}\varphi_{n-r}\otimes\varphi_{n-s}.$$

The bounds in this summation ensure that the formula determines a map from $k^0(k)_{(p)}$ into the completed tensor product $k^0(k)_{(p)}\widehat{\otimes}k^0(k)_{(p)}$, whose elements may be written as doubly infinite sums of the $\varphi_i \otimes \varphi_j$. It is unreasonable to expect a coproduct to map into the usual tensor product in this setting, so we will take the terms coalgebra, bialgebra and Hopf algebra to mean structures where the coproduct maps into the completed tensor product.

3. The ring structure of $k^0(k)_{(p)}$

In this section we write I for the augmentation ideal of the algebra $k^0(k)_{(p)}$ of stable operations of degree zero in p-local connective K-theory. Thus I/I^2 is the module of indecomposables. The ring of p-adic integers is denoted by $\mathbb{Z}_p$.

Theorem 3.1. (1) *The ring* $k^0(k)_{(p)}$ *is not Noetherian.*
(2) *Its module of indecomposables is isomorphic to* $\mathbb{Z}_p$.

To prove this theorem, we define a sequence of ideals of $k^0(k)_{(p)}$.

Definition 3.2. For $m \geqslant 0$ let

$$B_m = \left\{ \sum_{n\geqslant m} a_n\varphi_n : a_n \in \mathbb{Z}_{(p)} \right\}.$$

Thus $B_0 \supset B_1 \supset B_2 \supset \cdots$, with $B_0 = k^0(k)_{(p)}$ and $B_1 = I$. It is clear from Remark 2.3 that B_m consists of those operations which act as zero on the coefficient groups $\pi_{2i}(k_{(p)})$ for $0 \leqslant i < m$. Thus this filtration is independent of our choice of q.

Proposition 3.3. *For each $0 \leqslant n \leqslant m$, we have $B_n B_m = \varphi_n B_m$. In particular, B_m is an ideal of $k^0(k)_{(p)}$.*

Proof. Clearly $\varphi_n B_m \subseteq B_n B_m$, since $\varphi_n \in B_n$.

Suppose that $\alpha = \sum_{r \geqslant n} a_r \varphi_r \in B_n$ and $\beta = \sum_{s \geqslant m} b_s \varphi_s \in B_m$, then, using Proposition 2.8,

$$\alpha\beta = \sum_{\substack{r \geqslant n \\ s \geqslant m}} a_r b_s \varphi_r \varphi_s = \sum_{\substack{r \geqslant n \\ s \geqslant m}} a_r b_s \varphi_n \sum_{j=\max(r,s)}^{r+s-n} c_{r,s}^{n,j} \varphi_j = \varphi_n \sum_{j \geqslant m} \left(\sum_{\substack{r,s \leqslant j \\ r+s \geqslant j+n}} a_r b_s c_{r,s}^{n,j} \right) \varphi_j,$$

which belongs to $\varphi_n B_m$, as the inner summation is finite. □

Lemma 3.4. *For each $m \geqslant 1$, the quotient $B_m/\varphi_1 B_m$ is isomorphic to $\mathbb{Z}_p$.*

Proof. Define a $\mathbb{Z}_{(p)}$-module homomorphism $\pi_m : B_m \to \mathbb{Z}_p$ by

$$\pi_m : \left(\sum_{n \geqslant m} a_n \varphi_n \right) \mapsto \sum_{n \geqslant m} a_n (1-q^m)(1-q^{m+1})\ldots(1-q^{n-1}).$$

Note that the sum does indeed converge p-adically.

By Proposition 2.7, $\varphi_1 \varphi_n = (q^n - 1)\varphi_n + \varphi_{n+1}$. Hence $\pi_m(\varphi_1 \varphi_n) = 0$, if $n \geqslant m$, and $\ker \pi_m \supseteq \varphi_1 B_m$.

Now suppose $\alpha = \sum_{n \geqslant m} a_n \varphi_n \in \ker \pi_m$. We will show that there exists $\beta = \sum_{n \geqslant m} b_n \varphi_n \in B_m$ such that $\alpha = \varphi_1 \beta$. This is equivalent to showing that the equations

$$b_{n-1} + (q^n - 1) b_n = a_n \quad (n \geqslant m) \tag{3.5}$$

may be solved for $\{b_n \in \mathbb{Z}_{(p)} : n \geqslant m\}$, where $b_n = 0$ for $n < m$.

Suppose that we have found $b_r \in \mathbb{Z}_{(p)}$ satisfying (3.5) for $m \leqslant r < n$. It follows then that

$$0 = \pi_m(\alpha) = (a_n - b_{n-1})(1-q^m)\ldots(1-q^{n-1}) + a_{n+1}(1-q^m)\ldots(1-q^n) + \cdots.$$

Thus if $M = v_p((1-q^m)\ldots(1-q^{n-1}))$ and $N = v_p(1-q^n)$,

$$0 \equiv (a_n - b_{n-1}) p^M \bmod p^{M+N},$$

so that $a_n \equiv b_{n-1} \bmod p^N$, and (3.5) may be solved for $b_n \in \mathbb{Z}_{(p)}$.

This shows that $\ker \pi_m = \varphi_1 B_m$; it remains to prove that π_m is surjective.

Let $M = v_p((1-q)\ldots(1-q^{m-1}))$ and choose R such that $m \leqslant p^{R-1}(p-1)$.

For any $r \geqslant R$ we can write

$$\prod_{j=m}^{p^{r-1}(p-1)} (1-q^j) = u_r p^{\frac{p^r-1}{p-1} - M},$$

where $u_r \in \mathbb{Z}_{(p)}^{\times}$.

If $x \in \mathbb{Z}_p$, by grouping the p-adic digits of x we can write

$$x = x_0 + x_R p^{\frac{p^R-1}{p-1}-M} + x_{R+1} p^{\frac{p^{R+1}-1}{p-1}-M} + \cdots + x_r p^{\frac{p^r-1}{p-1}-M} + \cdots,$$

where $x_r \in \mathbb{N}$ for $r=0$, and for $r \geqslant R$. Now let

$$\alpha = x_0\varphi_m + \sum_{r\geqslant R} \frac{x_r}{u_r}\varphi_{p^{r-1}(p-1)+1}.$$

It is clear that $\alpha \in B_m$ and $\pi_m(\alpha) = x$. $\square$

Proposition 3.6. *For $m \geqslant 1$, the ideal B_m is not finitely generated.*

Proof. By Proposition 2.8, $k^0(k)_{(p)}$ acts on the quotient $B_m/\varphi_1 B_m = \mathbb{Z}_p$ via the augmentation $\varepsilon\colon k^0(k)_{(p)} \to \mathbb{Z}_{(p)}$ and the inclusion $\mathbb{Z}_{(p)} \subset \mathbb{Z}_p$. Thus if B_m were a finitely generated $k^0(k)_{(p)}$-ideal, then $\mathbb{Z}_p$ would be a finitely generated $\mathbb{Z}_{(p)}$-module. But a finite subset of $\mathbb{Z}_p$ can only generate a countable $\mathbb{Z}_{(p)}$-submodule of $\mathbb{Z}_p$. $\square$

Proof of Theorem 3.1. Part (1) follows immediately. For part (2) we note that Proposition 3.3 shows that $I^2 = \varphi_1 B_1$, so that $I/I^2 = B_1/\varphi_1 B_1 \cong \mathbb{Z}_p$ by Lemma 3.4. $\square$

It is interesting to note how far the augmentation ideal I is from being generated by $\varphi_1 = \Psi^q - 1$. Before carrying out the completion giving $k^0(k)_{(p)}$, the augmentation ideal in $\mathbb{Z}_{(p)}[\Psi^q - 1]$ is principal, and it is so again after p-completion; see Section 5. In contrast, we have

Corollary 3.7. *The quotient $I/\langle\varphi_1\rangle$ is isomorphic to $\mathbb{Z}_p/\mathbb{Z}_{(p)}$, where $\langle\varphi_1\rangle$ is the ideal of $k^0(k)_{(p)}$ generated by φ_1.*

Proof. It is clear that $k^0(k)_{(p)} = \mathbb{Z}_{(p)} + I$. $\square$

We remark that the abelian group $\mathbb{Z}_p/\mathbb{Z}_{(p)}$ is torsion free and divisible. It is thus a $\mathbb{Q}$-vector space.

The intersection of the ideal B_m with the polynomial subalgebra of $k^0(k)_{(p)}$ generated by Ψ^q is the principal ideal generated by $\varphi_m = \theta_m(\Psi^q)$. Theorem 2.2 exhibits $k^0(k)_{(p)}$ as the completion of $\mathbb{Z}_{(p)}[\Psi^q]$ with respect to the filtration by these ideals. But note that this filtration is not multiplicative, in the sense of [16], and hence there is no associated graded ring. It is for this reason that the completion fails to be Noetherian, and contains zero divisors, as we see in Section 4.

Of course the Adams operations Ψ^r, where r is a p-local unit, generate inside $k^0(k)_{(p)}$ a copy of the group ring $\mathbb{Z}_{(p)}[\mathbb{Z}_{(p)}^\times]$ which is dense in the B_m-filtration topology. Thus $k^0(k)_{(p)}$ is naturally a completion of $\mathbb{Z}_{(p)}[\mathbb{Z}_{(p)}^\times]$.

In the following theorem we identify the units of $k^0(k)_{(p)}$ in terms of our basis. This formulation is closely related to Theorem 1 of [12].

Theorem 3.8. *The element $\sum_{n\geqslant 0} a_n\varphi_n$ is a unit in the ring $k^0(k)_{(p)}$ if and only if $\sum_{n=0}^{i} a_n\theta_n(q^i)$ is a p-local unit for $i = 0, 1, \ldots, p-2$.*

Proof. If $\alpha = \sum_{n\geqslant 0} a_n\varphi_n$ is a unit, then, by Remark 2.3, $\sum_{n=0}^{i} a_n\theta_n(q^i)$ represents the action of α on $\pi_{2i}(k_{(p)})$, and so must be invertible for all i.

Conversely, assume $\sum_{n=0}^{i} a_n\theta_n(q^i) \in \mathbb{Z}_{(p)}^{\times}$ for $i = 0, 1, \ldots, p-2$, then, since $\theta_n(q^i) \equiv \theta_n(q^j) \bmod p$ if $i \equiv j \bmod p-1$ and $\theta_n(q^i) = 0$ if $n > i$, this holds for all $i \geqslant 0$.

Now suppose, inductively, that we have found $b_0, b_1, \ldots, b_{i-1} \in \mathbb{Z}_{(p)}$ such that

$$\left(\sum_{n\geqslant 0} a_n\varphi_n\right)(b_0 + b_1\varphi_1 + \cdots + b_{i-1}\varphi_{i-1}) \in 1 + B_i.$$

Then, using Proposition 2.7, we see that, for any b_i in $\mathbb{Z}_{(p)}$,

$$\left(\sum_{n\geqslant 0} a_n\varphi_n\right)(b_0 + b_1\varphi_1 + \cdots + b_i\varphi_i) \in 1 + B_i,$$

with the coefficient of φ_i having the form

$$b_i\left(\sum_{n=0}^{i} a_n\theta_n(q^i)\right) + \text{terms involving } b_1, \ldots, b_{i-1}.$$

Thus, by our assumption, it is possible to choose $b_i \in \mathbb{Z}_{(p)}$ so that

$$\left(\sum_{n\geqslant 0} a_n\varphi_n\right)(b_0 + b_1\varphi_1 + \cdots + b_i\varphi_i) \in 1 + B_{i+1}.$$

Repeating this process ad infinitum yields the required inverse. □

4. The Adams splitting and the Adams summand

Idempotent operations which split p-local K-theory (for p odd) into $p-1$ summands were constructed in [1]. We show how to write the connective version of these idempotents in terms of our basis elements.

Proposition 4.1. *If $\alpha \in \{0, 1, \ldots, p-2\}$, the Adams idempotent $e_\alpha \in k^0(k)_{(p)}$ is given by*

$$e_\alpha = \sum_{n\geqslant 0} c_{n,\alpha}\varphi_n,$$

where

$$c_{n,\alpha} = \frac{1}{\theta_n(q^n)} \sum (-1)^{n-i} q^{\binom{n-i}{2}} \begin{bmatrix} n \\ i \end{bmatrix},$$

the summation being over all integers i for which $0 \leqslant i \leqslant n$ and $i \equiv \alpha \bmod p-1$.

Proof. If an operation $\varphi \in k^0(k)_{(p)}$ acts on the coefficient group $\pi_{2i}(k_{(p)})$ by multiplication by λ_i, then $\langle \varphi, w^i \rangle = \lambda_i$. Hence Proposition 8 of [8] shows that

$$\langle \varphi, f_n(w) \rangle = \frac{1}{\theta_n(q^n)} \sum_{i=0}^{n} (-1)^{n-i} q^{\binom{n-i}{2}} \begin{bmatrix} n \\ i \end{bmatrix} \lambda_i .$$

The result now follows by duality, since e_α acts as the identity on $\pi_{2i}(k_{(p)})$ if $i \equiv \alpha \bmod p-1$, and as zero otherwise. □

The spectra $K_{(p)}$ and $k_{(p)}$ are each split by Adams's idempotents into $p-1$ suspensions of a multiplicative spectrum denoted by G and g, respectively.[1] The coefficient ring G_* can be identified with the subring $\mathbb{Z}_{(p)}[u^{p-1}, u^{-p+1}] \subset \mathbb{Z}_{(p)}[u, u^{-1}] = \pi_*(K_{(p)})$, and g_* is identified with $\mathbb{Z}_{(p)}[u^{p-1}] \subset \mathbb{Z}_{(p)}[u] = \pi_*(k_{(p)})$.

There is an algebra isomorphism $\iota : g^0(g) \to e_0 k^0(k)_{(p)}$, under which $\Psi^q \in g^0(g)$ maps to $e_0 \Psi^q$, but $e_0 k^0(k)_{(p)}$ is not a sub-bialgebra of $k^0(k)_{(p)}$. However, composing the projection $k^0(k)_{(p)} \to e_0 k^0(k)_{(p)}$ with the inverse of ι, exhibits $g^0(g)$ as a quotient bialgebra of $k^0(k)_{(p)}$. Thus $g^0(g)$ is a summand of $k^0(k)_{(p)}$ as an algebra, but not as a bialgebra.

Note that if $p > 3$, then, in contrast to the p-adic case [15,7], the algebra $k^0(k)_{(p)}$ is not isomorphic to $g^0(g) \widehat{\otimes} \mathbb{Z}_{(p)}[C_{p-1}]$. This can be seen by considering the action on the coefficient ring, which shows that $k^0(k)_{(p)}$ contains no elements of order $p-1$.

The results of Section 2 have analogues for the algebra $g^0(g)$ of degree zero stable operations on the Adams summand. We need first to adapt the ideas of [8] to provide a basis for $G_0(g)$. Let $z = w^{p-1} \in G_0(g)$.

Recalling that we have chosen q to be primitive modulo p^2, we write $\hat{q} = q^{p-1}$ and let

$$\hat{\theta}_n(X) = \prod_{i=0}^{n-1} (X - \hat{q}^i).$$

Proposition 4.2. *A $\mathbb{Z}_{(p)}$-basis for $G_0(g)$ is given by the elements*

$$\hat{f}_n(z) = \frac{\hat{\theta}_n(z)}{\hat{\theta}_n(\hat{q}^n)} \quad \text{for } n \geqslant 0.$$

Proof. We have

$$G_0(g) = \{ f(z) \in \mathbb{Q}[z] : f(1 + p\mathbb{Z}_{(p)}) \subseteq \mathbb{Z}_{(p)} \}.$$

The multiplicative group $1 + p\mathbb{Z}_p$ is topologically generated by $\hat{q}$, and $v_p(\hat{q}^n - 1) = 1 + v_p(n)$ for all $n \geqslant 1$. The rest of the proof parallels that of [8, Proposition 3]. □

We now identify the dual basis for the algebra $g^0(g) = G^0(g)$.

[1] The notations $E(1)$ and $e(1)$, or L and l, are also used.

Definition 4.3. Define $\hat{\varphi}_n \in g^0(g)$, for $n \geqslant 0$, by

$$\hat{\varphi}_n = \hat{\theta}_n(\Psi^q) = \prod_{i=0}^{n-1} (\Psi^q - \hat{q}^i).$$

To avoid any misunderstanding, we emphasise that the q indexing the Adams operation is hatless.

Theorem 4.4 (*Lellmann [13, Theorem 2.2]*). *The elements of $g^0(g)$ can be expressed uniquely as infinite sums*

$$\sum_{n \geqslant 0} a_n \hat{\varphi}_n,$$

where $a_n \in \mathbb{Z}_{(p)}$.

Proof. The proof is just as for Theorem 2.2: $g^0(g) = G^0(g)$ is $\mathbb{Z}_{(p)}$-dual to the bialgebra $G_0(g)$, and the $\hat{\varphi}_n$ are dual to the $\hat{f}_n(z)$, the action being given by $\hat{\varphi}_m \cdot \hat{f}_n(z) = \hat{q}^{m(m-n)} z^m \hat{f}_{n-m}(z)$. □

The formulas for the product and coproduct of the $\hat{\varphi}_n$ in $g^0(g)$ are, of course, just like those of Section 2 for the φ_n, but with the primitive element q replaced by $\hat{q}$. Similarly the proof of Theorem 3.1 generalises to show that $g^0(g)$ is not Noetherian.

The proof of Theorem 3.8 simplifies in the split context, since $\hat{\theta}_n(\hat{q}^i)$ is divisible by p for all $i \geqslant 0$ and all $n > 0$. Hence we have

Theorem 4.5. *An element of $g^0(g)$ is a unit if and only if its augmentation is a unit.*

This result was proved by Johnson [12]; it shows that $g^0(g)$ is a local ring. Johnson also showed in that paper that $g^0(g)$ is an integral domain.

5. Operations in p-adic K-theory

The p-adic completion of $g^0(g)$, which we denote by $g^0(g)_p$, is the ring of degree zero operations in p-adic K-theory. In fact, in this case the algebra of operations in the connective theory does not differ from the algebra of operations in the periodic theory; see [11] and Section 7 below. We give here an algebraic proof of the result due to Clarke [7] and Mitchell [15] that $g^0(g)_p$ is a power series ring.

Theorem 5.1. *$g^0(g)_p$ is the power series ring over $\mathbb{Z}_p$ generated by $\Psi^q - 1$.*

Proof. Retaining the notation of Section 4, we let $s(n,i), S(n,i) \in \mathbb{Z}_{(p)}$ be such that, for $n \geqslant 1$,

$$\hat{\theta}_n(X) = \sum_{i=1}^{n} s(n,i)(X-1)^i \quad \text{and} \quad (X-1)^n = \sum_{i=1}^{n} S(n,i)\hat{\theta}_i(X).$$

These constants are analogues of the Stirling numbers of the first and second kinds, respectively. (In the notation of Appendix A, $s(n,i) = A_{n,i}(\hat{\mathbf{q}}, \mathbf{1})$ and $S(n,i) = A_{n,i}(\mathbf{1}, \hat{\mathbf{q}})$, where $\hat{\mathbf{q}}$ is the sequence $(\hat{q}^{i-1})_{i \geqslant 1}$

and **1** is the constant sequence $(1)_{i\geqslant 1}$.) It is clear that $s(n,n)=S(n,n)=1$, $S(n,1)=(\hat{q}-1)^{n-1}$, and $s(n,1)=(1-\hat{q})(1-\hat{q}^2)\ldots(1-\hat{q}^{n-1})$. Moreover, the two lower triangular matrices $(s(n,i))_{n,i\geqslant 1}$ and $(S(n,i))_{n,i\geqslant 1}$ are mutually inverse.

In these cases, the recurrence of Proposition A.2 becomes

$$\begin{aligned} &s(n+1,i)=s(n,i-1)-(\hat{q}^n-1)s(n,i)\\ &\text{and}\quad S(n+1,i)=S(n,i-1)+(\hat{q}^i-1)S(n,i). \end{aligned}$$

Since $\hat{q}^n-1$ is divisible by p for all n, it follows easily that $v_p(s(n,i))\geqslant n-i$ and $v_p(S(n,i))\geqslant n-i$.

Let (p,Y) denote the maximal ideal of $\mathbb{Z}_p[[Y]]$. Since $\hat{\theta}_n(Y+1)=Y(Y+1-\hat{q})\ldots(Y+1-\hat{q}^{n-1})$, we have $\hat{\theta}_n(Y+1)\in(p,Y)^n$ for all n. As a result, there are ring homomorphisms forming the following commutative diagram:

$$\begin{array}{ccc} \mathbb{Z}_p[X]/(\hat{\theta}_n(X)) & \longrightarrow & \mathbb{Z}_p[[Y]]/(p,Y)^n \\ \uparrow & & \uparrow \\ \mathbb{Z}_p[X]/(\hat{\theta}_{n+1}(X)) & \longrightarrow & \mathbb{Z}_p[[Y]]/(p,Y)^{n+1} \end{array}$$

in which the horizontal maps are defined by $X\mapsto Y+1$. In the limit there is a ring homomorphism

$$g^0(g)_p=\varprojlim \mathbb{Z}_p[X]/(\hat{\theta}_n(X))\to\varprojlim \mathbb{Z}_p[[Y]]/(p,Y)^n=\mathbb{Z}_p[[Y]]$$

which, we will show, is an isomorphism. The variable X corresponds to Ψ^q, and thus Y to Ψ^q-1.

The kernel of $\mathbb{Z}_p[X]/(\hat{\theta}_n(X))\to\mathbb{Z}_p[[Y]]/(p,Y)^n$, which we denote by I_n, is the free $\mathbb{Z}_p$-module generated by the elements $\{[p^{n-i}(X-1)^i]:0\leqslant i\leqslant n-1\}$, where $[g(X)]$ denotes the coset of $g(X)$ in the quotient ring $\mathbb{Z}_p[X]/(\hat{\theta}_n(X))$. Under the homomorphism $I_{n+1}\to I_n$, the element $[p^{n+1-i}(X-1)^i]$ maps to $p[p^{n-i}(X-1)^i]$, if $i<n$, but $[p(X-1)^n]$ maps to

$$-\sum_{j=1}^{n-1}p\frac{s(n,j)}{p^{n-j}}[p^{n-j}(X-1)^j].$$

Note here that $s(n,j)/p^{n-j}\in\mathbb{Z}_{(p)}$ by the remarks above. This shows that the image of I_{n+1} lies in pI_n, and hence, since no non-zero element of I_n is infinitely divisible by p, that $\varprojlim I_n=0$. Thus $g^0(g)_p$ maps injectively into $\mathbb{Z}_p[[Y]]$.

To prove that it does so surjectively we define maps $\mathbb{Z}_p[[Y]]\to\mathbb{Z}_p[X]/(\hat{\theta}_n(X))$ by

$$\sum_{r\geqslant 0}c_rY^r\mapsto c_0+\sum_{i=1}^{n-1}\left(\sum_{j=i}^{\infty}S(j,i)c_j\right)[\hat{\theta}_i(X)].$$

It is here, of course, that we need to be working over $\mathbb{Z}_p$. The convergence of the infinite series is guaranteed since p^{j-i} divides $S(j,i)$.

It will turn out that we have a ring homomorphism. But at this point we need only to know that it is a homomorphism of $\mathbb{Z}_p$-modules, and this is trivial. It is also trivial that the maps factor through the projection

$$\mathbb{Z}_p[X]/(\hat{\theta}_{n+1}(X))\to\mathbb{Z}_p[X]/(\hat{\theta}_n(X))$$

and so define a $\mathbb{Z}_p$-module homomorphism

$$\mathbb{Z}_p[[Y]] \to \lim_{\leftarrow} \mathbb{Z}_p[X]/(\hat{\theta}_n(X)) = g^0(g)_p.$$

To verify that this is the inverse of the ring homomorphism constructed earlier we need to verify that the composition

$$\mathbb{Z}_p[[Y]] \to \mathbb{Z}_p[X]/(\hat{\theta}_n(X)) \to \mathbb{Z}_p[[Y]]/(p, Y)^n$$

coincides with the natural map. But since $S(j,i) \equiv 0 \bmod p^n$ for $j \geqslant i+n$, the composition sends $\sum_{r\geqslant 0} c_r Y^r$ to

$$c_0 + \sum_{i=1}^{n-1} \sum_{j=i}^{i+n-1} \sum_{k=1}^{i} S(j,i)s(i,k)c_j[Y^k] = c_0 + \sum_{k=1}^{n-1} \sum_{j=k}^{2n-2} \sum_{i=\max(k,j-n+1)}^{\min(j,n-1)} S(j,i)s(i,k)c_j[Y^k],$$

where $[Y^k]$ denotes the coset of Y^k in $\mathbb{Z}_p[[Y]]/(p,Y)^n$, and we note that $[Y^k]=0$ for $k \geqslant n$.

Now $[Y^k]$ has order p^{n-k} in $\mathbb{Z}_p[[Y]]/(p,Y)^n$, and $S(j,i)s(i,k)$ is divisible by p^{j-k}. This means that if $j \geqslant n$, then $S(j,i)s(i,k)[Y^k]=0$, and the image of $\sum_{r\geqslant 0} c_r Y^r$ is

$$c_0 + \sum_{k=1}^{n-1} \sum_{j=k}^{n-1} \sum_{i=k}^{j} S(j,i)s(i,k)c_j[Y^k] = c_0 + \sum_{k=1}^{n-1} c_k[Y^k],$$

since $\sum_{i=k}^{j} S(j,i)s(i,k) = \delta_{j,k}$. This completes the proof. $\square$

6. Operations in periodic K-theory

We now turn our attention to the periodic case. We let $K^0(K)_{(p)}$ denote the algebra of degree zero operations in p-local periodic K-theory. We show in this section how the results of Sections 2–4 extend to this context. Here the degree zero operations determine all stable operations, and $K^0(K)_{(p)}$ is a Hopf algebra so we determine the antipode as well as the other parts of the structure.

For each non-negative integer n, we define the polynomial $\Theta_n(X)$ by

$$\Theta_n(X) = \prod_{i=1}^{n} (X - \bar{q}_i),$$

where $\bar{q}_i$ is the ith term of the sequence

$$\bar{\mathbf{q}} = (1, q, q^{-1}, q^2, q^{-2}, q^3, q^{-3}, q^4, \ldots) = (q^{(-1)^i \lfloor i/2 \rfloor})_{i\geqslant 1},$$

i.e., $\Theta_n(X) = \theta_n(X; \bar{\mathbf{q}})$ in the notation of the appendix.

Definition 6.1. Define elements $\Phi_n \in K^0(K)_{(p)}$, for $n \geqslant 0$, by

$$\Phi_n = \Theta_n(\Psi^q).$$

Thus, for example, $\Phi_0 = 1$, $\Phi_1 = \Psi^q - 1$, $\Phi_2 = (\Psi^q - 1)(\Psi^q - q)$ and $\Phi_3 = (\Psi^q - 1)(\Psi^q - q)(\Psi^q - q^{-1})$.

Theorem 6.2. *The elements of $K^0(K)_{(p)}$ can be expressed uniquely as infinite sums*

$$\sum_{n\geqslant 0} a_n\Phi_n,$$

where $a_n \in \mathbb{Z}_{(p)}$.

Proof. The proof is analogous to that of Theorem 2.2. The polynomials $F_n(w) = w^{-\lfloor n/2\rfloor} f_n(w)$ form a $\mathbb{Z}_{(p)}$-basis for $K_0(K)_{(p)}$ according to Corollary 6 of [8], and the Φ_n are, modulo multiplication by units, dual to this basis. In fact the Kronecker pairing satisfies

$$\langle \Phi_n, F_j(w)\rangle = \begin{cases} q^{-n\lfloor n/2\rfloor} & \text{if } n=j,\\ 0 & \text{otherwise.}\end{cases}$$

As in the proof of Theorem 2.2, this follows from a study of the action of the dual on $K_0(K)_{(p)}$, but the details are a little more complicated and are given in the following result. □

Lemma 6.3.

(1) $\Phi_n \cdot F_j(w) = 0$ *if* $j < n$;
(2) $\Phi_n \cdot F_n(w) = \begin{cases} q^{-nk}w^{-k} & \textit{if } n=2k,\\ q^{-nk}w^{k+1} & \textit{if } n=2k+1;\end{cases}$
(3) $\Phi_n \cdot F_j(w)$ *is divisible by* $f_{j-n}(w)$ *for* $j > n$.

Proof. (1) Recall that Ψ^q acts as $\Psi^q \cdot f(w) = f(qw)$. If $j < n$ and $-\lfloor j/2\rfloor \leqslant i \leqslant \lceil j/2\rceil$, then $\Psi^q - q^i$ is a factor of Φ_n, so that $\Phi_n \cdot w^i = 0$. But $F_j(w)$ is a Laurent polynomial of codegree $-\lfloor j/2\rfloor$ and degree $j - \lfloor j/2\rfloor = \lceil j/2\rceil$.

(2) By the proof of (1), all monomials occurring in $F_n(w)$ are annihilated by Φ_n except one. If $n = 2k$, this is the lowest degree monomial w^{-k}, whose coefficient is $f_{2k}(0) = q^{\binom{2k}{2}}/\theta_{2k}(q^{2k})$, and we have

$$\Phi_{2k} \cdot F_{2k}(w) = \Phi_{2k} \cdot (f_{2k}(0)w^{-k}) = q^{\binom{2k}{2}}\frac{\Theta_{2k}(q^{-k})}{\theta_{2k}(q^{2k})}w^{-k} = q^{-2k^2}w^{-k}.$$

If $n = 2k+1$, it is the highest degree monomial w^{k+1} which must be considered. The leading coefficient is $1/\theta_{2k+1}(q^{2k+1})$, and

$$\Phi_{2k+1} \cdot F_{2k+1}(w) = \frac{\Theta_{2k+1}(q^{k+1})}{\theta_{2k+1}(q^{2k+1})}w^{k+1} = q^{-(2k+1)k}w^{k+1}.$$

(3) The proof is by (finite) induction on n. Note that $\Phi_0 \cdot F_j(w) = F_j(w)$ is certainly divisible by $f_j(w)$. Now assume that $\Phi_n \cdot F_j(w) = f_{j-n}(w)G_{n,j}(w)$ for some Laurent polynomial $G_{n,j}(w)$. Then, since $\Phi_{n+1} = (\Psi^q - q^i)\Phi_n$ for some i,

$$\begin{aligned}\Phi_{n+1} \cdot F_j(w) &= (\Psi^q - q^i) \cdot f_{j-n}(w)G_{n,j}(w)\\ &= f_{j-n}(qw)G_{n,j}(qw) - q^i f_{j-n}(w)G_{n,j}(w).\end{aligned}$$

But this is zero for $w = 1, q, q^2, \ldots, q^{j-n-2}$ and therefore divisible by $f_{j-n-1}(w)$. □

Remark 6.4. The action of the infinite sum $\sum_{n\geqslant 0} a_n\Phi_n$ on the coefficient group $\pi_{2i}(K_{(p)})$ is multiplication by the finite sum

$$\sum_{n=0}^{2|i|} a_n\Theta_n(q^i).$$

Clearly the augmentation sends $\sum_{n\geqslant 0} a_n\Phi_n$ to a_0.

We have the following analogue of Proposition 2.4.

Proposition 6.5. *For all $n\geqslant 0$,*

$$\Phi_n=\sum_{j=0}^{n}(-1)^{n-j}q^{e(n,j)}\begin{bmatrix} n\\ j\end{bmatrix}\Psi^{q^j},$$

where

$$e(n,j)=\begin{cases}-(n-j)(j-1)/2 & \text{if } n \text{ is even},\\ -(n-j)j/2 & \text{if } n \text{ is odd}.\end{cases}$$

Proof. Assuming, inductively, that the result holds for even $n=2k$, since $\Phi_{2k+1}=\Phi_{2k}(\Psi^q-q^{-k})$, the coefficient of Ψ^{q^j} in Φ_{2k+1} is

$$\begin{aligned}&(-1)^{j+1}\left(q^{-(2k-j+1)(j-2)/2}\begin{bmatrix} 2k\\ j-1\end{bmatrix}+q^{-k-(2k-j)(j-1)/2}\begin{bmatrix} 2k\\ j\end{bmatrix}\right)\\ &\quad=(-1)^{j+1}q^{-(2k+1-j)j/2}\left(q^{2k-j+1}\begin{bmatrix} 2k\\ j-1\end{bmatrix}+\begin{bmatrix} 2k\\ j\end{bmatrix}\right)\\ &\quad=(-1)^{j+1}q^{-(2k+1-j)j/2}\begin{bmatrix} 2k+1\\ j\end{bmatrix}.\end{aligned}$$

The argument to show that the odd case implies the next even case is similar. □

Conversely, the proof of Theorem 6.2 yields

Proposition 6.6. *If $j\in\mathbb{Z}_{(p)}^{\times}$,*

$$\Psi^j=\sum_{n\geqslant 0} q^{n\lfloor n/2\rfloor}j^{-\lfloor n/2\rfloor}\frac{\theta_n(j)}{\theta_n(q^n)}\Phi_n.$$

In particular, for $i\in\mathbb{Z}$,

$$\Psi^{q^i}=\sum_{n\geqslant 0} q^{(n-i)\lfloor n/2\rfloor}\begin{bmatrix} i\\ n\end{bmatrix}\Phi_n.$$

Note that this is a finite sum for $i\geqslant 0$.

We now consider the antipode χ of the Hopf algebra $K^0(K)_{(p)}$. In the dual $K_0(K)_{(p)}$ the antipode is given by $w \mapsto w^{-1}$(see [3]), while $\chi\Psi^j = \Psi^{j^{-1}}$.

Proposition 6.7. *The antipode in $K^0(K)_{(p)}$ is determined by*

$$\chi\Phi_n = \sum_{j \geqslant 2\lfloor (n-1)/2 \rfloor + 1} \left(\sum_{i=0}^{n} (-1)^{n-i} q^{(i+j)\lfloor j/2 \rfloor + e(n,i)} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} -i \\ j \end{bmatrix} \right) \Phi_j,$$

where $e(n,i)$ is defined in Proposition 6.5.

Proof. Proposition 6.5 shows that

$$\chi\Phi_n = \sum_{i=0}^{n} (-1)^{n-i} \begin{bmatrix} n \\ i \end{bmatrix} q^{e(n,i)} \Psi^{q^{-i}},$$

and, by Proposition 6.6,

$$\Psi^{q^{-i}} = \sum_{j \geqslant 0} q^{(i+j)\lfloor j/2 \rfloor} \begin{bmatrix} -i \\ j \end{bmatrix} \Phi_j.$$

The required formula, with summation over $j \geqslant 0$, now follows by substitution.

To see that in fact the coefficients are zero until $j = 2\lfloor (n-1)/2 \rfloor + 1$, we note that by duality the expansion of $\chi\Phi_n$ can also be obtained from expressing $F_j(w^{-1})$ as a linear combination of the $F_n(w)$. The remarks in the proof of Lemma 6.3 (1) show that the Laurent polynomial $F_j(w^{-1})$ can be written as a $\mathbb{Z}_{(p)}$-linear combination of the $F_n(w)$ for $n \leqslant j+1$ if j is odd, and for $n \leqslant j$ if j is even. Thus we may write $\chi\Phi_n$ as an infinite linear combination of the Φ_j for $j \geqslant n$ if n is odd, and for $j \geqslant n-1$ if n is even. □

The same approach, using Propositions 6.5 and 6.6, yields the following formulas for the product and coproduct.

Proposition 6.8. *For all $r, s \geqslant 0$,*

$$\Phi_r \Phi_s = \sum_{k=\max(r,s)}^{r+s} A_{r,s}^k \Phi_k,$$

where

$$A_{r,s}^k = \sum_{i=0}^{r} \sum_{j=0}^{s} (-1)^{r+s-i-j} q^{e(r,i)+e(s,j)+(k-i-j)\lfloor k/2 \rfloor} \begin{bmatrix} r \\ i \end{bmatrix} \begin{bmatrix} s \\ j \end{bmatrix} \begin{bmatrix} i+j \\ k \end{bmatrix}.$$

Proposition 6.9. *The coproduct in $K_0(K)_{(p)}$ satisfies*

$$\Delta\Phi_n = \begin{cases} \sum\limits_{\substack{r,s\geqslant 0\\ r+s\leqslant n}} C_n^{r,s}\ \Phi_{n-r}\otimes\Phi_{n-s} & \text{if } n \text{ is even},\\ \sum\limits_{\substack{r,s\geqslant 0\\ r+s\leqslant n}} C_n^{r,s}\ \Phi_{n-r}\otimes\Phi_{n-s} + \sum\limits_{\substack{r,s\geqslant 2\\ r+s=n+1}} C_n^{r,s}\ \Phi_{n-r}\otimes\Phi_{n-s} & \text{if } n \text{ is odd},\end{cases}$$

where

$$C_n^{r,s} = \sum_{k=0}^{\min(r,s)} (-1)^k q^{(e(n,n-k)+(k-r)\lfloor\frac{n-r}{2}\rfloor+(k-s)\lfloor\frac{n-s}{2}\rfloor)} \begin{bmatrix} n\\ k\end{bmatrix}\begin{bmatrix} n-k\\ n-r\end{bmatrix}\begin{bmatrix} n-k\\ n-s\end{bmatrix}.$$

The results of Section 3 also apply in the periodic case.

Theorem 6.10.

(1) *The ring $K^0(K)_{(p)}$ is not Noetherian.*
(2) *Its module of indecomposables is isomorphic to $\mathbb{Z}_p$.*

Proof. The proof exactly parallels that of Theorem 3.1. We define the family of ideals of $K^0(K)_{(p)}$

$$A_m = \left\{\sum_{n\geqslant m} a_n\Phi_n : a_n\in\mathbb{Z}_{(p)}\right\}.$$

Then we have $A_nA_m = \Phi_nA_m$ if $0\leqslant n\leqslant m$. Just as in Section 3, we show that, for $m\geqslant 1$, $A_m/\Phi_1A_m\cong\mathbb{Z}_p$, and thus A_m is not finitely generated. □

We can give criteria analogous to those of Theorem 3.8 for when a general element of $K^0(K)_{(p)}$ is a unit, but we omit the details.

We can now generalise the results of Section 4 to the periodic case.

Proposition 6.11. *If $\alpha\in\{0,1,\dots,p-2\}$, the Adams idempotent E_α in $K^0(K)_{(p)}$ is given by*

$$E_\alpha = \sum_{n\geqslant 0} C_{n,\alpha}\Phi_n,$$

where

$$C_{n,\alpha} = \frac{1}{\theta_n(q^n)}\sum(-1)^{\lceil n/2\rceil-i} q^{n\lfloor n/2\rfloor+\binom{\lceil n/2\rceil-i}{2}}\begin{bmatrix} n\\ \lfloor n/2\rfloor+i\end{bmatrix},$$

the summation being over all integers i for which $-n-1<2i\leqslant n+1$ and $i\equiv\alpha \bmod p-1$.

Recall that we let $\hat{q}=q^{p-1}$, where q is primitive modulo p^2.

Definition 6.12. Let

$$\hat{\Phi}_n = \prod_{i=1}^{n} (\Psi^q - \hat{q}^{(-1)^i \lfloor i/2 \rfloor}).$$

Theorem 6.13. *The elements of $G^0(G)$ can be expressed uniquely as infinite sums*

$$\sum_{n \geqslant 0} a_n \hat{\Phi}_n,$$

where $a_n \in \mathbb{Z}_{(p)}$.

It is possible to write down formulas generalising those above for the antipode, product and coproduct in $G^0(G)$, but we omit the details. We can also show easily that $G^0(G)$ is a non-Noetherian local ring.

7. The relation between the connective and periodic cases

We consider now the relation between the connective and periodic cases, focussing on the non-split setting, although it is clear that similar results hold for the relation between $g^0(g)$ and $G^0(G)$.

The covering map $k \to K$ leads to an inclusion

$$K^0(K)_{(p)} \hookrightarrow K^0(k)_{(p)} = k^0(k)_{(p)}$$

which is described by the following formula.

Proposition 7.1. *For $n > 0$,*

$$\Phi_n = \sum_{i=\lfloor n/2 \rfloor+1}^{n} \left(\sum_{j=i}^{n} (-1)^{n-j} q^{e(n,j)} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} j \\ i \end{bmatrix} \right) \varphi_i,$$

where $e(n, j)$ is as defined in Proposition 6.5.

Proof. We obtain the stated formula by combining Propositions 6.5 and 2.5, but with a summation from $i = 0$ to n.

To see that the coefficients are zero for $i = 0, \ldots, \lfloor n/2 \rfloor$, we note that $\Theta_n(X)$ is divisible by $\theta_{\lfloor n/2 \rfloor+1}(X)$, and the quotient is $\theta_{\lceil n/2 \rceil-1}(X, (q^{-1}, q^{-2}, \ldots))$, in the notation of Appendix A. Writing this quotient as a linear combination of the $\theta_r(X, (q^{\lfloor n/2 \rfloor+1}, q^{\lfloor n/2 \rfloor+2}, \ldots))$, and substituting $X = \Psi^q$, gives rise to the formula

$$\Phi_n = \sum_{i=\lfloor n/2 \rfloor+1}^{n} A_{n,i-\lfloor n/2 \rfloor-1} \left((q^{-1}, q^{-2}, \ldots), (q^{\lfloor n/2 \rfloor+1}, q^{\lfloor n/2 \rfloor+2}, \ldots) \right) \varphi_i. \qquad \square$$

An arbitrary element of $K^0(K)_{(p)}$ can then be written as

$$\sum_{n\geqslant 0} a_n\Phi_n = \sum_{i\geqslant 0}\left(\sum_{n=i}^{2i}\left(\sum_{j=i}^{n}(-1)^{n-j}q^{e(n,j)}\begin{bmatrix} n\\ j\end{bmatrix}\begin{bmatrix} j\\ i\end{bmatrix}\right)a_n\right)\varphi_i.$$

Note that the inner summations are finite.

Since they are polynomials in Ψ^q, the basis elements φ_i lie in the image of the inclusion and can be expressed in terms of our basis for $K^0(K)_{(p)}$ as follows.

Proposition 7.2.

$$\varphi_n = \sum_{i=0}^{n} q^{\lfloor (i+3)/2\rfloor(n-i)}\theta_{n-i}(q^{-i-1})\begin{bmatrix} n-\lceil i/2\rceil -1\\ n-i\end{bmatrix}\Phi_i.$$

Proof. It is merely necessary to verify that the given coefficients satisfy the recurrence given by Proposition A.2 for $A_{n,i}(\mathbf{q},\bar{\mathbf{q}})$, and this is routine. $\square$

Note that the summation here runs from $i=0$. It is not the case that the coefficients are zero for i small enough. So any attempt to write an arbitrary infinite sum $\sum a_n\varphi_n$ in $k^0(k)_{(p)}$ in terms of the Φ_i will lead to infinite sums for the coefficients. This reflects the fact that the map $K^0(K)_{(p)}\hookrightarrow k^0(k)_{(p)}$ is a *strict* monomorphism. However, if we complete p-adically, the highly p-divisible factor $\theta_{n-i}(q^{-i-1})$ ensures that these inner sums converge. Thus we recover the fact that, by contrast, $K^0(K)_p\to k^0(k)_p$ is an isomorphism, as discussed in [7,11].

8. 2-local operations

Here we provide a description of operations in 2-local K-theory. This is a little more complicated than for odd primes, essentially because instead of the single 'generator' Ψ^q one has to deal with both Ψ^3 and Ψ^{-1}.

Throughout this section and the next, the variable q occurring implicitly in the polynomial $\theta_n(X)$ will be set equal to 9, and we let $\bar\theta_n(X)=\prod_{i=0}^{n-1}(X-3^{2i+1})$. Thus, in the notation of the appendix, $\theta_n(X)=\theta_n(X;\mathbf{a})$ and $\bar\theta_n(X)=\theta_n(X;\mathbf{b})$, where $\mathbf{a}$ is the sequence of even powers of 3 and $\mathbf{b}$ is the sequence of odd powers of 3. These choices are related to the fact that $\{\pm 3^i : i\geqslant 0\}$ is dense in $\mathbb{Z}_2^\times$; see [8].

Definition 8.1. Define elements $\zeta_n\in k^0(k)_{(2)}$, for $n\geqslant 0$, by

$$\zeta_{2m+1}=(\Psi^{-1}-1)\bar\theta_m(\Psi^3),$$
$$\zeta_{2m}=\theta_m(\Psi^3)+\sum_{i=1}^{m}\frac{\theta_i(3)\theta_i(9^m)}{2\theta_i(9^i)}\zeta_{2m-2i+1}.$$

Theorem 8.2. *The elements of $k^0(k)_{(2)}$ can be expressed uniquely as infinite sums*

$$\sum_{n\geqslant 0} a_n\zeta_n,$$

where $a_n \in \mathbb{Z}_{(2)}$.

Proof. The proof mirrors that of Theorem 2.2. We show that the given elements form the dual basis to the basis $\{f_n^{(2)}(w) : n\geqslant 0\}$ obtained for $K_0(k)_{(2)}$ in [8, Proposition 20]. This may be proved by induction arguments using the following formulas, describing the action of $k^0(k)_{(2)}$ on $K_0(k)_{(2)}$.

$$\Psi^{-1}\cdot f_{2m}^{(2)}(w) = f_{2m}^{(2)}(w),$$
$$\Psi^{-1}\cdot f_{2m+1}^{(2)}(w) = f_{2m}^{(2)}(w) - f_{2m+1}^{(2)}(w),$$
$$\Psi^{3}\cdot f_{2m}^{(2)}(w) = 9^m f_{2m}^{(2)}(w) + f_{2m-2}^{(2)}(w),$$
$$\Psi^{3}\cdot f_{2m+1}^{(2)}(w) = 3^{2m+1} f_{2m+1}^{(2)}(w) - 9^m f_{2m}^{(2)}(w) + f_{2m-1}^{(2)}(w). \qquad \square$$

Remark 8.3. The operation ζ_n acts on the coefficient group $\pi_{2i}(k_{(2)})$ as multiplication by the values given in the following table.

	$n = 2m$	$n = 2m + 1$
i even	$\theta_m(3^i)$	0
i odd	$\overline{\theta}_m(3^i)$	$-2\overline{\theta}_m(3^i)$

Thus ζ_n acts as zero on $\pi_{2i}(k_{(2)})$ for all $i < n$.

The following proposition gives product formulas.

Proposition 8.4.

$$\zeta_{2m}\zeta_{2n} = \sum_{i=0}^{\min(m,n)} d_{m,n}^i \left(\zeta_{2m+2n-2i} - \frac{3^i-1}{2}\zeta_{2m+2n+1-2i}\right),$$

$$\zeta_{2m+1}\zeta_{2n} = \sum_{i=0}^{\min(m,n)} 3^i d_{m,n}^i \zeta_{2m+2n+1-2i},$$

$$\zeta_{2m+1}\zeta_{2n+1} = -2\sum_{i=0}^{\min(m,n)} 3^i d_{m,n}^i \zeta_{2m+2n+1-2i},$$

where $d_{m,n}^i = \dfrac{\theta_i(9^m)\theta_i(9^n)}{\theta_i(9^i)}$.

Proof. These formulas are proved by long but straightforward induction arguments. $\square$

Since $\zeta_{2n+1} = \zeta_1\zeta_{2n}$, the following proposition completely determines the coproduct.

Proposition 8.5.

$$\Delta\zeta_1 = \zeta_1 \otimes 1 + \zeta_1 \otimes \zeta_1 + 1 \otimes \zeta_1,$$
$$\Delta\zeta_{2n} = \sum_{\substack{r,s \geqslant 0 \\ r+s \leqslant n}} \frac{\theta_{r+s}(9^n)}{\theta_r(9^r)\theta_s(9^s)} \zeta_{2n-2r} \otimes \zeta_{2n-2s}.$$

Proof. These formulas may be deduced from the product formula for the dual basis of cooperations just as in the proof of Proposition 2.9. □

In principle, similar methods will give a description of the periodic case $K^0(K)_{(2)}$. However, this will be even more complicated, and we omit the details.

9. Operations in KO-theory

We consider *ko* and *KO* localised at $p = 2$. (For p odd, these spectra split as $(p-1)/2$ copies of g and G, respectively.) Proofs are omitted since the arguments are just the same as those in [8] and in earlier sections of this paper.

Just as in Section 8, the variable q used implicitly in the polynomials $\theta_n(X)$ and $\Theta_n(X)$ is set equal to 9.

Definition 9.1. Let $x = w^2 = u^{-2}v^2 \in KO_0(ko)$; see [3]. Let

$$h_n(x) = \frac{\theta_n(x)}{\theta_n(9^n)}.$$

Proposition 9.2.

(1) $\{h_n(x) : n \geqslant 0\}$ *is a* $\mathbb{Z}_{(2)}$*-basis for* $KO_0(ko)_{(2)}$.
(2) $\{x^{-\lfloor n/2 \rfloor} h_n(x) : n \geqslant 0\}$ *is a* $\mathbb{Z}_{(2)}$*-basis for* $KO_0(KO)_{(2)}$.

Theorem 9.3.

(1) *The elements of* $ko^0(ko)_{(2)}$ *can be expressed uniquely as infinite sums*

$$\sum_{n \geqslant 0} a_n \theta_n(\Psi^3),$$

where $a_n \in \mathbb{Z}_{(2)}$.

(2) *The elements of* $KO^0(KO)_{(2)}$ *can be expressed uniquely as infinite sums*

$$\sum_{n \geqslant 0} a_n \Theta_n(\Psi^3),$$

where $a_n \in \mathbb{Z}_{(2)}$.

Corollary 9.4. *There is an isomorphism of bialgebras*

$$k^0(k)_{(2)}/\langle \Psi^{-1} - 1\rangle \cong ko^0(ko)_{(2)}.$$

Proof. We have seen that the ideal I generated by $\zeta_1 = \Psi^{-1} - 1$ is a coideal. Note that $\zeta_{2n} \equiv \theta_n(\Psi^3)$ modulo I. By Theorem 9.3, the $\theta_n(\Psi^3)$ form a topological basis of $ko^0(ko)_{(2)}$, and the product and coproduct are respected. □

The methods of Section 5 may be adapted to show that the 2-adic completion $ko^0(ko)_2$ is the power series ring over $\mathbb{Z}_2$ generated by $\Psi^3 - 1$.

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Appendix A. Polynomial identities

Given two sequences $\mathbf{a} = (a_i)_{i\geqslant 1}$ and $\mathbf{b} = (b_i)_{i\geqslant 1}$ of elements of a commutative ring R, let

$$\theta_n(X;\mathbf{a}) = \prod_{i=1}^{n}(X - a_i) \quad \text{and} \quad \theta_n(X;\mathbf{b}) = \prod_{i=1}^{n}(X - b_i).$$

These polynomials provide two bases for $R[X]$ as an R-module and can therefore be written in terms of each other.

Definition A.1. Define $A_{n,r}(\mathbf{a},\mathbf{b}) \in R$ by

$$\theta_n(X;\mathbf{a}) = \sum_{r=0}^{n} A_{n,r}(\mathbf{a},\mathbf{b})\theta_r(X;\mathbf{b}).$$

Together with the identities $A_{0,0}(\mathbf{a},\mathbf{b}) = 1$, $A_{0,r}(\mathbf{a},\mathbf{b}) = 0$ for $r > 0$, and $A_{n,-1}(\mathbf{a},\mathbf{b}) = 0$ for $n\geqslant 0$, the coefficients $A_{n,r}(\mathbf{a},\mathbf{b})$ are determined by the following recurrence, which is used repeatedly in this paper.

Proposition A.2. *For* $n, r\geqslant 0$,

$$A_{n+1,r}(\mathbf{a},\mathbf{b}) = (b_{r+1} - a_{n+1})A_{n,r}(\mathbf{a},\mathbf{b}) + A_{n,r-1}(\mathbf{a},\mathbf{b}).$$

Proof. Since $\theta_{n+1}(X;\mathbf{a}) = (X - a_{n+1})\theta_n(X;\mathbf{a})$,

$$\begin{aligned}\theta_{n+1}(X;\mathbf{a}) &= \sum_{r=0}^{n} A_{n,r}(\mathbf{a},\mathbf{b})(X - a_{n+1})\theta_r(X;\mathbf{b})\\ &= \sum_{r=0}^{n} A_{n,r}(\mathbf{a},\mathbf{b})\left(\theta_{r+1}(X;\mathbf{b}) + (b_{r+1} - a_{n+1})\theta_r(X;\mathbf{b})\right),\end{aligned}$$

and the result follows using $A_{n,-1}(\mathbf{a},\mathbf{b}) = A_{n,n+1}(\mathbf{a},\mathbf{b}) = 0$. □

In most cases which we have considered, direct substitution in the recurrence allows considerable simplification. However, it is possible to give an explicit formula for the coefficients $A_{n,r}(\mathbf{a}, \mathbf{b})$ as polynomials in the a_i and b_j.

Proposition A.3. *For $n, r \geqslant 0$,*

$$A_{n,r}(\mathbf{a}, \mathbf{b}) = \sum_{\substack{J \subset \{1,\dots,n\} \\ |J|=r}} \prod_{\substack{1 \leqslant i \leqslant n \\ i \notin J}} (b_{\sigma(i,J)} - a_i),$$

where $\sigma(i, J) = 1 + |\{j \in J : j < i\}|$.

Proof. Let

$$\widetilde{A}_{n,r} = \sum_{\substack{J \subset \{1,\dots,n\} \\ |J|=r}} \prod_{\substack{1 \leqslant i \leqslant n \\ i \notin J}} (b_{\sigma(i,J)} - a_i).$$

We have $\widetilde{A}_{0,0} = 1$, $\widetilde{A}_{0,r} = 0$ for $r > 0$, and $\widetilde{A}_{n,-1} = 0$ for $n \geqslant 0$. It is therefore only necessary to verify that $\widetilde{A}_{n,r}$ satisfies the recurrence $\widetilde{A}_{n+1,r} = (b_{r+1} - a_{n+1})\widetilde{A}_{n,r} + \widetilde{A}_{n,r-1}$ of Proposition A.2.

Break the sum defining $\widetilde{A}_{n+1,r}$ into two parts by dividing the subsets $J \subset \{1, \dots, n+1\}$ such that $|J| = r$ according to whether $n + 1 \in J$ or not. If $n + 1 \notin J$, then $J \subset \{1, \dots, n\}$ and the corresponding summand in $\widetilde{A}_{n,r}$ occurs in $\widetilde{A}_{n+1,r}$ multiplied by the factor $(b_{r+1} - a_{n+1})$ since $\sigma(n + 1, J) = r + 1$.

If $n + 1 \in J$, let $I = J \setminus \{n + 1\} \subset \{1, \dots, n\}$, then $\sigma(i, J) = \sigma(i, I)$ for all $i \notin J$, and

$$\prod_{\substack{1 \leqslant i \leqslant n+1 \\ i \notin J}} (b_{\sigma(i,J)} - a_i) = \prod_{\substack{1 \leqslant i \leqslant n \\ i \notin I}} (b_{\sigma(i,I)} - a_i)$$

is a summand in both $\widetilde{A}_{n+1,r}$ and $\widetilde{A}_{n,r-1}$.

It is clear that in both cases the process is reversible. □

We show finally how essentially the same coefficients arise in formulas for products of the $\theta_n(X, \mathbf{a})$. Given a sequence $\mathbf{c} = (c_i)_{i \geqslant 1}$, we write $\mathbf{c}[m]$ for the shifted sequence $(c_{m+i})_{i \geqslant 1}$.

Proposition A.4. *If $r \geqslant m \geqslant 0$ and $s \geqslant 0$, then*

$$\theta_r(X; \mathbf{c})\theta_s(X; \mathbf{c}) = \theta_m(X; \mathbf{c}) \sum_{j=s}^{r+s-m} A_{r-m,j-s}(\mathbf{c}[m], \mathbf{c}[s])\theta_j(X; \mathbf{c}).$$

Proof. Setting $\mathbf{a} = \mathbf{c}[m]$ and $\mathbf{b} = \mathbf{c}[s]$ in Definition A.1, we have

$$\theta_{r-m}(X; \mathbf{c}[m]) = \sum_{j=s}^{r+s-m} A_{r-m,j-s}(\mathbf{c}[m], \mathbf{c}[s])\theta_{j-s}(X; \mathbf{c}[s]).$$

Multiplying by $\theta_m(X;\mathbf{c})\theta_s(X;\mathbf{c})$, and using the identities

$$\theta_r(X;\mathbf{c}) = \theta_m(X;\mathbf{c})\theta_{r-m}(X;\mathbf{c}[m]) \quad \text{and} \quad \theta_j(X;\mathbf{c}) = \theta_s(X;\mathbf{c})\theta_{j-s}(X;\mathbf{c}[s]),$$

now gives the result. □

Writing $A^{m,j}_{r,s} = A_{r-m,j-s}(\mathbf{c}[m],\mathbf{c}[s])$ for the coefficient in the above expansion for $\theta_r(X;\mathbf{c})\theta_s(X;\mathbf{c})$, the recurrence of Proposition A.2 takes the form

$$A^{m,j}_{r+1,s} = (c_{j+1} - c_{r+1})A^{m,j}_{r,s} + A^{m,j-1}_{r,s}. \tag{A.5}$$

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