ANNALS OF MATHEMATICAL LOGIC – Volume 2, no. 4 (1971) 379–417

ADMISSIBLE ORDINALS AND LATTICES OF α-R.E. SETS

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Received 20 May 1970

§0. Introduction

This paper concerns the theory of recusion on initial segments of the ordinal numbers which was originated by Kripke [4] and Platek [12]. For any admissible ordinal α, let R_α denote the lattice of α-r.e. sets and let $A(R_\alpha)$ denote the Boolean algebra generated by R_α whose elements are finite unions of differences of α-r.e. sets. Denote the quotients of R_α, $A(R_\alpha)$ by the ideal of finite sets by $R_\alpha{}^*$, $A(R_\alpha{}^*)$ and by the ideal of bounded sets by $R_\alpha{}^\#$, $A(R_\alpha{}^\#)$ respectively. Also, let Q_α denote the lattice of α-r.e. subsets of ω, and let $A(Q_\alpha)$, $Q_\alpha{}^*$, $A(Q_\alpha{}^*)$ denote, respectively, the Boolean algebra generated by Q_α, and the quotients of Q_α and of $A(Q_\alpha)$ by the ideal of finite sets. Note that Q_{ω_1} is the lattice of Π^1_1 sets.

We shall consider the first order language with function symbols $\cap$, $\cup$, $'$ and with unary predicate symbols E, L, and in which quantifiers are restricted to ranging over the domain of the predicate L. The language will always be interpreted in a Boolean algebra generated by a lattice; $\cap$, $\cup$, $'$ will be interpreted as meet, join, complementation, respectively, $E(x)$ will be interpreted as "x is the zero element of the lattice", and $L(x)$ will be interpreted as "x is an element of the lattice".

[1]) This paper is based on a portion of the author's Ph.D. thesis (M.I.T. 1969) supervised by Gerald E. Sacks and supported by an N.S.F. Graduate Fellowship.

We are concerned with determining which sentences of this language are true when interpreted in the Boolean algebras and lattices which were defined in the preceding paragraph.

Lachlan [5] has given a decision procedure for the AE-sentences of this language when interpreted in $A(R_\omega{}^*)$. The principal goal of this paper is to show that the same decision procedure works for $A(R_\alpha{}^*)$, $A(R_\alpha{}^\#)$, and $A(Q_\alpha{}^*)$ for all admissibles α which are projectible into ω. As an immediate consequence we get that any two-quantifier sentence is true in $A(R_\omega{}^*)$ if and only if it is true in $A(R_\alpha{}^*)$, $A(R_\alpha{}^\#)$, and $A(Q_\alpha{}^*)$ for any admissible α projectible into ω. In other words, the lattices $R_\alpha{}^*$, $R_\alpha{}^\#$, and $Q_\alpha{}^*$ are equivalent with respect to two-quantifier sentences. We shall also show, using results of Sacks and Owings, that this result is best possible. Note that this result gives a general criterion for taking theorems of ordinary recursion theory and "lifting" them to generalized recursion theory.

This paper follows Lachlan [5] as closely as possible and leans heavily on that paper. This paper is self-contained to the extent that all the definitions and theorems of [5] are stated, but those proofs and sections of proofs which are identical to those in [5] have been omitted. For the sake of completeness, some sections of [5] have been copied exactly except for minor typographic variations.

§1. Notation and definitions

The reader is assumed to be familiar with the basics of recursion theory on the ordinals less than an admissible ordinal as originated by Kripke [4] and Platek [12]. Refs. [3, 7–11] are other excellent sources of material on the basics of this subject. We now list some definitions taken principally from [3].

If E is a finite set of equations of Kripke's equation calculus, S_β^E will denote the set of equations resulting from β-many applications of the deduction rules to E. An ordinal α is *admissible* if $S_\alpha^E = S_{\alpha+1}{}^E$ for every finite set of equations E. A partial function f from α to α is a *partial α-recursive* function if for some finite set of equations E, $f(\gamma) = \delta$ if and only if the equation $g(\gamma) = \delta$ is in $S_\alpha{}^E$. An *α-recursive* function is a function which is partial α-recursive and total. A subset of α is *α-recursively enumerable* (*α-r.e.*) if it is the domain of a partial α-recursive function. A subset A of α is *α-recursive* if both A and A' are α-r.e. A subset of α is *α-finite* if it is α-recursive and if it is bounded below α. A subset A of α is *regular* if $A \cap M$ is α-finite for every α-finite subset M of α. The *projectum* of an admissible α is the least ordinal β such that some one–one α-recursive function maps α into β. α is *projectible into* ω if the projectum of α is ω.

We shall always regard a Boolean algebra as a complemented distributive lattice with least and greatest elements; the least element will be denoted by ϕ.

The algebraic structures which are of particular relevance are d-lattices. A *d-lattice* is a pair (L, A) where A is a Boolean algebra, L is a sublattice of A containing the least and greatest elements, and where L generates A. Note that A is determined to within isomorphism by L.

A d-lattice (L_1, A_1) is said to be a *sub-d-lattice* of a d-lattice (L, A) if A_1 is a subalgebra of A and $L_1 = L \cap A_1$. A d-lattice (L_1, A_1) is said to be *isomorphic to* a d-lattice (L, A) if there is an isomorphism of A_1 onto A which maps L_1 onto L. An *embedding of* (L_1, A_1) *in* (L, A) is an isomorphism of (L_1, A_1) onto a sub-d-lattice of (L, A).

Let (L, A) be a d-lattice and let b be in L and $\neq \phi$, we denote by $(L, A)|b$ the pair (L_1, A_1) where L_1, A_1 are $L|b$ and $A|b$ respectively, that is the restriction of L, A respectively to elements $\leq b$. It is easy to show that $(L, A)|b$ is a d-lattice. A *component* of (L, A) is a pair $((L_1, A_1), b)$ such that b is in L, b' is in L, $b \neq \phi$, and such that (L_1, A_1) is $(L, A)|b$. We shall sometimes suppress b and speak of (L_1, A_1) as a component.

Let (L_1, A_1), (L_2, A_2) be two d-lattices, their *direct union* $(L_1, A_1) \times (L_2, A_2)$ is defined to be $(L_1 \times L_2, A_1 \times A_2)$ where $L_1 \times L_2$ is the direct union of L_1, L_2 and $A_1 \times A_2$ is the direct union of A_1, A_2. It is easy to show that the direct union is a d-lattice. A sequence $(L_1, A_1), ..., (L_k, A_k)$ of d-lattices is a *decomposition of* (L, A) if $k > 1$ and

$$(1) \qquad (L, A) = (L_1, A_1) \times ... \times (L_k, A_k)\,.$$

A d-lattice is called *indecomposable* if it has no decomposition. The following is obtained easily by the same reasoning ([1], p. 68) which gives the corresponding result for partially ordered sets:

Lemma 1.1. *Every finite decomposable d-lattice has a decomposition into a finite number of indecomposable d-lattices; this decomposition is unique to within order and isomorphism of the components.*

A d-lattice is called *finite* if the order of its algebra is finite.

Let (L, A) be any finite d-lattice. We define a partial ordering of the atoms of A as follows. If b, c are atoms of A we say *b is within c* written $b \prec c$ if $b \neq c$ and if for every d in L, $c \leq d$ implies $b \leq d$. This is a partial ordering; for suppose $b \prec c$ in (L, A), then since A is generated by the elements of L, there exist c_1, c_2 in L such that c_1 is the least member of L containing c, $c_2 \leq c_1$, and $c = c_1 - c_2$. Now c_2 contains b but not c, hence $\prec$ is irreflexive. The transitivity of $\prec$ is immediate. We write $b \preccurlyeq c$ just if $b = c$ or $b \prec c$. The atom b is said to be *just within* the atom c in (L, A) if $b \prec c$ and there is no atom d such that $b \prec d \prec c$. An atom b is said to be *outermost, innermost* respectively, in (L, A) if there exists no atom c such that $b \prec c$, $c \prec b$ respectively.

We now define the rank and characteristic of a finite d-lattice. A *path* in a finite d-lattice (L, A) is a finite sequence of atoms $b_1, b_2, ..., b_k$ such that b_1 is outermost and such that b_{i+1} is just within b_i for $1 \leq i < k$. A path is said to *end in* a member b of A if the last member of the path is $\leq b$. The *rank* of a finite d-lattice is defined to be $(n_1, n_2, ..., n_k)$ where n_i is the number of paths of length i and where k is the greatest i such that $n_i \neq 0$. The possible ranks are well-ordered by the definition: $(n_1, ..., n_k)$ *is less than*, written $<$, $(m_1, ..., m_h)$ if either $k < h$, or $k = h$ and

$$(\mathrm{E}x)(1 \leq x \leq h \;\&\; (\mathrm{A}y)(x < y \leq h \rightarrow n_y = m_y) \;\&\; n_x < m_x)\,.$$

The *characteristic sequence* (or *characteristic* for short) of a finite d-lattice (L, A) is now defined as follows. Let (1) be the decomposition of (L, A) into indecomposable components arranged so that the components on the right have decreasing rank. Then the characteristic of

(L, A) is the sequence $r_1, r_2, ..., r_k$ where for $1 \leq i \leq k$, r_i is the rank of (L_i, A_i). The possible characteristics are well-ordered by: $(r_1, ..., r_k)$ *is less than*, written $\prec$, $(s_1, ..., s_m)$ if either

$$(\mathrm{E}x)(1 \leq x \leq k \;\&\; 1 \leq x \leq m \;\&\; (\mathrm{A}y)(1 \leq y < x \to r_y = s_y) \;\&\; r_x < s_x)\,,$$

or $k < m$ and $r_x = s_x$ for $1 \leq x \leq k$.

The lattices R_α, $A(R_\alpha)$, etc. and our formal language for d-lattices were introduced on the first page of this paper.

For our formal language, a predicate P is called *consistent* if there is some d-lattice (L, A) and some assignment of values in L to the free variables of P under which P is true. A sentence is *valid* if it is true in every d-lattice.

Let P, Q be predicates all of whose free variables are contained in $\{x_1, ..., x_m\}$. We say that P *implies* Q written $P \vdash Q$ just if $(\mathrm{A}x_1) ... (\mathrm{A}x_m)(P \to Q)$ is valid. Let P be a consistent quantifierless (q-less) predicate containing just the variables $x_1, ..., x_m$. We say that P is *complete* if for every q-less predicate Q whose variables are contained in $\{x_1, ..., x_m\}$ we have either $P \vdash Q$ or $P \vdash \sim Q$. A complete q-less predicate is called a *diagram*. With any diagram D we associate a finite d-lattice (L, A) as follows. Consider the set T of terms containing only variables from D. We define an equivalence relation $\approx$ on T by:

$$s \approx t \quad \text{iff} \quad D \vdash E((s \cap t') \cup (s' \cap t))\,.$$

The members of A are the equivalence classes into which $\approx$ splits T; there can only be a finite number of these. Meet, join, and complementation in A are to be the operations induced by the respective formal symbols $\cap$, $\cup$, $'$. The members of L are the equivalence classes which have some representative t for which $D \vdash L(t)$. Thus every diagram gives rise to a unique finite d-lattice; and it is easy to see conversely that given any finite d-lattice (L, A) there is some diagram whose associated d-lattice is isomorphic to (L, A). The d-lattice of D is uniquely determined by the property that D is true when for $1 \leq i \leq m$, x_i is mapped into the equivalence class which it represents. In referring to the d-lattice of a diagram below we shall often not distinguish between terms and the corresponding equivalence classes. We can now define for diagrams the concepts defined for finite d-lattices. Two diagrams are *isomorphic* if their d-lattices are isomorphic. Let D, D_1, D_2 be diagrams containing just the variables $x_1, ..., x_m$ then we say that D is the *direct union* of D_1 and D_2 provided there are terms t_1, t_2 containing just $x_1, ..., x_m$ such that $D \vdash E((t_1 \cap t_2) \cup (t_1' \cap t_2'))$, and such that for $i = 1, 2$

$$D \vdash \sim E(t_i) \mathbin{\&} \sim E(t_i') \mathbin{\&} L(t_i) \mathbin{\&} L(t_i'),$$

and such that for any term s containing only $x_1, ..., x_m$,

$$D_i \vdash E(s) \text{ iff } D \vdash E(s \cap t_i); D_i \vdash L(s) \text{ iff } D \vdash L(s \cap t_i).$$

A diagram is *decomposable* if it can be expressed as the direct union of two other diagrams, and *indecomposable* otherwise. Every diagram has a decomposition into indecomposable diagrams, and this decomposition corresponds precisely to the decomposition of the d-lattice of the diagram. The *rank* and *characteristic* of a diagram are to be the rank and characteristic of its d-lattice respectively.

Two predicates P, Q are said to be *equivalent* if $P \vdash Q$ and $Q \vdash P$. If P, Q are q-less predicates then we can tell effectively whether or not $P \vdash Q$, and thus whether or not P, Q are equivalent. There are only a finite number of equivalence classes of q-less predicates containing only a fixed set of variables $x_1, ..., x_m$. We find it convenient on occasion to consider q-less predicates modulo equivalence. Thus we can say that, if P is any q-less predicate, then P is equivalent to the disjunction of *all* diagrams D whose variables are just those of P and which imply P (meaning that we select one diagram D for each equivalence class). Similarly, when we are considering terms constructed from some fixed finite set of variables.

§2. Separated d-lattices and existential statements

In this section we shall show that a sentence

$$(\mathrm{E}x_1) \ldots (\mathrm{E}x_m)\, P(x_1, ..., x_m),$$

where P is a q-less predicate is true in the d-lattices $(R_\alpha, A(R_\alpha))$, $(R_\alpha^*, A(R_\alpha^*))$, $(R_\alpha^\#, A(R_\alpha^\#))$ for any admissible α and in $(Q_\alpha, A(Q_\alpha))$, $(Q_\alpha^*, A(Q_\alpha^*))$ for α projectible into ω provided only that P is consistent. We shall rely heavily on the theorem that if A, B are α-r.e. sets then there exist disjoint α-r.e. sets A_1, B_1 such that $A_1 \subseteq A$, $B_1 \subseteq B$, and $A_1 \cup B_1 = A \cup B$. This suggests the following. Call a d-lattice (L, A) *separated* if for any pair x, y of elements of L there exists a disjoint pair x_1, y_1 of elements of L such that $x_1 \leq x$, $y_1 \leq y$, and $x_1 \cup y_1 = x \cup y$. For a finite d-lattice the property of being separated can be expressed in another way.

Lemma 2.1. *A finite d-lattice (L, A) is separated if and only if there exist elements $b_1, \ldots, b_m$ of L such that*

$$(2) \qquad (\mathrm{A}x)_{1 \leqslant x \leqslant m} (\mathrm{A}y)_{1 \leqslant y \leqslant m} (b_x \cap b_y = \phi \text{ or } b_x \leqslant b_y)$$

and such that every element of L is the union of some subset of $b_1, \ldots, b_m$.

For a proof of Lemma 2.1 see Lachlan ([5], p. 127). We say that an element of a lattice can be *split non-trivially* if it can be expressed as the union (join) of two non-zero disjoint elements of the lattice. For any finite separated d-lattice the elements of the lattice $\neq \phi$ which cannot be split non-trivially in the lattice are called *canonical generators*; it follows from Lachlan's proof of Lemma 2.1 that any element of the lattice can be expressed as a union of canonical generators. Let (L, A) be a finite separated d-lattice with canonical generators $b_1, \ldots, b_m$. Any atom of A is of the form

$$\cup\{b_i : i \in S\} - \cup\{b_i : i \in T\}$$

where S, T are disjoint sets whose union is $\{1, 2, \ldots, m\}$. From (2) we may also suppose that S, T have the respective forms $\{j\}$, $\{i : b_j \nleq b_i\}$. Thus there is a one–one correspondence between canonical generators and atoms. Let c_i, c_j be the atoms corresponding to b_i, b_j respectively; note that $c_i \preccurlyeq c_j$ if and only if $b_i \leq b_j$. Thus c_i is just within c_j if and only if b_j is minimal with respect to properly containing b_i. There can only be one such j for each i from (2). Hence for each atom there is exactly one path in (L, A) which ends in that atom. This gives a one–one correspondence between canonical generators and paths under which the ordering of canonical generators in the lattice corresponds to the ordering of paths by extension. Thus the rank and characteristic of a finite separated d-lattice are determined by the partial ordering of its canonical generators.

The next two lemmas are proved in Lachlan ([5], p. 128).

Lemma 2.2. *Let (L, A) be a finite d-lattice, then b is in $A - L$ if and only if there exist atoms c_1, c_2 such that $c_1 \leq b$, $c_2 \not\leq b$, and c_2 is just within c_1.*

Lemma 2.3. *Every finite d-lattice (L, A) can be embedded in a finite separated d-lattice which has the same characteristic.*

Before we can proceed with Lachlan's work, there are two results of ordinary recursion theory which must be proved in abstract recursion theory. They are given in the next two lemmas.

Lemma 2.4. *Let α be any admissible ordinal. Let A, B be α-r.e. sets such that $B \subseteq A$ and $A - B$ is not α-r.e. Then there exists an α-r.e. set C such that $C \subseteq B$, $\mathrm{lub}(C) = \mathrm{lub}(B)$ $C \cup (A - B)$ is not α-r.e. and $B - C$ is not α-r.e. If B is regular, then C can be taken to be regular.*

Proof. Let β be the least ordinal $\leq \alpha$ such that $B \cap (\alpha - \beta)$ is α-finite. Find an α-recursive set D such that $D \subseteq (B \cap \beta)$ and $\mathrm{lub}(D) = \mathrm{lub}(B \cap \beta)$: for example let D be enumerated by enumerating $B \cap \beta$ and placing in D all elements which are greater than all previously enumerated elements of $B \cap \beta$. Now because of the condition imposed on β, there will be a one–one α-recursive function mapping α into D. Then, we can find a subset C_1 of D such that C_1 is α-r.e., C_1 is not α-recursive, and $\mathrm{lub}(C_1) = \mathrm{lub}(D)$. If B is regular, hence unbounded in α, we can take C_1 to be regular. Now we let $C = C_1 \cup (B \cap (\alpha - \beta))$. Clearly $\mathrm{lub}(C) = \mathrm{lub}(B)$ and $C \subseteq B$.

Let $D_1 = D \cup (B \cap (\alpha - \beta))$ and note that D_1 is α-recursive and $C \subseteq D_1$.

$C \cup (A - B)$ cannot be α-r.e., because if so $A - B = (C \cup (A - B)) \cap (\alpha - D_1)$ is α-r.e. contradicting our hypothesis. $B - C$ cannot be α-r.e., because if so $D - C_1 = (B - C) \cap D$ is α-r.e. contradicting our choice of C_1. If B is regular then $C = C_1$ is regular. This completes the proof of the lemma.

Lemma 2.5. *Let α be any admissible ordinal. Let A be an α-r.e. set which is not α-recursive. Then there exist disjoint α-r.e. sets B, C such that $A = B \cup C$ and such that for any α-r.e. set R, if $R - A$ is not α-r.e. then $R - B$ and $R - C$ are both not α-r.e. Moreover, if A is regular, then B, C can be taken to be regular.*

Proof. (This, of course, is really just Friedberg's splitting theorem [2].) Let α^* be the projectum of α and let $\{R_\beta\}_{\beta<\alpha^*}$ be an enumeration of all the α-r.e. sets. We will enumerate A along with $\{R_\beta\}_{\beta<\alpha^*}$. Every time we enumerate a member of A we will put it into either B or C. β is said to be *satisfied* when R_β intersects both B and C.

Let γ be enumerated in A at stage σ. If every β such that γ has been enumerated in R_β by stage σ is satisfied, put γ in B and go to stage $\sigma+1$. Otherwise, *attack* the first unsatisfied β such that γ has been enumerated in R_β by stage σ. If that R_β intersects neither B nor C, put γ in B. Otherwise, put γ in B or C accordingly as $R_\beta \cap B$ or $R_\beta \cap C$ is empty. Go to stage $\sigma+1$.

Now assume $R - B$ is α-r.e. for some α-r.e. set R, and that $R - B = R_{\beta'}$. No β is attacked more than twice; after two attacks a β is satisfied; and only an unsatisfied β is ever attacked. Moreover, no partial α-r.e. function can map an unbounded α-r.e. set one–one into a proper initial segment of α^* (any α-r.e. set bounded below α^* is α-finite). Therefore, there is a stage σ' after which no $\beta \leq \beta'$ is ever attacked. Thus, after stage σ', no member of A is enumerated which has previously been enumerated in $R_{\beta'}$. This allows us to enumerate $R - A$. Similarly, for C. It is clear from the construction that if A is regular, then B, C will both be regular. This completes the proof of the lemma.

We now return to Lachlan's paper by introducing some special terminology. Let P be any relation of order $n \geq 1$ defined on R_α. For an

n-tuple A of α-r.e. subsets of B we say that $P(\mathrm{A})$ *holds in* B just if for some one–one α-recursive function F mapping α onto B we have $P(F^{-1}(\mathrm{A}))$. Thus in this paper "A is α-recursive in B" with A a subset of B will not have its usual meaning, but will mean simply that $B - A$ is α-r.e.

Theorem 2.6. *Let α be any admissible ordinal. Every finite separated d-lattice is embeddable in* $(R_\alpha, A(R_\alpha))$, $(R_\alpha{}^*, A(R_\alpha{}^*))$, and $(R_\alpha{}^\#, A(R_\alpha{}^\#))$.

Proof. Let (L, A) be a finite separated d-lattice. It is sufficient to show that (L, A) can be embedded in $(R_\alpha, A(R_\alpha))$ such that the images of the atoms of A are unbounded. For we then take equivalence classes modulo finite sets and modulo bounded sets to get the result.

We define a map of the canonical generators $b_1, ..., b_m$ of (L, A) into R_α which preserves disjointness and inclusion. Moreover, the images of the canonical generators will be regular, unbounded sets. Let $m_1, ..., m_j$ be the maximal canonical generators and let $N_1, ..., N_j$ be regular, unbounded, pairwise disjoint α-recursive sets whose union is α. Map m_x to N_x for $1 \leq x \leq j$. The definition of the map now proceeds by induction "downwards" with respect to the order of the lattice. Let b be a canonical generator not yet mapped such that all of those which properly contain it have already been mapped. Let c be the least canonical generator which properly contains b, and let $n_1, ..., n_k$ be the canonical generators which are maximal with respect to being properly contained in c among which must occur b. Assume for induction that none of $n_1, ..., n_k$ has been mapped. Let C be the image of c, choose an α-r.e. set B as follows:

Case 1. If c is a maximal canonical generator, let B be any subset of C which is α-r.e., not α-recursive, and regular (hence unbounded).

Case 2. Otherwise, let d be the least canonical generator which properly contains c and let D be the image of d. Let B be any α-r.e. subset of C which is regular, unbounded, not α-recursive in C, and such that $B \cup (D - C)$ is not α-r.e. We assume as part of the induction hypothesis that C is not α-recursive in D, therefore by Lemma 2.4, such a B exists.

Having chosen B we decompose it into α-r.e., regular, unbounded, pairwise disjoint sets $B_1, ..., B_k$ such that each like B is not α-recursive in C. This is possible by Lemma 2.5. We now define the images of

$n_1, \ldots, n_k$ to be $B_1, \ldots, B_k$ respectively. This completes the definition of the mapping; the definition is a good one because the induction hypothesis can be discharged.

By inspection of the definition of the mapping, we have that the mapping of the canonical generators induces an isomorphism of A into $A(R_\alpha)$ such that the images of atoms of A are regular, unbounded sets. Let A^* be the image of A; then the lattice L is mapped onto the sublattice L^* of R_α whose canonical generators are the images $B_1, \ldots, B_m$ of $b_1, \ldots, b_m$ respectively. Let d be any member of $A - L$; then by Lemma 2.2 there exist atoms e, f such that $e \leq d, f \nleq d$, and such that f is just within e. Let b, c be the canonical generators corresponding to e, f respectively. Then c is maximal with respect to the property of being properly contained in b. Let B, C be the images of b, c respectively; by the construction the image of e is $B - G$ where G is an α-r.e. subset of B not α-recursive in B. Suppose that D, the image of d, is α-r.e.; then D contains $B - G$ and so intersects C, otherwise we should have $B - C$ equal to the α-r.e. set $(B \cap D) \cup (G - C)$ contrary to the provision in the construction which makes C not α-recursive in B. If c were minimal, then $c = f$, and we have a contradiction. Otherwise, by the construction there is an α-r.e. subset H of C such that $H \cup (B - C)$ is not α-r.e. and such that f is mapped into $C - H$. Since D cannot intersect the image of f we have $D \cap C \subseteq H$, therefore $H \cup (B - C)$ is α-r.e. which is a contradiction. Therefore the image of any member of $A - L$ is not α-r.e. Thus (L^*, A^*) is a sub-d-lattice of $(R_\alpha, A(R_\alpha))$ with the required properties, and the theorem is proved.

Corollary. 2.7. *Let α be an admissible ordinal projectible into ω. Every finite, separated d-lattice is embeddable in $(Q_\alpha, A(Q_\alpha))$ and $(Q_\alpha^*, A(Q_\alpha^*))$;*

Proof. It is enough to show the result for $(Q_\alpha, A(Q_\alpha))$ with the images of atoms being infinite. To do this simply replace "regular, unbounded" throughout the proof of Theorem 2.6 by "contained in ω". Note the the corollary is false for α not projectible into ω.

We wish to show that all possible existential sentences are true in the d-lattices mentioned in the statement of Theorem 2.6 and in the statement of Corollary 2.7. Consider any sentence

(3) $(\mathrm{E}x_1) \ldots (\mathrm{E}x_m)\, P(x_1, \ldots, x_m)$

where P is a consistent q-less predicate containing just the variables $x_1, \ldots, x_m$. Since P can be expressed as the disjunction of all the diagrams containing just $x_1, \ldots, x_m$ which imply P, to show that any such sentence (3) is true in some d-lattice it is sufficient to show that any sentence (3) holds when P is a diagram. Let (L, A) be the d-lattice associated with P; this can be embedded in a finite separated d-lattice, which in turn can be embedded in all the d-lattices under consideration. Let $B_1, \ldots, B_m$ be the images in the lattice under consideration of the elements in (L, A) represented by $x_1, \ldots, x_m$ respectively; then $P(B_1, \ldots, B_m)$ holds in the lattice under consideration, and so (3) is true in that lattice. We have proved:

Theorem 2.8. *Any sentence* $(\mathrm{E}x_1) \ldots (\mathrm{E}x_m)P(x_1, \ldots, x_m)$ *where* P *is a consistent q-less predicate containing just the variables* $x_1, \ldots, x_m$ *is true in all the d-lattices mentioned in the statements of* Theorem 2.6 *and* Corollary 2.7.

There is a sense in which Theorem 2.8 is the best possible result. If sentence complexity is measured in terms of the number of alternations of quantifiers in the prefix when the sentence is in prenex normal form, then the class of one-quantifier (no alternations) sentences is the largest class of sentences for which the lattices covered by Theorem 2.6 could be equivalent. This is because it has been shown by Sacks [9] that there are countable admissible ordinals α such that $R_\alpha{}^*$ has no maximal elements, and elsewhere by Sacks [3], that if α is an admissible projectible into ω then $R_\alpha{}^*$ does have maximal elements. The sentence expressing the existence of a maximal element is a two-quantifier sentence, and therefore it cannot be that exactly the same two-quantifier sentences are true in all the lattices covered by Theorem 2.6. As has already been mentioned, Corollary 2.7 is false if α is not projectible into ω (the sentence $(\mathrm{E}x) \sim L(x)$ is false in $(Q_\alpha, A(Q_\alpha))$ for such α). Therefore the d-lattices covered by Theorem 2.8 is the largest class of such d-lattices for which the theorem is true.

§3. A preliminary reduction

In this section we begin to generalize Lachlan's decision procedure for AE-sentences to all admissible ordinals projectible into ω. To that end, for the remainder of this paper α will always denote an admissible ordinal projectible into ω unless it is specified to be otherwise. Also, for the remainder of this paper $(\mathbf{R}, A(\mathbf{R}))$ will stand for any one of $(R_\alpha^*, A(R_\alpha^*))$, $(R_\alpha^\#, A(R_\alpha^\#))$, or $(Q_\alpha^*, A(Q_\alpha^*))$. Thus any assertion about $(\mathbf{R}, A(\mathbf{R}))$ is an assertion about all three of the d-lattices. We first narrow down the class of sentences we need consider to those in a special form. To obtain the special form we need the following:

Theorem 3.1. *Let (L, A) be any finite sub-d-lattice of $(\mathbf{R}, A(\mathbf{R}))$ with characteristic c; there exists a finite separated sub-d-lattice (L^*, A^*) of $(\mathbf{R}, A(\mathbf{R}))$ which is an extension of (L, A) and whose characteristic is $\leq c$. Also, given (L, A) we can effectively enumerate a finite number of isomorphism types, together with for each type an isomorphism of (L, A) into it, so that (L^*, A^*) can be found in one of these types with the isomorphism picking out (L, A).*

Proof. Note: Lachlan has proved this theorem for any separated d-lattice in the place of $(\mathbf{R}, A(\mathbf{R}))$; therefore, the proof below is presented solely for heuristic reasons.

It is sufficient to prove the theorem for R_α instead of R_α^* and $R_\alpha^\#$, and for Q_α instead of Q_α^*. Lachlan proves this theorem in three steps, which we state as follows:

Lemma 3.2. *It is sufficient to suppose that (L, A) is indecomposable.*

Lemma 3.3. *Suppose that (L, A) is indecomposable. Then there is a map T of the paths of (L, A) into R_α or Q_α (depending on whether we started with (L, A) in $(R_\alpha, A(R_\alpha))$ or in $(Q_\alpha, A(Q_\alpha))$ respectively) satisfying the following three conditions:*

(i) *For B in L, $B = \bigcup \{T(p)$: p is a path in (L, A) ending in $B\}$.*

(ii) *If p is a path extending a path q, then $T(p) \subseteq T(q)$.*

(iii) *If neither p nor q extends the other, then $T(p) \cap T(q) = \phi$.*

Lemma 3.4. *For (L, A) indecomposable, the sets $T(p)$ from* Lemma 3.3 *generate the required sub-d-lattice (L^*, A^*).*

The only non-algebraic fact which Lachlan uses in the proof of Lemma 3.2 is that if B is an infinite recursive set then $(R_\omega, A(R_\omega))$ is isomorphic to $(R_\omega, A(R_\omega))|B$. Lachlan uses this fact to establish that, assuming the theorem holds in $(R_\omega, A(R_\omega))$ for (L, A) indecomposable, then it holds in $(R_\omega, A(R_\omega))|B$ for (L, A) indecomposable. For the case where we start originally with a sub-d-lattice of $(R_\alpha^\#, A(R_\alpha^\#))$ or $(Q_\alpha^*, A(Q_\alpha^*))$ and B is an unbounded α-recursive set or an infinite α-recursive subset of ω respectively, it is certainly the case that $(R_\alpha, A(R_\alpha))$ is isomorphic to $(R_\alpha, A(R_\alpha))|B$ or $(Q_\alpha, A(Q_\alpha))$ is isomorphic to $(Q_\alpha, A(Q_\alpha))|B$ respectively. For the case where we start originally with a sub-d-lattice of $(R_\alpha^*, A(R_\alpha^*))$ and B is an infinite α-recursive set, we have that $(R_\alpha, A(R_\alpha))|B$ is isomorphic to either $(R_\alpha, A(R_\alpha))$ or $(Q_\alpha, A(Q_\alpha))$ depending on whether B is unbounded or not. In any case, since we are assuming the theorem for both R_α and Q_α in the case when (L, A) is indecomposable, Lachlan's proof of Lemma 3.2 now applies.

Lachlan's proof of Lemma 3.4 is purely algebraic; therefore, we now need only prove Lemma 3.3 to prove the theorem.

Proof of Lemma 3.3. Suppose that (L, A) is indecomposable and that $a_1, a_2, ..., a_m$ are the atoms of (L, A). For $1 \leq i \leq m$ let B_i be the least member of L containing a_i. We carry out a construction as follows. Suppose we are given "boxes" $S_1, S_2, ..., S_m$ corresponding to $a_1, a_2, ..., a_m$ respectively. In the course of the construction we shall place ordinals $< \alpha$ or natural numbers (depending on whether we are in R_α or in Q_α) in these boxes; henceforth we shall refer to the contents of these boxes as *elements*. We shall also move elements from one box to another. A particular element can be in at most one box at a time. At each step we enumerate one element in one of $B_1, B_2, ..., B_m$ in such a way that in the course of the construction each B_i has each of its members enumerated an unbounded set of times. At the beginning of the construction, no element is in any box.

Step σ: Suppose n is enumerated in B_i at this step. *Case 1.* a_i is outermost and n is not in any box, then put n in S_i. *Case 2.* If n is currently in S_j and a_i is just within a_j, then move n from S_j into S_i. Otherwise, do nothing.

Consider an element n of a_k. At some stage n will be placed in one of the boxes. For there exists i such that $a_k \preccurlyeq a_i$ and a_i is outermost, hence n will eventually be enumerated in B_i and placed in S_i if not in any box earlier. Also, if n is in S_j at some stage, then $a_k \preccurlyeq a_j$ since $B_k \subseteq B_j$. If n is moved from S_p into S_q then a_q is just within a_p. Thus, if n ever reaches S_k, it will remain there. However, if n never reaches S_k, then n remains in S_j at all sufficiently large steps. In this case there exists a_p such that a_p is just within a_j and such that $a_k \preccurlyeq a_p$. Since n is in a_k, it is in B_p and so n cannot remain in S_j forever. Thus if n is in a_k, it eventually reaches S_k and stays there.

Let $a_{i_1}, a_{i_2}, ..., a_{i_q}$ be any path in (L, A). We say that n follows this path if in the above construction n is placed first in S_{i_1}, next in S_{i_2}, ..., and next in S_{i_q}; n need not remain in S_{i_q}. For any path p in (L, A) we let $T(p)$ be the set of elements which follow p. It is easy to see that $T(p)$ is α-r.e. for every p. Moreover, the mapping: p goes to $T(p)$ clearly satisfies the three conditions stated in Lemma 3.3. This completes the proof of the lemma and of the theorem.

Consider now any sentence $(\mathrm{Ax})(\mathrm{Ey})P(\mathrm{x}, \mathrm{y})$ where x is $x_1, ..., x_m$ and y is $y_1, ..., y_n$ and where P is a q-less predicate. In any d-lattice this is equivalent to the conjunction of the sentences

$$(4) \qquad (\mathrm{Ax})(\mathrm{Ey})(D(\mathrm{x}) \to P(\mathrm{x}, \mathrm{y})),$$

where D runs through the finite number of isomorphism types of diagrams containing just the variables x. A sentence of the form (4) is called *primitive*; its *characteristic* is defined to be the characteristic of D. By the last theorem, given D we can enumerate a finite sequence of pairs $((L_1, A_1), F_1), ..., ((L_k, A_k), F_k)$ satisfying the following conditions:

(i) (L_i, A_i) is a finite separated d-lattice with characteristic $\leq$ that of D.

(ii) F_i is an isomorphism of the d-lattice D into (L_i, A_i).

(iii) If G is an isomorphism of the d-lattice of D into $(\mathbf{R}, A(\mathbf{R}))$, then for some j, $1 \leq j \leq k$, there is an isomorphism H of (L_j, A_j) into $(\mathbf{R}, A(\mathbf{R}))$ such that $G = HF_j$. We can find a natural number p and diagrams $D^1, ..., D^k$ containing just the variables $z_1, ..., z_p$ such that for $1 \leq i \leq k$ the d-lattice of D^i is isomorphic to (L_i, A_i) under a map K_i say, of the former into the latter. Further, for each i, $1 \leq i \leq k$, we can find terms $X_1{}^i, ..., X_m{}^i$ containing just $z_1, ..., z_p$ such that mapping x_j into $X_j{}^i$ for $1 \leq j \leq m$ induces the isomorphism $K_i^{-1} F_i$ of D into D^i. Now we see that (4) is equivalent in $(\mathbf{R}, A(\mathbf{R}))$ to the conjunction of the sentences

$$(\mathrm{Az})(\mathrm{Ey})\,(D^i(\mathrm{z}) \to P(X_1{}^i, ..., X_m{}^i, \mathrm{y}))\,,$$

where z is $z_1, ..., z_p$, and where i runs from 1 to k.

Call a primitive AE-sentence *separated* if it has the form (4) with D a separated diagram. We conclude that to obtain a decision procedure for AE-sentences in $(\mathbf{R}, A(\mathbf{R}))$ *it is sufficient to reduce the decision problem for a separated* AE-*sentence to the decision problem for primitive* AE-*sentences of lower characteristic.* This is the program to be carried out below.

§4. A necessary condition

In this section we establish a certain condition as being necessary for a separated AE-sentence to be true in $(\mathbf{R}, A(\mathbf{R}))$. We need five facts about α-r.e. sets which can be expressed as follows:

(i) The lattice of α-r.e. subsets of an α-r.e. set which is not α-finite is isomorphic to the lattice of α-r.e. sets.

(ii) There exists B in R, $B \neq \phi$, such that B' is in R and $B' \neq \phi$.

(iii) R has maximal elements.

(iv) For any α-r.e. set B which is not α-recursive there exist disjoint α-r.e. sets C, D such that $B = C \cup D$ and such that for any α-r.e. set R, if $R - B$ is not α-r.e. then $R - C$ and $R - D$ are both not α-r.e. Moreover, if B is regular then C and D can be taken to be regular.

(v) For any regular α-r.e. set B which is not α-recursive or for any α-r.e. subset of ω which is not α-recursive there exists an α-r.e. subset C of B which is regular or not respectively, and which has the following properties:

(a) $\text{lub}(B) = \text{lub}(B - C)$;

(b) for every α-r.e. set R such that $B \cup R \supseteq \text{lub}(B)$, $\text{lub}(B) - (C \cup R)$ is finite;

(c) for any α-r.e. sets R, S such that $S \supseteq R \cap (B - C)$, $S \cup (R - B)$ is α-r.e.

Note that any α-r.e. set C satisfying (a) and (b) is called a *major* subset of B.

The first two facts are too elementary to require any comment. The third fact was proved by Sacks [3]. The fourth fact was proved in Lemma 2.5. We now prove the fifth fact.

Theorem 4.1. *Fact* (v) *above is true.*

Proof. Let B be enumerated one element at a time without repetitions and call $B_\sigma = \{b_\iota\}_{\iota<\sigma}$ the set containing the first σ elements enumerated in B. We shall α-effectively enumerate the required set C in steps $0, 1, ..., \sigma, ...$ letting C_σ be the set of elements which have been placed in C before step σ. For $i < \omega$, let X_i run through all α-r.e. sets; enumerate the sets X_i simultaneously letting $X_{i,\sigma}$ be the set of elements enumerated in X_i by step σ in this enumeration. Similarly, for $i < \omega$, let (R_i, S_i) run through all pairs of α-r.e. sets; enumerate the sets R_i, S_i simultaneously letting $R_{i,\sigma}$, $S_{i,\sigma}$ be the set of elements enumerated in R_i or S_i respectively, by step σ in this enumeration.

To the end of satisfying condition (b) we define two helpful auxiliary functions:

$$F(i, \sigma) = \min(\min(B_{\sigma+1}' \cap X_{i,\sigma}'), \text{lub}(B)) ,$$

and for χ in B we define $W(\chi, i)$ by letting

$$W(b_\sigma, i) = \Sigma \{ 2^{i-j} : j \leq i \ \& \ b_\sigma \in X_{j,\sigma} \ \& \ b_\sigma < F(j, \sigma)\}.$$

Note that $F(i, \sigma)$ is increasing in σ for fixed i and that $\lim_\sigma F(i, \sigma) = \text{lub}(B)$ if and only if $B \cup X_i \supseteq \text{lub}(B)$.

To satisfy condition (c) we shall enumerate simultaneously with C α-r.e. sets $T_0, T_1, ..., T_i, ...$ for $i < \omega$ letting $T_{i,\sigma}$ denote the set of elements which have been placed in T_i before step σ. It is our intention that, if $S_i \supseteq R_i \cap (B - C)$, then $T_i \cup S_i$ shall be $S_i \cup (R_i - B)$. Because of conflict with the need to satisfy condition (b), this intention will not be fulfilled, but we shall succeed sufficiently to make $S_i \cup (R_i - B)$ α-r.e. whenever $S_i \supseteq R_i \cap (B - C)$. We say that χ is *protected by i at step* σ if χ is in $(B_\sigma \cap T_{i,\sigma}) - (C_\sigma \cup S_{i,\sigma})$.

The requirements which we want C to meet are

$$P_{2i}\colon B \cup X_i \supseteq \mathrm{lub}(B) \to \mathrm{lub}(P) - (C \cup X_i) \text{ finite}, \; i < \omega;$$

and

$$P_{2i+1}: S_i \supseteq R_i \cap (B - C) \to S_i \cup (R_i - B) \;\; \alpha\text{-r.e.}, \; i < \omega.$$

In the construction we give these requirements priority according to the sequence $P_0, P_1, \ldots$ In step σ we define a one–one finite function D_σ. Let f be an α-recursive function mapping α one–one into ω.

Step 0. Define $D_0(0) = b_0$ and $D_0(x)$ to be undefined otherwise.

Step σ ($\sigma > 0$). *Case 1.* There exist z, x, y such that $z \le x < y \le f(\sigma)$; such that $\lim_{\tau<\sigma} D_\tau(x)$ and $\lim_{\tau<\sigma} D_\tau(y)$ are both defined; such that

$$W(\lim_{\tau<\sigma} D_\tau(x), z) < W(\lim_{\tau<\sigma} D_\tau(y), z)\;;$$

and such that $\lim_{\tau<\sigma} D_\tau(x)$ is not protected by any $i < z$. Choose the least possible x, and the least possible y for that x, enumerate $\lim_{\tau<\sigma} D_\tau(x)$ in C and let

$$D_\sigma(w) = \begin{cases} \lim_{\tau<\sigma} D_\tau(w) & \text{if defined, } w < f(\sigma), \text{ and } w \ne x, y, \\ \lim_{\tau<\sigma} D_\tau(y) & \text{if } w = x\,, \\ b_\sigma^* & \text{if } w = y\,, \end{cases}$$

and let $D_\sigma(w)$ be undefined otherwise, where b_σ^* is the least member of $B_{\sigma+1} - C_\sigma$ such that $b^* \ne \lim_{\tau<\sigma} D_\tau(w)$ for any $w < f(\sigma)$.

Case 2. Otherwise. Let x be the least number such that $\lim_{\tau<\sigma} D_\tau(x)$ is undefined, if any. Let $y = \min(x, f(\sigma))$; let $D_\sigma(w) = \lim_{\tau<\sigma} D_\tau(w)$ for $w < y$; let $D_\sigma(y) = b_\sigma^*$ if $y = x$; and let $D_\sigma(y) = \lim_{\tau<\sigma} D_\tau(y)$ otherwise. For each $i < \omega$ let $M(i)$ be the greatest number $m \le y$ such that

$$S_{i,\sigma} \supseteq R_{i,\sigma} \cap \{D_\sigma(0), D_\sigma(1), \ldots, D_\sigma(m)\}\,,$$

and enumerate in T_i all members θ of $R_{i,\sigma} - B_{\sigma+1}$ such that $f(\theta) < M(i)$.
This completes the construction. Let $D = \lim_{\sigma<\alpha} D_\sigma$.

Proposition 4.2. $\mathrm{Lim}_{\tau<\sigma} D_\tau$ *is well-defined (i.e. either* $\lim_{\tau<\sigma} D_\tau(x)$ *is defined or* $D_\tau(x)$ *is undefined for all sufficiently large* $\tau < \sigma$, *for all* $x < \omega$) *for all* $\sigma \leq \alpha$, *and* $\mathrm{dom}(D) = \omega$.

Proof. By induction on σ and x. Assume the proposition false and let σ be least and x least for that σ such that $\lim_{\tau<\sigma} D_\tau(x)$ is not well-defined. It cannot be that $D_\tau(x)$ is undefined for all sufficiently large $\tau < \sigma$. From our assumption on σ and the fact that there are only finitely many $\mu < \alpha$ such that $f(\mu) \leq x$ it follows that $D_\tau(x)$ is defined for all sufficiently large $\tau < \sigma$. From the definition of W we have for any ρ, η in B that

$$[z \leq x \ \&\ W(\rho, z) < W(\eta, z)] \rightarrow [W(\rho, x) < W(\eta, x)] \ .$$

Therefore, by examining Case 1 of the construction we see that in order for $\lim_{\tau<\sigma} D_\tau(x)$ not to exist, $W(D_\tau(x), x)$ must take on arbitrarily large integer values as τ approaches σ. But this contradicts $W(D_\tau(x), x) < 2^{x+1}$ which follows immediately from the definition of W. Therefore $\lim_{\tau<\sigma} D_\tau$ is well-defined for all $\sigma \leq \alpha$.

From the fact that there are only finitely many $\mu < \alpha$ such that $f(\mu) \leq x$ for any $x < \omega$ and from the result of the preceding paragraph it follows by induction on x that for all $x < \omega$, $D(x)$ is defined. Thus $\mathrm{dom}(D) = \omega$. Also, observe that since D_σ is one–one for all $\sigma < \alpha$, D too is one–one.

Proposition 4.3. $C = B - \mathrm{rng}(D)$ *and* $B - C$ *has order type* ω.

Proof. First we show that for any β in B there is a σ such that for all $\tau > \sigma$

$$\beta \cap (B_{\tau+1} - (\mathrm{rng}(D_\tau) \cup C_{\tau+1})) = \phi \ . \tag{5}$$

Assume that this is false and let β^* be the least β for which it fails; let $\beta^\#$ be the least β in B such that $\beta^\# < \beta^*$ and such that for all σ there is a

$\tau > \sigma$ such that

$$\beta^{\#} \in (\beta^* \cap (B_{\tau+1} - (\mathrm{rng}(D_\tau) \cup C_{\tau+1}))) .$$

Let σ' be such that for all $\tau > \sigma'$, b_τ is greater than β^*. That σ' exists follows from the assumptions on B in the statement of the theorem. Now for any $\tau' > \sigma'$ such that

$$\beta^{\#} \in (\beta^* \cap (B_{\tau'+1} - (\mathrm{rng}(D_{\tau'}) \cup C_{\tau'+1}))) ,$$

there is a $\sigma'' > \tau'$ such that $\beta^{\#}$ is $b_\sigma{}^*$ for all $\tau' < \sigma < \sigma''$ and $\beta^{\#}$ is $D_{\sigma''}(x)$ for some $x < \omega$. Thus, for all $\tau > \sigma'$ there is a $\mu > \tau$ such that $D_\mu(x) = \beta^{\#}$ for some $x < \omega$. From our minimality assumption on $\beta^{\#}$ it follows that if $\mu' > \mu > \sigma'$ such that $D_\mu(x) = \beta^{\#}$ and $D_{\mu'}(y) = \beta^{\#}$ then $y \leq x$, and if there is a τ, $\mu < \tau < \mu'$, such that $D_\tau(w) \neq \beta^{\#}$ for any $w < \omega$ then $y = x$. Therefore, we have deduced that for some $x < \omega$, $\beta^{\#} = D_\mu(x)$ for all sufficiently large μ. But this contradicts our assumptions about $\beta^{\#}$; therefore, for any β in B there is a σ such that for all $\tau > \sigma$ (5) holds.

It follows directly from the result of the preceding paragraph that rng(D) has order type ω. It also follows directly that $C \subseteq (B - \mathrm{rng}(D))$. Let b_σ be a member of $B - C$, and let σ' be such that for all $\tau \geq \sigma'$, $b_\tau > b_\sigma$ and (5) holds with $\beta = b_{\sigma'}$. Thus there exist $x(\sigma'), x(\sigma'+1), \ldots$ such that for all $\mu > \sigma'$, $D_\mu(x(\mu)) = b_\sigma$. By inspection of Step σ of the construction we have that $x(\sigma') \geq x(\sigma'+1) \geq \ldots$. Hence b_σ is in rng(D). This completes the proof of the proposition.

Consider a fixed natural number e. Let

$$G = \{i\colon i \leq e \;\&\; X_i \cup B \supseteq \mathrm{lub}(B)\}.$$

Let $w = \sum\{2^{e-i}\colon i \in G\}$. Then for each i in G we have $\lim_\sigma F(i, \sigma) = \mathrm{lub}(B)$, and for each $i \leq e$ in G' we have $\lim_\sigma F(i, \sigma) < \mathrm{lub}(B)$. Let H be the α-r.e. set

$$\{\mu\colon (\mathrm{E}\sigma)(\mu \in B_\sigma \;\&\; (\mathrm{A}i)(i \in G \to (\mu \in X_{i,\sigma} \;\&\; \mu < F(i, \sigma))))\}.$$

Clearly lub(B) $- B$ is contained in H, and since B is not α-recursive,

$\text{lub}(B) = \text{lub}(B \cap H)$. Thus the set

$$B \cap H \cap \{\mu\colon (\text{A}i)\,(i \in G \;\&\; i \leq e \to \mu \geq \lim\nolimits_\sigma F(i, \sigma))\}$$

also has lub = lub(B). We conclude that for each σ there is a $\tau \geq \sigma$, such that $W(b_\tau, e) = w$. We also observe that for an α-finite set of σ, $W(b_\sigma, e) > w$.

Proposition 4.4. *For infinitely many* $x < \omega$, $W(D(x), e) = w$.

Proof. From our last remark it is sufficient to show that $W(D(x), e) \geq w$ for infinitely many x. Suppose to the contrary that $W(D(x), e) < w$ for all $x \geq y$, where y is fixed $\geq e$. Then there exists σ such that $D_\mu(x) = D(x)$ for all $x < y$ and $\mu \geq \sigma$. Hence we can choose σ and z such that $W(b_\sigma, e) = w$, $D_\sigma(z) = b_\sigma$, and $z \geq y$. It is easily shown by induction that for all $\mu \geq \sigma$

$$(\text{E}v)\,(y \leq v \leq z \;\&\; W(D_\mu(v), e) \geq w)\,.$$

Thus $W(D(v), e) \geq w$ for some v, $y \leq v \leq z$. This is a contradiction and the proposition is proved.

Proposition 4.5. *For each* $i < \omega$ *there exists* $\nu(i)$ *such that for each* $\nu > \nu(i)$ *and all sufficiently large* σ, ν *is not protected by* i *at step* σ.

Proof. We may suppose $S_i \supseteq R_i \cap (B - C)$ for otherwise there exists x such that $D_\sigma(x)$ is in $R_{i,\sigma} - S_{i,\sigma}$, for all sufficiently large σ, whence only a finite number of elements are ever enumerated in T_i. Supposing that $S_i \supseteq R_i \cap (B - C)$, only a ν in $B \cap R_i$ can ever be protected by i, thus eventually any such ν will either be in $S_{i,\sigma}$ or C_σ, and thereafter it can never be protected by i.

Proposition 4.6. $W(D(x), e) = w$ *for all sufficiently large* x.

Proof. If not, choose $x \geq e$ such that

$$W(D(x), e) < w,\; D(x) > \text{lub}\{\nu(i)\colon i < e\}\,,$$

and choose $y > x$ such that $W(D(y), e) = w$. From Case 1, at all sufficiently large steps σ we have $D_\sigma(z) \neq \lim_{\tau<\sigma} D_\tau(z)$ for some $z \leq x$, which is a contradiction.

Proposition 4.7. *If $S_e \supseteq R_e \cap (B - C)$, then $S_e \cup (R_e - B)$ is α-r.e.*

Proof. By the separation property there exist disjoint α-recursive sets B^*, H^* included in B, H respectively such that $B^* \cup H^* = \text{lub}(B)$ and such that B^* and H^* are regular if B is regular. It is clearly sufficient to show that, if we assume the hypothesis of the proposition, then

$$(S_e \cup (R_e - B)) \cap H^* = (S_e \cup T^*) \cap H^*$$

for a set T^* which differs only α-finitely from T_e. The set of steps at which case 2 occurs must be unbounded, otherwise the domain of D would be finite. Hence T_e certainly includes $R_e - B$. Consider a step σ at which a member $\beta = \lim_{\tau<\sigma} D_\tau(x)$ of H^* is enumerated in C, then $W(\beta, e) \geq w$. Thus either z in Case 1 is $> e$, in which case we say β enters C in the *first way*, or $z \leq e$ and $W(\beta, e) > w$ in which case we say β enters C in the *second way*.

Let $T^* = T_e - \{\beta\colon \beta \text{ enters } C \text{ in the second way}\}$. Then $T^* \supseteq R_e - B$ and $R_e \subseteq T^*$. If ν is in $T^* - (R_e - B)$ then ν is in B and either ν is not in C or ν enters C in the first way; if ν is not in C then ν is in $R_e \cap (B - C)$, hence ν is in S_e; if ν enters C in the first way then ν was not protected by e when it entered C, and since ν cannot be enumerated in T_e after it enters C, ν must be in S_e. Therefore,

$$(S_e \cup T^*) \cap H^* = (S_e \cup (R_e - B)) \cap H^*.$$

$T_e - T^*$ is certainly an α-recursive set. There is only an α-finite set of members β of B satisfying $W(\beta, e) > w$; therefore, this set has an upper bound $\xi < \text{lub}(B)$. Since rng(D) has order type ω, there are only finitely many x such that $D(x) < \xi$. Therefore, there is a step σ in the construction after which no member of B can enter C in the second way. We conclude that the set of members of B which enter C in the second way is α-finite. Thus $T_e - T^*$ is α-finite, and the proposition is proved.

From the preceding proposition, P_{2e+1} is certainly met. From proposition 4.6, $W(D(x), e) = w$ for all sufficiently large x whence $D(x)$ is in X_e if $B \cup X_e \supseteq \mathrm{lub}(B)$. From Proposition 4.3 it follows that $\mathrm{lub}(B) - (C \cup X_e)$ is finite if $B \cup X_e \supseteq \mathrm{lub}(B)$, which proves that P_{2e} is also met. Finally, that $\mathrm{lub}(B) = \mathrm{lub}(B - C)$ follows from Propositions 4.2 and 4.3, and that C is regular if B is follows from Proposition 4.3. This completes the proof of the theorem.

The necessary condition mentioned at the beginning of this section will be deduced from the following:

Theorem 4.8. *Let* (L, A) *be a finite separated d-lattice. There exists a sub-d-lattice* (L^*, A^*) *of* $(R_\alpha, A(R_\alpha))$ *in which all the atoms are unbounded and regular, and there exists a sub-d-lattice* (L^*, A^*) *of* $(Q_\alpha, A(Q_\alpha))$ *in which all the atoms are infinite, such that each* (L^*, A^*) *is isomorphic to* (L, A) *and satisfies the following four conditions:*

(A1) *Let* A *be an outermost atom in* (L^*, A^*) *which is not innermost, then* A' *is maximal.*

(A2) *Let* A_1, A_2 *be atoms in* (L^*, A^*) *such that* A_1 *is just within* A_2 *and such that* A_1 *is not innermost, then for any α-r.e. set* R *such that* $R \supseteq A_2$, $A_1 - R$ *is finite.*

(A3) *Let* A_1, A_2 *be atoms in* (L^*, A^*) *such that* A_1 *is just within* A_2, *then for any α-r.e. set* R, $\mathrm{lub}(R \cap A_2) = \mathrm{lub}(A_2)$ *implies* $\mathrm{lub}(R \cap A_1) = \mathrm{lub}(A_2)$.

(A4) *Let* A *be an atom of* (L^*, A^*) *and let* B *be the least canonical generator containing* A; *if* R, S *are α-r.e. sets such that* $S \supseteq R \cap A$, *then* $S \cup (R - B)$ *is α-r.e.*

Proof. To prove this theorem we use a refinement of the construction used in the proof of Theorem 2.6. Let $m_1, \ldots, m_j$ be the maximal canonical generators of (L, A), and let $N_1, \ldots, N_j$ be pairwise disjoint α-recursive sets whose union is α or ω respectively, and which are regular and unbounded when the union is α. We map m_x to N_x for $1 \leq x \leq j$. The definition of the map of the canonical generators of (L, A) now proceeds downwards with respect to the inclusion of the lattice. Let b be a canonical generator not yet mapped such that all those which properly contain it have already been mapped. Let c be the least canonical generator which properly contains b, and let $n_1, ..., n_k$ be the canonical generators which are maximal with respect to the property of being properly contained in c; among them must occur b. Assume for induction that none of $n_1, ..., n_k$ has already been mapped. Let C be the image of c, and choose an element C of **R** as follows:

Case 1. If c is a maximal canonical generator, let B be any α-r.e. subset of C which is maximal in C and which is regular and unbounded if C is.

Case 2. Otherwise, let d be the least canonical generator which properly contains c and let D be the image of d. Let B be an α-r.e. subset of C which has the following properties:

(i) $\mathrm{lub}(C) = \mathrm{lub}(C - B)$;

(ii) for every α-r.e. subset R of D such that $C \cup R = D$, $D - (B \cup R)$ is finite;

(iii) for any α-r.e. subsets R, S of D such that $S \supseteq R \cap (C - B)$, $S \cup (R - C)$ is α-r.e.;

and such that B is regular and unbounded if C is. Such a B can be found by facts (i) and (v) above provided C is not α-recursive in D, which we assume as part of the induction hypothesis.

Having chosen B we decompose it into pairwise disjoint α-r.e. sets $B_1, ..., B_k$ such that each B_i is regular and unbounded if B is and such that for any α-r.e. subset R of C, $R - B$ not α-r.e. implies $R - B_i$ not α-r.e. for each i. We let the images of $n_1, ..., n_k$ be $B_1, ..., B_k$ respectively. It is clear from the proof of Theorem 2.6 that this gives a well-defined mapping which is an isomorphism of (L, A) onto a sub-d-lattice (L^*, A^*) of $(\mathbf{R}, A(\mathbf{R}))$ because this construction is a refinement of that given in the proof of Theorem 2.6. Also, (A1) is clearly satisfied through Case 1. To prove (A2) consider atoms A_1, A_2 of (L^*, A^*) such that A_1 is just within A_2 and such that A_1 is not innermost. By the correspondence between atoms and canonical generators there are canonical generators B_1, B_2 in L^* and members C_1, C_2 of L^* such that $C_1 \subseteq B_1 \subseteq C_2 \subseteq B_2$, $A_1 = B_1 - C_1$, $A_2 = B_2 - C_2$, and such that $C_2 - B_1$ is α-r.e. and regular if C_2 is regular. Let R be any α-r.e. set $\supseteq A_2$ then $R \cup (C_2 - B_1)$ is an α-r.e. set which is complementary to B_1 in B_2. By Case 2 we have C_1 a major subset of B_1 in B_2 whence $(B_1 - C_1) - (R \cup (C_2 - B_1))$ is finite, whence $A_1 - R$ is finite. Thus (A2) is satisfied.

With the same notation let R now be an α-r.e. set such that lub$(R \cap A_2) =$ lub(A_1). Then $(R \cap B_2) - C_2 = R \cap A_2$ is not α-r.e. since A_2 is clearly immune, and the construction of B_1 makes $(R \cap B_2) - B_1$ not α-r.e. also. If A_1 is innermost, i.e. if $A_1 = B_1$, it follows that lub$(R \cap B_1) =$ lub(A_2), since we can replace R by the portion of R above a level arbitrarily high below lub(A_2) and still have the preceding sentence hold. Otherwise, then C_1 is constructed by facts (i) and (v). Note that from the construction it follows that any atom which is not innermost has order type ω. If lub$(R \cap A_1) \neq$ lub(A_2) then $R \cap A_1$ is finite, hence α-r.e., whence $(R \cap A_1) \cup ((R \cap B_2) - B_1)$ is α-r.e. (from peroperty (c) in fact (v)), which is impossible. Thus in any case lub$(R \cap A_1) =$ lub(A_2), and (A3) is proved.

Let A be an atom of (L^*, A^*) and let B be the least canonical generator containing A. Let C be the union of the canonical generators properly contained in B; then $A = B - C$. Let R, S be α-r.e. sets such that $S \supseteq R \cap A$. We wish to prove that $S \cup (R - B)$ is α-r.e. This is trivial if A is either innermost or outermost, thus suppose A is neither innermost nor outermost, and without loss of generality, suppose $S \subseteq B$. Let $B = B_1, \ldots, B_p$ be a maximal increasing sequence of canonical generators. Then B_p is α-recursive. Since C is constructed by (i) and (v) in Case 2 we have that

$$S \cup ((R - B_1) \cap B_2)$$

is α-r.e. Now

$$S \cup ((R - B_1) \cap B_2) \supseteq R \cap (B_2 - C_2)$$

where C_2 is the union of the canonical generators properly included in B_2. Thus we may repeat the argument if $p > 2$ to see that

$$(S \cup ((R - B_1) \cap B_2)) \cup ((R - B_2) \cap B_3) = S \cup ((R - B_1) \cap B_3)$$

is α-r.e., and so on. Thus $S \cup ((R - B_1) \cap B_p)$ is α-r.e., whence $S \cup (R - B)$ is α-r.e. This completes the proof of (A4) and of the theorem.

Consider now the separated AE-sentence

$$(\mathrm{Ax})(\mathrm{Ey})(D(\mathrm{x}) \to P(\mathrm{x},\mathrm{y})) .$$

Let (L, A) be the d-lattice of the separated diagram D. From Theorem 4.8 we can find a sub-d-lattice (L^*, A^*) of $(R_\alpha, A(R_\alpha))$ or $(Q_\alpha, A(Q_\alpha))$ in which all the atoms are unbounded and regular or infinite, respectively, and which is isomorphic to (L, A) and satisfies (A1)–(A4). Now x is $x_1, ..., x_m$, and we let $B_1, ..., B_m$ be the images in L^* of the elements represented by $x_1, ..., x_m$ respectively in L. Clearly, $D(\{B_1\}, ..., \{B_m\})$ holds in $(\mathrm{R}, A(\mathrm{R}))$. Suppose that the separated AE-sentence we are considering holds in $(\mathrm{R}, A(\mathrm{R}))$, then there exist α-r.e. sets $C_1, ..., C_n$ such that $P(\{B_1\}, ..., \{B_m\}, \{C_1\}, ..., \{C_n\})$ holds in $(\mathrm{R}, A(\mathrm{R}))$. Let Q be the unique diagram containing just the variables x, y such that $Q(\{B_1\}, ..., \{B_m\}, \{C_1\}, ..., \{C_n\})$ holds in $(\mathrm{R}, A(\mathrm{R}))$. Then certainly Q implies P. Let (L_1, A_1) be the d-lattice of Q, then (L, A) is a sub-d-lattice of (L_1, A_1). From (A1)–(A4) we can deduce the following:

(B1) An outermost atom of (L, A) which is not innermost is an atom of (L_1, A_1).

(B2) If a_1 is an atom of (L_1, A_1) contained in an atom a of (L, A) such that a is just within b and not innermost, then there is an atom b_1 of (L_1, A_1) contained in b such that $a_1 \preccurlyeq b_1$.

(B3) If b_1 is an atom of (L_1, A_1) contained in an atom b of (L, A) and if a is just within b, then there exists an atom a_1 of (L_1, A_1) contained in a such that $a_1 \preccurlyeq b_1$.

(B4) If a_1, b_1 are atoms of (L_1, A_1) contained in the respective atoms a, b of (L, A), and if a_1 is just within b_1, then $a = b$ or a is just within b.

We have (B1) directly from (A1). Suppose the hypothesis of (B2). For each b_1 contained in b let t_1 be the least member of L_1 containing b_1. From (A2) the union of all the t_1 covers a since it covers b, whence for some b_1 we have $a_1 \preccurlyeq b_1$. This proves (B2). Suppose the hypothesis of (B3), and let t_1 be the least element of L_1 containing b_1. From (A3), $t_1 \cap a \neq \emptyset$, and so $a_1 \preccurlyeq b_1$ for some a_1 contained in a. This proves (B3). Finally, suppose the hypothesis of (B4) and that $a \neq b$. Since L is a sublattice of L_1, $a \preccurlyeq b$. Let c be the atom of (L, A) such that a is just within c, and let t_1 be the least element of L_1 containing b_1. For each atom c_1

of (L_1, A_1) contained in $t_1 \cap c$ let s_1 be the least element of L_1 containing c_1. Then the union s_1^* of all the s_1 is $\geq t_1 \cap c$. From (A4), $s_1^* \cup (t_1 - u)$ is in L_1 where u is the least element of L containing c. Hence $t_1 \leq s_1^* \cup (t_1 - u)$, whence $t_1 \cap u \leq s_1^*$. Thus a_1 is contained in some s_1, and so there exists c_1 contained in c such that $a_1 \preccurlyeq c_1 \preccurlyeq b_1$. This completes the proof of (B4).

Thus for $(\mathrm{Ax})(\mathrm{Ey})(D(\mathrm{x}) \to P(\mathrm{x}, \mathrm{y}))$ to be true in $(\mathbf{R}, A(\mathbf{R}))$ where D is a separated diagram *it is necessary that there exists a diagram* Q *containing just the variables* x, y *such that* $Q(\mathrm{x}, \mathrm{y})$ *implies* $D(\mathrm{x})$ *and* $P(\mathrm{x}, \mathrm{y})$ *and such that* (B1)–(B4) *hold.*

§5. The Decision Procedure for AE-sentences

Let x, y stand for $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ respectively. Call the separated sentence $(\mathrm{Ax})(\mathrm{Ey})(D(\mathrm{x}) \to P(\mathrm{x}, \mathrm{y}))$ *potentially true* in $(\mathbf{R}, A(\mathbf{R}))$ if it satisfies the condition shown to be necessary in the last section. In the statement of the following theorem "holds" means "holds in $(\mathbf{R}, A(\mathbf{R}))$", and $\{\mathbf{B}\}$, $\{\mathbf{C}\}$ abbreviate $\{B_1\}, \ldots, \{B_m\}$ and $\{C_1\}, \ldots, \{C_n\}$ respectively.

Theorem 5.1. *Let* $(\mathrm{Ax})(\mathrm{Ey})(D(\mathrm{x}) \to P(\mathrm{x}, \mathrm{y}))$ *be a separated sentence which is potentially true in* $(\mathbf{R}, A(\mathbf{R}))$. *Let* $Q(\mathrm{x}, \mathrm{y})$ *be a diagram which implies* $D(\mathrm{x})$ *and* $P(\mathrm{x}, \mathrm{y})$ *and such that* (B1)–(B4) *are true when* (L, A), (L_1, A_1) *are the d-lattices of* D, Q *respectively. For any m-tuple* $\mathbf{B}$ *such that* $D(\{\mathbf{B}\})$ *holds either there exists an n-tuple* $\mathbf{C}$ *such that* $Q(\{\mathbf{B}\}, \{\mathbf{C}\})$ *holds or there exists an* α*-recursive set* C *and a diagram* $D^*(\mathrm{x}, y)$ *of characteristic less than that of* D *such that* $D^*(\{\mathbf{B}\}, \{C\})$ *holds.*

Proof. Suppose the hypothesis of the theorem holds. We attempt to construct an n-tuple $\mathbf{C}$ such that $Q(\{\mathbf{B}\}, \{\mathbf{C}\})$ holds. We may suppose without loss of generality that the m-tuple $\mathbf{B}$ generates a sub-d-lattice (L^*, A^*) of $(R_\alpha, A(R_\alpha))$ or $(Q_\alpha, A(Q_\alpha))$ which is isomorphic to (L, A) and which has only infinite atoms or unbounded atoms whichever is appropriate. For $\mathbf{C}$ to make $Q(\{\mathbf{B}\}, \{\mathbf{C}\})$ hold in $(\mathbf{R}, A(\mathbf{R}))$ the following conditions are sufficient by Lemma 2.2:

(C1) For each atom a_1 in (L_1, A_1), $a_1(\mathbf{B}, \mathbf{C})$ is infinite or unbounded, whichever is appropriate; where for any term t containing only the variables x, y we let $t(\mathbf{B}, \mathbf{C})$ denote the interpretation of t when x, y are interpreted as **B**, **C** respectively.

(C2) For each pair of atoms a_1, b_1 of (L_1, A_1) such that a_1 is just within b_1 and for each α-r.e. set R, if $R \supseteq b_1(\mathbf{B}, \mathbf{C})$ then $R \cap a_1(\mathbf{B}, \mathbf{C}) \neq \phi$.

(C3) For each term t_1 which is equal to ϕ in (L_1, A_1), $t_1(\mathbf{B}, \mathbf{C}) = \phi$.

(C4) For each term t_1 in L_1, $t_1(\mathbf{B}, \mathbf{C})$ is α-r.e.

Our construction of the α-r.e. sets **C** attempts to meet (C1)–(C4). Before we state the construction we need the following:

Lemma 5.2. *There is a function C, defined on each pair (a_1, a) such that a is an atom of (L, A) and a_1 is an atom of (L_1, A_1) not contained in an innermost atom of (L, A), for which the following hold:*

(D1) *If a, b are atoms of (L, A) such that $a \preccurlyeq b$, then $C(a_1, a)$, $C(a_1, b)$ are atoms of (L_1, A_1) such that $C(a_1, a) \preccurlyeq C(a_1, b)$.*

(D2) *If a^* is an atom of (L, A) containing a_1, then $C(a_1, a^*) = a_1$.*

(D3) $C(a_1, a) \leq a$.

Proof. This is a purely algebraic lemma, therefore we omit its proof and refer the reader to Lachlan ([5], p. 141).

Before continuing with the proof of the theorem, we separate the cases for **R** equal to $R_\alpha{}^*$, $R_\alpha{}^\#$, and $Q_\alpha{}^*$. We shall deal first with the cases $\mathbf{R} = R_\alpha{}^*$ and $\mathbf{R} = Q_\alpha{}^*$. The construction we give will have the property that if all the sets **B** are subsets of ω then so will be **C** and C. The case of $\mathbf{R} = R_\alpha{}^\#$ will be dealt with later.

Before giving the construction we can dispose of some of the subconditions which go to make up (C1), (C2). For each innermost atom a of (L, A) we choose an infinite α-recursive subset W of $a(\Sigma)$. Then $(R_\alpha, A(R_\alpha))|W$ will be isomorphic to $(R_\alpha, A(R_\alpha))$ or $(Q_\alpha, A(Q_\alpha))$ depending on whether W is unbounded or bounded. By Theorem 2.6 and Corollary 2.7 we can choose the intersections of the sets **C** with W so that the sub-d-lattice of $(R_\alpha, A(R_\alpha))|W$ generated by them has no finite atoms and is isomorphic to (L_1, A_1) under the map which takes **B**, **C**, W

into x, y, a respectively. This is enough to satisfy (C1) when a_1 is contained in an innermost atom of (L, A) and (C2) when both a_1 and b_1 are contained in an innermost atom of (L, A). At the same time (C3) and (C4) are satisfied as regards the intersections of the sets C with W.

The rest of (C1) can be split into conditions $(a_1 ; i)$ where a_1 runs through all the atoms of A_1 not contained in innermost atoms of (L, A) and i runs through all natural numbers. This condition requires that the cardinality of $a_1(\mathbf{B}, \mathbf{C})$ be at least i. The rest of (C2) can be split into conditions $(a_1, b_1 ; i)$ where (a_1, b_1) runs through all pairs of atoms of A_1 such that a_1 is just within b_1 and such that b_1 is not contained in an innermost atom of (L, A), and where i runs through all natural numbers. This condition requires that if X_i is the i-th α-r.e. set then $X_i \supseteq b_1(\mathbf{B}, \mathbf{C})$ implies $X_i \cap a_1(\mathbf{B}, \mathbf{C}) \neq \phi$. We arrange all the conditions $(a_1 ; i)$, $(a_1, b_1 ; i)$ in an effective one–one correspondence with the natural numbers. When we speak of *the k-th condition* we mean the one corresponding to the natural number k.

Let T be the union of all the α-recursive sets W as a runs through the innermost atoms of (L, A). Just as in the proof of Lemma 3.3 let $A^1, A^2, \ldots, A^p$ be all the atoms of (L^*, A^*) and let $a^1, a^2, \ldots, a^p$ be the atoms of (L, A) so that $A^i = a^i(\mathbf{B})$, and let $S^1, S^2, \ldots, S^p$ be respective corresponding boxes. At each step σ of the construction we may place one member of T' in an outermost box, or we may move an element from one box to another just within it. (We carry over the partial ordering $\preccurlyeq$ from atoms to boxes via the correspondence.) From the proof of Lemma 3.3 this can be done so that for $1 \leq i \leq p$ any member of $T' \cap A^i$ eventually reaches S^i and remains there. Our procedure for this will be that of Lemma 3.3 except that we may temporarily restrain some elements from leaving outermost boxes. These restraints will be made explicit below.

We let $S_1{}^1, S_1{}^2, \ldots, S_1{}^q$ be boxes corresponding respectively to the atoms $a_1{}^1, a_1{}^2, \ldots, a_1{}^q$ of (L_1, A_1). At each step, if an element is placed in an S-box or moved from one S-box to another it will also be placed in an S_1-box or moved from one S_1-box to another respectively. This will be done so that at any stage, if ξ is in a box S corresponding to an atom a of A, then at the same stage ξ is in just one S_1-box and that S_1-box corresponds to an atom a_1 of A_1 which is $\leq a$. Simultaneously with the rest of the construction we shall be enumerating all the α-r.e. sets X_i; we let $X_{i,\sigma}$ denote the set of elements which have already been enumerated in X_i at the beginning of step σ. If Y denotes a box, then $Y(\sigma)$ denotes the set of elements which are in Y at the beginning of step σ.

The auxiliary functions f, G are defined as follows. If the k-th condition is $(a_1; i)$ where a_1 corresponds to the box S_1, then for all elements χ let $f(k, \chi)$ be the least σ such that $\chi \in S_1(\sigma)$ if one exists and undefined otherwise: let $G(k, \tau)$ be the i-th member of $\{f(k,\chi): \chi \in S_1(\tau)\}$ in order of magnitude if one exists and let it be τ otherwise. If the k-th condition is $(a_1, b_1; i)$ where a_1, b_1 correspond to boxes R_1, S_1 respectively, then for all elements χ let $f(k, \chi)$ be the least σ such that

$$\chi \in (R_1(\sigma) \cap X_{i,\sigma}) \cup (S_1(\sigma) \cap X_{i,\sigma}')$$

if one exists and undefined otherwise; let $G(k, \tau)$ be the least member of

$$\{f(k, \chi): \chi \in (R_1(\tau) \cap X_{i,\tau}) \cup (S_1(\tau) \cap X_{i,\tau}')\}$$

if one exists and let it be τ otherwise. Since G is not quite convenient for the construction, we define F from it by double induction; but first let g be any α-recursive function such that $g(k, \sigma)$ is increasing in σ for k fixed, strictly increasing in k for σ fixed, $\lim_\sigma g(k, \sigma) = g(k)$ exists for all natural numbers k, and such that $\lim_k g(k) = \alpha$:

$$F(0, 0) = G(0, 0)\,,$$

$$F(k+1, 0) = \max\{F(k, 0)+1,\ G(k+1, 0)\}\,;$$

$$F(0, \sigma) = \max\{\mathrm{lub}(\{F(0,\tau)\}_{\tau<\sigma}),\ G(0, \sigma)\}\,,\ \sigma > 0\,;$$

$$F(k+1, \sigma) = \max\{F(k, \sigma)+1,\ G(k+1, \sigma),\ g(k+1, \sigma),$$
$$\mathrm{lub}(\{F(k+1, \tau)\}_{\tau<\sigma})\}\,,\ \sigma > 0\,.$$

It is also useful to make the convention that $F(-1, \sigma) = -1$. Now $F(k, \sigma)$ is increasing in σ for k fixed, strictly increasing in k for σ fixed, and $\geq G(k, \sigma)$. Further, $\mathrm{lub}(\{F(k, \sigma)\}_{\sigma<\alpha}) < \alpha$ if and only if $\mathrm{lub}(\{G(y, \sigma)\}_{\sigma<\alpha}) < \alpha$

for all $y \leq k$, and for all τ there exist k, σ such that $F(k, \sigma) > \tau$. Finally, let $H(\xi, \sigma)$ equal the least number k such that $\xi \leq F(k, \sigma)$ if such a number exists and let it equal -1 otherwise.

For any element ξ let $h(\xi)$ be the first step σ such that ξ is enumerated in some set of L^* corresponding to an atom of A^* (see proof of Lemma 3.3). If at step σ the procedure in the proof of Lemma 3.3 would move the element ξ from S^i to S^j where S^i is outermost and $H(h(\xi), \sigma) = -1$ then in our present procedure we do nothing and go to step $\sigma + 1$. This is the restraint mentioned above, and is the only change in the procedure of Lemma 3.3. If at step σ an element ξ is moved from S^i to S^j where S^i is outermost, then at that step we *assign* ξ to the $H(h(\xi), \sigma)$-th condition and ξ remains assigned to the $H(h(\xi), \sigma)$-th condition for the rest of the construction.

A condition $(a_1 ; i)$ is said to be *of the first kind*; a condition $(a_1, b_1 ; i)$ is said to be *of the second kind* if a_1, b_1 are contained in the same atom of A, and to be *of the third kind* otherwise.

If at step σ of the construction an element is put in some S-box or is moved from one S-box to another, the remainder of the σ-th step of the construction consists of the following two parts:

Part 1. Let k be the least number, if any, such that the k-th condition is of the second kind, $(a_1, b_1 ; i)$ say, and such that there exists ν in $X_{i,\sigma}$ where ν has been assigned to the k-th condition and where ν is currently in the box of b_1. If the k exists, move the least such ν from the box of b_1 into the box of a_1.

Part 2. *Case 1.* At step σ suppose that ξ is placed in the outermost box S corresponding to the atom a of A. Place ξ in the outermost box S_1 corresponding to the atom a_1 of A_1 which is equal to a in A_1. If Case 1 does not hold, then at step σ some ξ is moved from box S^p to S^q and ξ is now or already has been assigned to some condition.

Case 2. ξ is or has been assigned to a condition $(a_1 ; i)$ of the first kind. If ξ is in the box of $C(a_1, a^p)$, we move it from that one to the box of $C(a_1, a^q)$. Otherwise, ξ is in a box $S_1{}^k$ say, then we move ξ from $S_1{}^k$ into the box of $C(a_1{}^k, a^q)$.

Case 3. ξ is or has been assigned to a condition $(a_1, b_1 ; i)$ of the second kind. If ξ is in the box of $C(b_1, a^p)$, we move it from that one to the box of $C(b_1, a^q)$. Otherwise, ξ is in a box $S_1{}^k$ say, then we move ξ from $S_1{}^k$ into the box of $C(a_1{}^k, a^q)$.

Case 4. ξ is or has been assigned to a condition $(a_1, b_1; i)$ of the third kind. If ξ is in the box of $C(b_1, a^p)$ and $C(b_1, a^p) \neq b_1$, then we move ξ from that box into the box of $C(b_1, a^q)$. If ξ is in the box of b_1 and a^q contains a_1, we move ξ from the box of b_1 into the box of a_1. Otherwise, proceed as in Case 3.

This completes the construction. For $1 \leq i \leq n$ define $C_i \cap T'$ to consist of all elements ξ such that there exists a box S_1, corresponding to an atom a_1 of A_1 which is $\leq y_i$, and a σ such that ξ is in $S_1(\sigma)$. This defines the sets **C** completely, because their intersections with T have already been chosen. It is clear that all the sets **C** are α-r.e. As stated above, if at any stage ξ is in box S corresponding to the atom a of A, then at the same stage ξ is in a box S_1 corresponding to an atom a_1 of A_1 such that a_1 is included in a. This follows from (D3). From (D1) we see that if ξ is in $S_1{}^i(\sigma)$ and ξ is in $S_1{}^j(\tau)$ and $\sigma < \tau$, then $a_1{}^j \preccurlyeq a_1{}^i$. It follows that for each atom a_1 of A_1, $a_1(\mathbf{B}, \mathbf{C}) \cap T'$ consists of just those elements which eventually come into S_1, the box corresponding to a_1, and remain there for the rest of the construction. From this we see that (C3) is satisfied, because every element in T' is in $a_1(\mathbf{B}, \mathbf{C})$ for some term a_1 which is $\neq \phi$ in A_1. It also follows that if t_1 is any term in L_1, then $t_1(\mathbf{B}, \mathbf{C})$ is α-r.e. For if a_1, b_1 are any atoms of A_1 such that $a_1 \preccurlyeq b_1$ then $b_1 \leq t_1$ implies $a_1 \leq t_1$, whence

$$t_1(\mathbf{B}, \mathbf{C}) \cap T' = \{\chi : (\mathrm{E}a_1)(\mathrm{E}s_1)(\mathrm{E}\sigma)\,(\text{box } S_1 \text{ corresponds to atom } a_1 \text{ of } A_1 \;\&\; \chi \text{ is in } S_1(\sigma) \;\&\; a_1 \leq t_1)\}.$$

The set on the right is clearly α-r.e. and so (C4) is satisfied.

There are two possibilities to be considered. First, suppose that $\mathrm{lub}(\{F(k, \sigma)\}_{\sigma<\alpha}) < \alpha$ for every k. If the k-th condition is $(a_1; i)$ then from the fact that $\mathrm{lub}(\{F(k, \sigma)\}_{\sigma<\alpha}) < \alpha$ it follows that $\mathrm{lub}(\{G(k, \sigma)\}_{\sigma<\alpha}) < \alpha$ and thus that at least i elements eventually reach S_1, the box corresponding to a_1, and remain there. Thus $a_1(\mathbf{B}, \mathbf{C}) \cap T'$ has cardinality $\geq i$. If the k-th condition is $(a_1, b_1; i)$, then we see that

$$\mathrm{lub}_\sigma(\min\{f(k, \chi) : \chi \in (R_1(\sigma) \cap X_{i,\sigma}) \cup (S_1(\sigma) \cap X_{i,\sigma}{}')\}) < \alpha$$

where R_1, S_1 are the boxes corresponding to a_1, b_1 respectively. Therefore

$$(a_1(\mathbf{B},\mathbf{C}) \cap X_i) \cup (b_1(\mathbf{B},\mathbf{C}) \cap X_i') \neq \phi .$$

In either event, the k-th condition is satisfied, and therefore (C1)–(C4) are satisfied.

The other possibility is that $\text{lub}(\{F(k,\sigma)\}_{\sigma<\alpha}) = \alpha$ for some k. Choose the least such k and choose σ_0 such that

$$(\mathrm{A}\sigma)(\mathrm{A}t)((\sigma < \sigma_0 \;\&\; t < k) \rightarrow F(t,\sigma) = F(t,\sigma_0)) .$$

Now the k-th condition is concerned with a particular indecomposable component of (L, A) in the sense that there is a unique term t_k containing just the variables x such that t_k, t_k' are both in L, $t_k \neq \phi$ in A, such that no t which is $< t_k$ in L also has these properties, and such that in A_1 the atom(s) of the k-th condition is (are) contained in t_k. We define C to be the union of $t_k'(\mathbf{B})$ and $\{\chi\colon \chi$ is assigned to some condition other than the k-th$\}$, then C is clearly α-r.e. Also at step $\sigma > \sigma_0$ if

$$F(k-1,\sigma_0) < h(\xi) \leq F(k,\sigma)$$

then we know that subsequently ξ cannot be assigned to any condition but the k-th. As χ increases $F(k,\chi)$ increases without bound (below α). Thus we can α-effectively enumerate C' as well as C. Therefore C is α-recursive.

Suppose that the k-th condition is $(a_1\,; i)$. Let a^* be the atom of A which contains a_1. Consider ξ in $C' \cap a^*(\mathbf{B})$; we suppose for the sake of argument that such ξ exists. Then in the construction ξ follows a path $a^{i_1}, ..., a^{i_p} = a^*$ in (L, A). From Cases 1, 2 of step σ we see that in the S_1-system ξ occupies in turn the boxes corresponding to the respective atoms $C(a_1, a^{i_1}), ..., C(a_1, a^{i_p}) = a_1$ and that ξ remains in the last of these boxes indefinitely. This is because if $p > 1$, then ξ is assigned to the k-th condition when it is moved from S^{i_1} to S^{i_2}. It follows at once that $C' \cap a^*(\mathbf{B})$ is finite; otherwise we should have $a_1(\mathbf{B},\mathbf{C})$ infinite whence $G(k,\sigma)$ would be bounded as σ increases, whence $\text{lub}(\{F(k,\sigma)\}_{\sigma<\alpha}) < \alpha$.

Suppose the k-th condition is of the second kind, $(a_1, b_1; i)$ say, where a_1, b_1 are contained in the same atom a^* of A. Consider the least ξ in $C' \cap a^*(\mathbf{B})$ as before; then again ξ follows a path $a^{i_1}, ..., a^{i_p} = a^*$ in (L, A). From Cases 1 and 3 in step σ we see that in this case ξ occupies in turn the boxes of $C(b_1, a^{i_1}), ..., C(b_1, a^{i_p}) = b_1$. But ξ does not necessarily remain indefinitely in the box of b_1. From Part 1 of step σ we see that in fact ξ remains in the box of b_1 just if ξ is in X_i'; if ξ is in X_i then ξ eventually gets moved from the box of b_1 into the box of a_1 where it remains indefinitely. Thus if any such ξ exists $G(k, \sigma)$ is bounded as σ increases, whence $\mathrm{lub}(\{F(k, \sigma)\}_{\sigma<\alpha}) < \alpha$. Thus from our hypothesis that k is the least number such that $\mathrm{lub}(\{F(k, \sigma)\}_{\sigma<\alpha}) = \alpha$ we again deduce that $C' \cap a^*(\mathbf{B})$ is finite.

Suppose finally that the k-th condition is $(a_1, b_1; i)$ and of the third kind. Let a_1, b_1 be contained in the respective atoms a^*, b^* of A. Consider ξ in $C' \cap b^*(\mathbf{B})$. From Cases 1 and 4 of step σ we see that while ξ follows a path $a^{i_1}, ..., a^{i_p} = b^*$ in (L, A), in the S_1-system ξ occupies in turn the boxes of $C(b_1, a^{i_1}) = a^{i_1}, ..., C(b_1, a^{i_p}) = b_1$ and ξ remains indefinitely in the last of these. Hence $C' \cap b^*(\mathbf{B})$ is the same as $C' \cap b_1(\mathbf{B}, \mathbf{C})$. If ξ is in $T' \cap C' \cap a^*(\mathbf{B})$, then ξ follows the path $a^{i_1}, ..., a^{i_p} = b^*, a^*$ in (L, A), and in the S_1-system ξ occupies in turn the boxes of $C(b_1, a^{i_1})$, ..., $C(b_1, a^{i_p}) = b_1, a_1$ and remains indefinitely in the last of these. Thus $T' \cap C' \cap a^*(\mathbf{B})$ is the same as $T' \cap C' \cap a_1(\mathbf{B}, \mathbf{C})$. Also, since $\mathrm{lub}(\{F(k, \sigma)\}_{\sigma<\alpha}) = \alpha$, we cannot have $\mathrm{lub}(\{G(k, \sigma)\}_{\sigma<\alpha}) < \alpha$ whence $X_i \cap b_1(\mathbf{B}, \mathbf{C})$ and $X_i \cap T' \cap a_1(\mathbf{B}, \mathbf{C}) = \phi$. It follows that $X_i \supseteq b^*(\mathbf{B}) \cap C'$ and that $X_i \cap T' \cap a^*(\mathbf{B}) = \phi$. Let u, v, w be terms in L such that $u \leq v \leq w$, $b^* = w - v$, and $a^* = v - u$. Then $C' \cap (b^*(\mathbf{B}) \cup u(\mathbf{B}))$ is α-r.e. since it can be expressed in the form

$$C' \cap ((X_i \cap T' \cap w(\mathbf{B})) \cup u(\mathbf{B})) .$$

At this point we need the following lemma:

Lemma 5.3. *Let (K, B), (K_1, B_1) be finite d-lattices. Let F be a one–one map of the atoms of B into the atoms of B_1 such that if $a \preccurlyeq b$ in (K, B), then $F(a) \preccurlyeq F(b)$ in (K_1, B_1). Then the characteristic of (K, B) is less than or equal to the characteristic of (K_1, B_1) and the characteristics are equal only if F is onto and induces an isomorphism of (K, B) onto (K_1, B_1).*

Proof. Since this is a purely algebraic proposition, we omit the proof and refer the reader to Lachlan ([5], p. 145).

Now suppose that C has been adjusted so that the sub-algebra of $(R_\alpha, A(R_\alpha))$ generated by $\mathbf{B}$, C has no finite atoms; this adjustment consists of adding one finite set to C and subtracting another from it. This saves taking equivalence classes modulo finite sets, because after the adjustment the sub-d-lattice of $(\mathbf{R}, A(\mathbf{R}))$ generated by $\{\mathbf{B}\}$, $\{C\}$ is isomorphic to the sub-d-lattice of $(R_\alpha, A(R_\alpha))$ generated by $\mathbf{B}$, C. We now compare the sub-d-lattice $D = (L^*, A^*)$ of $(R_\alpha, A(R_\alpha))$ generated by $\mathbf{B}$ with $D(C)$ generated by $\mathbf{B}$, C. Outside the indecomposable component $C_k(\mathbf{B})$ of D which is obtained by restricting to subsets of $t_k(\mathbf{B})$ the indecomposable components of D and $D(C)$ are exactly the same, because $C \supseteq t_k'(\mathbf{B})$ by the definition of C. Thus $D(C)$ differs from D in that the component $C_k(\mathbf{B})$ of D is replaced by the two components of $D(C)$ obtained by restricting to subsets of $C \cap t_k(\mathbf{B})$, $C' \cap t_k(\mathbf{B})$ respectively. Denote these components by $C(C)$, $C(C')$ respectively; they may or may not be indecomposable.

Define a map of the atoms of $C(C)$ into the atoms of $C_k(\mathbf{B})$ by mapping each atom of $C(C)$ to the unique atom of $C_k(\mathbf{B})$ which contains it. This map clearly preserves the relation $\preccurlyeq$, and so Lemma 5.3 may be applied. An outermost atom of $C_k(\mathbf{B})$ is an outermost atom of D and any element of an outermost atom of D does not get assigned to any condition; therefore, any element of an outermost atom of $C_k(\mathbf{B})$ is in C'. Thus an outermost atom of $C_k(\mathbf{B})$ is not the image of any atom of $C(C)$, whence the characteristic of $C(C)$ is less than that of $C_k(\mathbf{B})$.

Define a map F of the atoms of $C(C')$ into the atoms of $C_k(\mathbf{B})$ by mapping each atom of $C(C')$ to the unique atom of $C_k(\mathbf{B})$ which contains it. We may apply Lemma 5.3. If the k-th condition is of the first or second kinds we know that $C' \cap a^*(\mathbf{B})$ is finite for some atom a^* of A, whence F is not onto in this case. If the k-th condition is of the third kind $(a_1, b_1 : i)$ say, where a_1, b_1 are contained in the respective atoms a^*, b^* of A, then we know that $C' \cap (b^*(\mathbf{B}) \cup u(\mathbf{B}))$ is α-r.e. Because this set contains $C' \cap b^*(\mathbf{B})$ while excluding $a^*(\mathbf{B})$, we cannot have $C' \cap a^*(\mathbf{B}) \preccurlyeq C' \cap b^*(\mathbf{B})$ in $C(C')$. Thus although F can be one–one onto in this case, it cannot induce an isomorphism of $C(C')$ onto $C_k(\mathbf{B})$. In any event, $C(C')$ has characteristic less than that of $C_k(\mathbf{B})$.

Let r be the rank of $C_k(\mathbf{B})$; then the characteristic of $C_k(\mathbf{B})$ is $\{r\}$. Let $\{r_1, ..., r_j\}$ be the characteristic of $C(C) \times C(C')$. It follows that $r_i < r$ for $1 \leq i \leq j$, and from that it follows that the characteristic of $D(C)$ is less than that of D. Letting $D^*(\mathrm{x}, y)$ be the diagram such that $D^*(\mathbf{B}, C)$ is true in $(R_\alpha, A(R_\alpha))$, the proof of the theorem is complete for the cases where $\mathbf{R}$ is R_α^* or Q_α^*.

We now show how to modify the preceding proof to handle the case of $\mathbf{R} = R_\alpha^{\#}$. We begin by having the α-recursive subset of $a(\mathbf{B})$ for each innermost atom a of (L, A) be unbounded instead of merely infinite. Then $(R_\alpha, A(R_\alpha)) \mid W$ will be isomorphic to $(R_\alpha, A(R_\alpha))$ and we may proceed as before. The next change has the condition $(a_1 ; i)$ require that there be an element in $a_1(\mathbf{B}, \mathbf{C})$ which is greater than $g(i) = \lim_\sigma g(i, \sigma)$. Then if the k-th condition is $(a_1 ; i)$ we define $f(k, \sigma)$ as before but we alter the definition of $G(k, \sigma)$ as follows: let $G(k, \tau)$ be the least member of $\{f(k, \chi): \chi \in S_1(\tau) \;\&\; \chi > g(k, \tau)\}$ if one exists and let it be τ otherwise. The rest of that paragraph remains unchanged.

The statement of the construction and the definition of the sets $\mathbf{C}$ remain unchanged. Likewise, the proofs that (C3) and (C4) are satisfied need no change. The proof that, assuming $\mathrm{lub}(\{F(k, \sigma)\}_{\sigma<\alpha}) < \alpha$ for all k, (C1) and (C2) are satisfied needs to be changed only for the case of conditions of the first kind; in that case it follows from $\mathrm{lub}(\{G(k, \sigma)\}_{\sigma<\alpha}) < \alpha$ that $a_1(\mathbf{B}, \mathbf{C}) \cap T'$ has an element greater than $g(i)$ where the k-th condition is $(a_1 ; i)$.

We now give the changes in the proof following the assumption that $\mathrm{lub}(\{F(k, \sigma)\}_{\sigma<\alpha}) = \alpha$ for some k. The first change is that if the k-th condition is $(a_1 ; i)$ then it follows that $C' \cap a^*(\mathbf{B})$ is bounded, instead of being finite. The next change is that we cannot conveniently avoid taking equivalence classes modulo bounded sets, so we do not, we compare the sub-d-lattice D of $(\mathbf{R}, A(\mathbf{R}))$ generated by $\{\mathbf{B}\}$ with $D(C)$ generated by $\{\mathbf{B}\}$, $\{C\}$. The only remaining change is to note that if the k-th condition is of the first or second kinds we know that $C' \cap a^*(\mathbf{B})$ is bounded for some atom a^* of A, whence our map of atoms F is not onto in this case. We have then made all the changes needed to show that if $D^*(\mathrm{x}, y)$ is the diagram such that $D^*(\{\mathbf{B}\}, \{C\})$ is true in $(\mathbf{R}, A(\mathbf{R}))$, the proof of the theorem is complete for the case $\mathbf{R} = R_\alpha^{\#}$.

We are now ready to give the long-awaited decision procedure. By the conclusion of Section 3 it is sufficient to reduce the decision problem for a separated AE-sentence to the decision problem for primitive AE-sentences of lesser characteristic. Now, the separated AE-sentence

$$(\mathrm{Ax})(\mathrm{Ey})(D(\mathrm{x}) \to P(\mathrm{x}, \mathrm{y}))$$

is false in (**R**, A(**R**)) unless it is potentially true; and if it is potentially true then from Theorem 5.1 it is true in (**R**, A(**R**)) just if each sentence

$$(\mathrm{Ax})(\mathrm{A}y)(\mathrm{Ey})(D^*(\mathrm{x}, y) \to P(\mathrm{x}, \mathrm{y}))$$

is true in (**R**, A(**R**)) where D^* runs through all diagrams containing just the variables x, y which are consistent with D and which have characteristic less than that of D. This constitutes an effective decision procedure, proving the following:

Theorem 5.4. *There is an effective decision procedure which for any* AE-*sentence tells whether that sentence is true in the lattices* $(R_\alpha^*, A(R_\alpha^*))$, $(R_\alpha^\#, A(R_\alpha^\#))$, $(Q_\alpha^*, A(Q_\alpha^*))$ *for* α *any admissible ordinal projectible into* ω, *or whether the sentence is false in all of these lattices.*

§6. Conclusions

As an immediate corollary to Theorem 5.4 we have the following:

Corollary 6.1. *Let* α *be any admissible ordinal projectible into* ω *and let* (**R**, A(**R**)) *be any of the three lattices* $(R_\alpha^*, A(R_\alpha^*))$, $(R_\alpha^\#, A(R_\alpha^\#))$, $(Q_\alpha^*, A(Q_\alpha^*))$. *Then exactly the same two-quantifier sentences are true in all the lattices* (**R**, A(**R**)).

Using a result of Owings [8] we can show that Corollary 6.1 is the best possible result in that there is no larger class of sentences for which all the lattices (**R**, A(**R**)) are equivalent, where sentence complexity is measured in terms of the number of alternations of quantifiers. Let A be a maximal element in **R**; following Owings [8] we say that A is *of type 1* if whenever B is maximal in A there is a maximal element C such that $B = A \cap C$; otherwise, we say A is *of type 2*. It is a result of Lachlan [6] that every maximal element in $R_\omega{}^*$ is of type 1. Owings has shown that if $\alpha > \omega$ and α is projectible into ω then $R_\alpha{}^*$ has maximal elements of both types and $Q_\alpha{}^*$ has maximal elements of type 2 only. The existence of maximal elements of type 1 can be asserted by a four-quantifier sentence, and the existence of maximal elements of type 2 can be asserted by a three-quantifier sentence. Therefore, there is a three-quantifier sentence true in some (**R**, A(**R**)) and false in some (**R**, A(**R**)).

It is not known at present of what types are the maximal elements of $R_\alpha{}^\#$ for $\alpha > \omega$ and projectible into ω. It seems likely that they are all of type 1. Obviously, knowledge of their types would provide valuable information about the lattices $R_\alpha{}^\#$. Ideally, one would like to have three-quantifier sentences which distinguished between all pairs of lattices among $R_\alpha{}^*$, $R_\alpha{}^\#$, $Q_\alpha{}^*$ for a fixed $\alpha > \omega$. At present we have no such sentences.

As was mentioned at the beginning of this paper, Corollary 6.1 is actually a metatheorem which gives a criterion for "lifting" a fairly large class of theorems of ordinary recursion theory to generalized recursion theory. Needless to say, any other such criteria would be a welcome addition to the subject of generalized recursion theory.

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