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# On $\alpha$-convex operators ☆

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**Abstract**

In this paper, the existence and uniqueness of positive fixed points for a class of convex operators is obtained by means of the properties of cone, concave operators and the monotonicity of set-valued maps. In the end, we give a simple application to certain integral equations.



*Keywords:* $\alpha$-Convex operator; Normal cone; Positive fixed point

## 1. Introduction

It is well known for some time that concave and convex operators defined on a cone in a Banach space play an important role in theory of positive operators (see, for instance, [6, Chapter 6]). In [7] A.J.B. Potter introduces the definitions of $\alpha$-concave operators and $\alpha$-convex operators, and shows that for $\alpha \geqslant 0$, increasing $\alpha$-concave and decreasing $(-\alpha)$-convex mappings have contraction ratios less than or equal to $\alpha$ and gives the existence of solutions to the nonlinear eigenvalue problem $Ax = \lambda x$. The method is based upon Hilbert's projective metric (see [1] for details). In [8] the author improves the corresponding results presented in [7] by using contraction mapping theorem. In [3] Guo Dajun widens the conditions and removes the hypotheses of continuation for operators,

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and then extends the results of fixed points, eigenvectors for $\alpha$-concave ($-\alpha$-convex) operators. However, they restrict their attention to $0 < |\alpha| \leqslant 1$, while for the remaining cases $\alpha > 1$ and $\alpha < -1$, the research proceeds slowly and appears difficult because the Hilbert's projective metric is useless for these cases. Up to now, pleasant results are seldom obtained.

The aim of this paper is to obtain the existence and uniqueness of positive fixed points for $\alpha$-convex ($\alpha > 1$) operators. Our method is based upon the properties of cone, $\alpha$-concave ($0 < \alpha < 1$) operators and the monotonicity of set-valued maps. To demonstrate the applicability of our results, we give in the final section of the paper a simple application to certain integral equations.

## 2. Preliminaries

In this section we summarize some basic concepts in real Banach spaces.

Suppose that $E$ is a real Banach space which is partially ordered by a cone $P \subset E$, i.e., $x \leqslant y$ if and only if $y - x \in P$. If $x \leqslant y$ and $x \neq y$, then we denote $x < y$ or $y > x$. By $\theta$ we denote the zero element of $E$. Recall that a nonempty closed convex set $P \subset E$ is a cone if it satisfies

$$x \in P, \quad \lambda \geqslant 0 \quad \Rightarrow \quad \lambda x \in P, \tag{2.1}$$

$$x \in P, \quad -x \in P \quad \Rightarrow \quad x = \theta. \tag{2.2}$$

Putting $\mathring{P} = \{x \in P\colon\ x \text{ is an interior point of } P\}$, a cone $P$ is said to be solid if its interior $\mathring{P}$ is nonempty. Moreover, $P$ is called normal if there exists a constant $N > 0$ such that, for all $x, y \in E$, $\theta \leqslant x \leqslant y$ implies $\|x\| \leqslant N\|y\|$; in this case $N$ is called the normality constant of $P$. In the case $y - x \in \mathring{P}$, we write $x \ll y$. If $x_1, x_2 \in E$, the set $[x_1, x_2] = \{x \in E\colon\ x_1 \leqslant x \leqslant x_2\}$ is called the order interval between $x_1$ and $x_2$. We say that an operator $A : E \to E$ is increasing (decreasing) if $x \leqslant y$ implies $Ax \leqslant Ay$ ($Ax \geqslant Ay$).

For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leqslant y \leqslant \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$ (i.e., $h \geqslant \theta$ and $h \neq \theta$), we denote by $P_h$ the set

$$P_h = \bigl\{x \in E\colon \text{ there exist } \lambda(x),\ \mu(x) > 0 \text{ such that } \lambda(x)h \leqslant x \leqslant \mu(x)h\bigr\},$$

and it is easy to see that $P_h \subset P$.

All the concepts discussed above can be found in [4,5].

Let $D$ be a subset of $E$ and $c$ be a real number we denote $cD = \{cx\colon\ x \in D\}$. Recall the following definition from [7].

**Definition 2.1.** Let $A$ be a positive mapping on $\mathring{P}$ and let $\alpha \in R$. Then we say $A$ is $\alpha$-concave ($\alpha$-convex) if and only if $A(tx) \geqslant t^\alpha Ax (A(tx) \leqslant t^\alpha Ax)$ for all $x \in \mathring{P}$ and $t \in (0, 1]$.

Let $A$ be a positive mapping on $\mathring{P}$ which is $\alpha$-concave ($\alpha \in [0, 1]$). Choose $h \in \mathring{P}$, then $P_h = \mathring{P}$. So there exist $\lambda, \mu \in R^+$ ($R^+$ denote the positive reals) such that $\lambda h \leqslant Ax \leqslant \mu h$ and $A(tx) \geqslant t^\alpha Ax \geqslant tAx$ for $t \in (0, 1]$. Thus $\alpha$-concave mappings are concave in the sense of Krasnoselskii [6, p. 185]. Similar remarks apply to $\alpha$-convex mappings. Note also $A$ is $\alpha$-concave ($\alpha$-convex) if and only if $A(sx) \leqslant s^{\alpha} Ax (A(sx) \geqslant s^{\alpha} Ax)$ for all $x \in \mathring{P}$ and $s \geqslant 1$.

## 3. Main results

In this section we consider, for $\alpha > 1$, the existence and uniqueness of fixed points for $\alpha$-convex operators. The main theorems, under reasonable conditions, show that increasing $\alpha$-convex operators have a unique fixed point in order intervals or totally ordered sets.

**Theorem 3.1.** *Let $E$ be a real Banach space and $P$ be a normal, solid cone, and let $\alpha > 1$. Suppose that $A : P \to P$ is an increasing $\alpha$-convex operator which satisfies the following assumptions*:

(i) *there exist $u_0, v_0 \in \mathring{P}$ such that $\theta \ll Au_0 \leqslant u_0 < v_0 \leqslant Av_0$;*
(ii) *there exists a linear operator $L : E \to E$ which has an increasing inverse $L^{-1} : E \to E$ such that*

$$Ay - Ax \leqslant L(y - x), \quad \textit{for } \forall y \geqslant x \geqslant \theta. \tag{3.1}$$

*Then $A$ has a unique fixed point in $[u_0, v_0]$.*

**Proof.** Firstly, for $\forall x \in P$, by (3.1), we have $\theta \leqslant Ax - A\theta \leqslant Lx$, i.e., $Lx \geqslant \theta$. Thus, $Lx \in P$. Further, for $\theta \leqslant x \leqslant y$, we have

$$\theta \leqslant Ay - Ax \leqslant L(y - x) = Ly - Lx.$$

This implies $Ly \geqslant Lx$, so $L : P \to P$ is increasing.

Consider the operator

$$Bx = L^{-1}(Lx + x - Ax) \quad \text{for } \forall x \in P,$$

then $B$ is increasing in $P$. In fact, for $x \in P$, we have by (ii),

$$Bx = L^{-1}(Lx + x - Ax) \geqslant L^{-1}(x) \geqslant L^{-1}(\theta) = \theta.$$

By (3.1), we obtain

$$Ly - Ay \geqslant Lx - Ax \geqslant \theta \quad \text{for } y \geqslant x \geqslant \theta. \tag{3.2}$$

Consequently, by (ii) and (3.2), we have

$$By = L^{-1}(Ly + y - Ay) \geqslant L^{-1}(Lx - Ax + y) \geqslant L^{-1}(Lx - Ax + x) = Bx.$$

Hence, $B$ is increasing in $P$.

Secondly, by (i), we obtain

$$Bu_0 = L^{-1}(Lu_0 + u_0 - Au_0) \geqslant L^{-1}(Lu_0 + Au_0 - Au_0) = u_0, \tag{3.3}$$

$$Bv_0 = L^{-1}(Lv_0 + v_0 - Av_0) \leqslant L^{-1}(Lv_0 + Av_0 - Av_0) = v_0. \tag{3.4}$$

Then (3.3), (3.4) together implies $B([u_0, v_0]) \sqsubseteq [u_0, v_0]$.

Since $\theta \ll Au_0 \leqslant u_0 < v_0 \leqslant Av_0$, and then $Au_0 \leqslant u_0 < v_0 \leqslant Lv_0 + v_0$, there exist $r, \xi_0 \in (0,1)$ such that

$$u_0 \geqslant r v_0 \quad \text{and} \quad \xi_0(Lv_0 + v_0) \leqslant Au_0. \tag{3.5}$$

Consider the following function:

$$f(t) = \frac{1 - t^{1-\gamma}}{1 - t^{\alpha-\gamma}}, \quad \forall t \in (0,1), \text{ where } \gamma \in (0,1).$$

It is easy to prove that $f$ is decreasing in $(0,1)$, thus

$$\frac{1 - t^{1-\gamma}}{1 - t^{\alpha-\gamma}} = f(t) \leqslant f(r) = \frac{1 - r^{1-\gamma}}{1 - r^{\alpha-\gamma}}, \quad \forall t \in [r, 1).$$

Further

$$\lim_{\gamma \to 1^-} \frac{1 - r^{1-\gamma}}{1 - r^{\alpha-\gamma}} = 0.$$

So there exists $\gamma_0 \in (0,1)$ such that $\frac{1-r^{1-\gamma}}{1-r^{\alpha-\gamma}} < \xi_0$, $\forall \gamma \in [\gamma_0, 1)$. In particular,

$$\frac{1 - r^{1-\gamma_0}}{1 - r^{\alpha-\gamma_0}} < \xi_0.$$

Hence

$$\frac{1 - t^{1-\gamma_0}}{1 - t^{\alpha-\gamma_0}} \leqslant \frac{1 - r^{1-\gamma_0}}{1 - r^{\alpha-\gamma_0}} < \xi_0, \quad \forall t \in [r, 1). \tag{3.6}$$

Consider (3.5) and (3.6), for $x \in [u_0, v_0]$, $t \in [r, 1)$, we have

$$\frac{1 - t^{1-\gamma_0}}{1 - t^{\alpha-\gamma_0}}(Lx + x) \leqslant \frac{1 - r^{1-\gamma_0}}{1 - r^{\alpha-\gamma_0}}(Lv_0 + v_0) \leqslant \xi_0(Lv_0 + v_0) \leqslant Au_0 \leqslant Ax.$$

Then we obtain

$$t^{\gamma_0}(Lx + x - Ax) \leqslant L(tx) + tx - t^{\alpha}Ax \leqslant L(tx) + tx - A(tx), \quad \forall t \in [r, 1).$$

Applying the monotonicity of $L^{-1}$, we have

$$B(tx) = L^{-1}\bigl[L(tx) + tx - A(tx)\bigr] \geqslant L^{-1}\bigl[t^{\gamma_0}(Lx + x - Ax)\bigr] = t^{\gamma_0}Bx.$$

That is

$$B(tx) \geqslant t^{\gamma_0}Bx \quad \text{for } \forall x \in [u_0, v_0],\ t \in [r, 1),\ \gamma_0 \in (0,1).$$

Finally, we show that $B$ has a unique fixed point $x^*$ in $[u_0, v_0]$. Denote $u_n = Bu_{n-1}$, $v_n = Bv_{n-1}$ $(n = 1, 2, \ldots)$, and by the monotonicity of $B$, we have

$$u_0 \leqslant u_1 \leqslant u_2 \leqslant \cdots \leqslant u_n \leqslant \cdots \leqslant v_n \leqslant \cdots \leqslant v_2 \leqslant v_1 \leqslant v_0.$$

Note that $r^{\gamma_0^n} \in [r, 1)$ $(n = 0, 1, 2, \ldots)$ and $rv_0 \leqslant u_0 < v_0$. It follows that

$$u_n \geqslant r^{\gamma_0^n} v_n \quad (n = 0, 1, 2, \ldots),$$

and for any natural number $p$ we have

$$0 \leqslant u_{n+p} - u_n \leqslant v_n - u_n, \qquad 0 \leqslant v_n - v_{n+p} \leqslant v_n - u_n. \tag{3.7}$$

Further

$$v_n - u_n \leqslant v_n - r^{\gamma_0^n} v_n = \left(1 - r^{\gamma_0^n}\right) v_n \leqslant \left(1 - r^{\gamma_0^n}\right) v_0.$$

Since $P$ is normal, we have

$$\|v_n - u_n\| \leqslant N\left(1 - r^{\gamma_0^n}\right)\|v_0\| \to 0 \quad (\text{as } n \to \infty). \tag{3.8}$$

Here $N$ is the normal constant. So (3.7) and (3.8) together implies that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences. Because $E$ is complete, there exist $u^*, v^* \in [u_0, v_0]$ such that $u_n \to u^*$, $v_n \to v^*$ as $n \to \infty$. By (3.8), we know that $u^* = v^*$. Evidently,

$$0 \leqslant Bu_n = u_{n+1} \leqslant Bu^* \leqslant Bv_n = v_{n+1}. \tag{3.9}$$

Passing the limit in (3.9), we have $Bu^* = u^*$, which implies $u^*$ is a fixed point of $B$, and it is unique in $[u_0, v_0]$. In fact, suppose $\bar{u}$ is a fixed point of $B$ in $[u_0, v_0]$ with $\bar{u} \neq u^*$, then $u_0 \leqslant \bar{u} \leqslant v_0$. By the monotonicity of $B$, we have $u_n \leqslant \bar{u} \leqslant v_n$, letting $n \to \infty$ yields $\bar{u} = u^*$. This is a contradiction. Therefore, $B$ has a unique fixed point $x^*$ in $[u_0, v_0]$. Obviously, $Bx = x \Leftrightarrow Ax = x$. Thus, $A$ also has a unique fixed point $x^*$ in $[u_0, v_0]$. □

**Corollary 3.2.** *Let $E$ be a real Banach space and $P$ be a normal, solid cone, and let $\alpha > 1$. Suppose that $A : P \to P$ is an increasing $\alpha$-convex operator which satisfies the following assumptions*:

(i) *there exist $u_0, v_0 \in \mathring{P}$ such that $\theta \ll Au_0 \leqslant u_0 < v_0 \leqslant Av_0$;*
(ii) *there exists $M > 0$ such that*

$$Ay - Ax \leqslant M(y - x) \quad \textit{for } \forall y \geqslant x \geqslant \theta.$$

*Then $A$ has a unique fixed point in $[u_0, v_0]$.*

**Proof.** Put $L = MI$, where $I$ is the identity operator in $E$. Then we have that $L^{-1} = \frac{1}{M} I : E \to E$ is increasing. Hence, the conclusion follows from Theorem 3.1. □

**Remark 3.3.** Up to now, we have not seen such results as discussed above in literature available.

The following statements play a very important role in the proofs of Theorems 3.4 and 3.5.

For $C, B \subset E$, we write $C \leqslant^s B$ ($C \geqslant^s B$) if $c \leqslant b$ ($c \geqslant b$) for any $c \in C$, $b \in B$. A set-valued map $A : S(\subset E) \to 2^E$ (here $2^E$ denotes the family of nonempty subsets of $E$) is said to be increasing if $x \leqslant y$ implies $Ax \leqslant^s Ay$. As a direct consequence, we have: if $C \leqslant^s C$, then $C$ is a singleton.

**Theorem 3.4.** *Let $E$ be a real Banach space and $P$ be a normal cone in $E$. Let $A : P_h \to P_h$ be an increasing $\alpha$-convex operator ($\alpha > 1$), i.e., $A(tx) \leqslant t^\alpha Ax$ for all $x \in P_h$ and $t \in$*

$(0,1]$. *In addition, assume that there exists a nonempty, totally ordered set* $S \subset P_h$ *such that*

(i) $\lambda S \subset S(\lambda \in (0,1))$, $AS = S$;
(ii) $Av_0 \geqslant v_0$, *for certain* $v_0 \in S$.

*Then* $A$ *has exactly one positive fixed point in* $S$.

**Proof.** We divide the proof into several steps.

**Step 1.** Consider operator $A : S \to S$. For $\forall y \in S$, set $A^{-1}y = \{x \in S\colon\ Ax = y\}$. Then $A^{-1} : S \to 2^S$ is a set-valued mapping, and we have the following conclusions.

(a) $A^{-1}$ is increasing in the sense of set-valued mappings.

In fact, if $y_1 > y_2$ for $\forall y_1, y_2 \in S$, then $A^{-1}y_1 \geqslant^s A^{-1}y_2$. Suppose that is not the case, then we have $x_1 < x_2$ for $\forall x_1 \in A^{-1}y_1, x_2 \in A^{-1}y_2$. Using the monotonicity of $A$, we have $Ax_1 \leqslant Ax_2$. That is to say, $y_1 \leqslant y_2$. This is a contradiction.

(b) $A^{-1}(sy) \geqslant^s s^{1/\alpha}A^{-1}y$ for $y \in S$, $s \in (0,1)$.

For $x \in A^{-1}y$, then $Ax = y$ and $A(tx) \leqslant t^\alpha Ax$ for $t \in (0,1)$. Let $s = t^\alpha$, we have $A(s^{1/\alpha}x) \leqslant sAx$. So (a) implies $A^{-1}(A(s^{1/\alpha}x)) \leqslant^s A^{-1}(sAx)$. Thus

$$\left\{s^{1/\alpha}x\right\} \leqslant^s A^{-1}(sy), \quad \forall x \in A^{-1}y.$$

By the arbitrariness of $x$, one obtains that

$$\left\{s^{1/\alpha}x\colon\ x \in A^{-1}y\right\} \leqslant^s A^{-1}(sy),$$

namely, $s^{1/\alpha}A^{-1}y \leqslant^s A^{-1}(sy)$.

**Step 2.** For $v_0 \in S$, $t \in (0,1)$, we have $A(tv_0) \leqslant t^\alpha Av_0$. Since $Av_0 \in S \subset P_h$, there exist $\lambda, \mu > 0$ such that $\lambda v_0 \leqslant Av_0 \leqslant \mu v_0$. So $A(tv_0) \leqslant t^\alpha Av_0 \leqslant \mu t^\alpha v_0$, and we can choose $t_1$ sufficiently small satisfying $\mu t^\alpha < t$, then $A(t_1 v_0) \leqslant t_1 v_0$, $t_1 \in (0,1)$.

Now we write $u_0 = t_1 v_0$, then $Au_0 \leqslant u_0$ and $u_0 < v_0$. Take $\lambda_0 = t_1^2$, then $\lambda_0 \in (0,1)$, $u_0 = t_1 v_0 \geqslant t_1^2 v_0 = \lambda_0 v_0$.

**Step 3.** By Step 2 and (ii), we know that

$$u_0, v_0 \in S, \quad u_0 < v_0, \quad u_0 \geqslant \lambda_0 v_0, \quad Au_0 \leqslant u_0, \quad Av_0 \geqslant v_0.$$

By Step 1, we have $A^{-1}(Au_0) \leqslant^s A^{-1}u_0$, then there exists $u_1 \in A^{-1}u_0$ such that $u_0 \leqslant u_1$, if $u_0 = u_1$ then $Au_0 = Au_1 = u_0$, i.e., $u_0$ is a fixed point of $A$. So without loss of generality we can assume $u_0 < u_1$, therefore, $A^{-1}u_0 \leqslant^s A^{-1}u_1$, so there is $u_2 \in A^{-1}u_1$ such that $u_1 \leqslant u_2$, if $u_1 = u_2$, then $u_1$ is the fixed point of $A$. Also, we can assume that $u_1 < u_2$, repeating this process, we can obtain an increasing sequence as follows:

$$u_0 \leqslant u_1 \leqslant \cdots \leqslant u_n \leqslant \cdots.$$

Similarly

$$v_0 \geqslant v_1 \geqslant \cdots \geqslant v_n \geqslant \cdots.$$

Evidently, $v_n \geqslant u_n$, where $u_n \in A^{-1}u_{n-1}$, $v_n \in A^{-1}v_{n-1}$, $n = 1, 2, \ldots$.

Since $u_0 > \lambda_0 v_0$, we have $A^{-1}u_0 \geqslant^s A^{-1}(\lambda_0 v_0) \geqslant^s {\lambda_0}^{1/\alpha} A^{-1}v_0$, which implies $u_1 \geqslant {\lambda_0}^{1/\alpha} v_1$. Then $A^{-1}u_1 \geqslant^s {\lambda_0}^{1/\alpha^2} A^{-1}v_1$. So we have $u_2 \geqslant {\lambda_0}^{1/\alpha^2} v_2$. Repeating this process, we can obtain $u_n \geqslant {\lambda_0}^{1/\alpha^n} v_n$. Therefore

$$\theta \leqslant v_n - u_n \leqslant v_n - {\lambda_0}^{1/\alpha^n} v_n = \big(1 - {\lambda_0}^{1/\alpha^n}\big) v_n < \big(1 - {\lambda_0}^{1/\alpha^n}\big) v_0.$$

By the normality of cone $P$, we have

$$\|v_n - u_n\| \leqslant N\big(1 - {\lambda_0}^{1/\alpha^n}\big)\|v_0\| \to 0 \quad (n \to \infty),$$

here $N$ is the normal constant.

Further

$$\theta \leqslant u_{n+p} - u_n \leqslant v_n - u_n, \qquad \theta \leqslant v_n - v_{n+p} \leqslant v_n - u_n \quad (p \in N).$$

Thus

$$\|u_{n+p} - u_n\| \leqslant N\|v_n - u_n\| \to 0 \quad (n \to \infty),$$
$$\|v_n - v_{n+p}\| \leqslant N\|v_n - u_n\| \to 0 \quad (n \to \infty).$$

So we can claim that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences. Then there exists $u^*$ such that $u_n \to u^*$ as $n \to \infty$ and $v_n \to u^*$ as $n \to \infty$. It follows that $u_n \leqslant u^* \leqslant v_n$. Thus

$$A^{-1}u_n \leqslant^s A^{-1}u^* \leqslant^s A^{-1}v_n.$$

Since $u_{n+1} \in A^{-1}u_n$, $v_{n+1} \in A^{-1}v_n$, we have $u_{n+1} \leqslant u \leqslant v_{n+1}$ for $\forall u \in A^{-1}u^*$. Passing the limit, we obtain $u = u^*$. That is, $A^{-1}u^* = \{u^*\}$, which implies $Au^* = u^*$. Then $u^*$ is a positive fixed point of $A$.

**Step 4.** In the following we prove that $u^*$ is the unique fixed point of $A$ in $S$.

In fact, suppose $\bar{u} \in S$ is a positive fixed point of $A$ with $\bar{u} \neq u^*$. Evidently, $\bar{u}, u^* > \theta$. Since $S$ is a totally ordered set, without loss of generality we assume that $\bar{u} > u^*$. Write $x_1 = u^*$, $x_2 = \bar{u}$, then $x_2 > x_1$. The fact that $x_1, x_2 \in P_h$ shows that there exists $\mu_0 > 0$ such that $x_1 \geqslant \mu_0 x_2$. Obviously, $\mu_0 < 1$.

If $x_1 = \mu_0 x_2$, then $Ax_1 = A(\mu_0 x_2) \leqslant {\mu_0}^{\alpha} Ax_2$, i.e., $x_1 \leqslant {\mu_0}^{\alpha} x_2 < \mu_0 x_2$, which is a contradiction. Thus $x_1 > \mu_0 x_2$, and so $A^{-1}x_1 \geqslant^s A^{-1}(\mu_0 x_2) \geqslant^s {\mu_0}^{1/\alpha} A^{-1}x_2$. This shows $x_1 \geqslant {\mu_0}^{1/\alpha} x_2$. If $x_1 = {\mu_0}^{1/\alpha} x_2$, then $A(x_1) = A({\mu_0}^{1/\alpha} x_2) \leqslant \mu_0 x_2$. This is a contradiction. Hence $x_1 > {\mu_0}^{1/\alpha} x_2$. Repeating this process, we obtain $x_1 > {\mu_0}^{1/\alpha^n} x_2$. Consequently

$$\theta < x_2 - x_1 < x_2 - {\mu_0}^{1/\alpha^n} x_2 = \big(1 - {\mu_0}^{1/\alpha^n}\big) x_2,$$

by the normality of cone $P$, we have

$$\|x_2 - x_1\| \leqslant N\big(1 - {\mu_0}^{1/\alpha^n}\big)\|x_2\| \to 0 \quad (n \to \infty),$$

thus $x_1 = x_2$, which is a contradiction. This completes the proof. □

**Remark.** The condition of $\lambda \in (0,1)$, $\lambda S \subset S$ is very important to prove our conclusions and can assure that we discuss in $S$. If it were removed, we can not seek certain $u_0 \in S$ such that $u_0 < v_0$, $Au_0 \leqslant u_0$, with the result that neither the existence nor the uniqueness of fixed points could be obtained.

**Theorem 3.5.** *Let $E$ be a real Banach space and $P$ be a normal cone in $E$. Let $A : P_h \to P_h$ be an increasing $\alpha$-convex operator ($\alpha > 1$), i.e., $A(tx) \leqslant t^\alpha Ax$ for all $x \in P_h$ and $t \in (0,1]$. In addition, assume that there exists a nonempty, totally ordered set $S \subset P_h$ such that $\lambda S \subset S$ ($\lambda > 0$), $AS = S$. Then $A$ has exactly one positive fixed point in $S$.*

**Proof.** From $\lambda S \subset S$ for $\lambda > 0$, it follows that $\lambda S \subset S$ for $\lambda \in (0,1)$. Therefore, the condition (i) of Theorem 3.4 is satisfied.

In the following, we show the condition (ii) of Theorem 3.4 is also satisfied.

For $\forall x_0 \in S$, $s > 1$, we have $A(sx_0) \geqslant s^\alpha Ax_0$. Since $Ax_0 \in S \subset P_h$, there exist $\mu, \lambda > 0$ such that $\lambda x_0 \leqslant Ax_0 \leqslant \mu x_0$, thus $A(sx_0) \geqslant s^\alpha \lambda x_0$. So we can choose $s_0$ large enough such that $s_0^\alpha \lambda \geqslant s_0$, hence, $A(s_0x_0) \geqslant s_0x_0$.

Now we write $v_0 = s_0x_0$, then $v_0 \in S$, $Av_0 \geqslant v_0$. Thus, the conclusion follows from Theorem 3.4. □

**Remark 3.6.**

(i) Under the hypotheses of this paper, the tool—Hilbert's projective metric used in papers [1,2,7]—cannot be used.
(ii) By using the properties of inverse mapping (set-valued mapping), we give similar results to $\alpha$-concave operators ($0 < \alpha < 1$), so our results compliment the theory of concave and convex operators. Moreover, the method is new and different from previous ones.

## 4. Applications

Theorem 3.4 or Theorem 3.5 can be used to discuss the solution of the following special integral equation:

$$x(t) = \int_0^1 k(t,s)x^\alpha(s)\,ds, \quad \alpha > 1. \tag{$*$}$$

Suppose that $k(t,s) = h(t)f(s)$, and $h, f$ are nonnegative continuous functions with $f(t) > 0$, $h(t) > 0$ for $t \in [0,1]$.

Then Eq. $(*)$ has exactly one positive continuous solution.

**Proof.** First some notation. Put $X = C[0,1]$ (the space of continuous functions defined on $[0,1]$ endowed with supremum norm). Let $P$ be the cone of nonnegative functions in $X$. So $P$ is normal and $\mathring{P}$ is the set of positive functions in $X$. Note that $P$ is a closed solid, the norm is monotonic.

Consider the integral operator $A : \mathring{P} \to X$ defined by

$$Ax(t) = \int_0^1 k(t,s)x^\alpha(s)\,ds, \quad x \in \mathring{P}.$$

Therefore

$$Ax(t) = h(t)\int_0^1 f(s)x^\alpha(s)\,ds = ah(t),$$

where $a = \int_0^1 f(s)x^\alpha(s)\,ds = x^\alpha(\xi)\int_0^1 f(s)\,ds$, for certain $\xi \in (0,1)$.

Evidently, $a > 0, ah \in \mathring{P}$, so $A : \mathring{P} \to \mathring{P}$. Let $S = \{x\colon x(t) = ah(t),\ a \in R^+\}$, then we have $S \subset \mathring{P}$ and $S$ is a totally ordered set with $\lambda S \subset S(\lambda > 0)$.

In the following we prove $AS = S$.

For $\forall y \in S$, $y = ah$, there is $x = (a/H)^{1/\alpha}h \in S$ such that $Ax = y$, where $H = \int_0^1 f(s)h^\alpha(s)\,ds$. Thus $AS = S$.

Since $A$ is increasing in $\mathring{P}$, Theorem 3.5 implies that $A$ has exactly one fixed point $x^*$ in $S$. Further, $x^* = H^{\frac{1}{1-\alpha}}h$. In fact, let $x^* = a_0 h$, then $Ax^* = A(a_0 h) = {a_0}^\alpha Ah = {a_0}^\alpha Hh = a_0 h$. So we obtain $a_0 = H^{\frac{1}{1-\alpha}}$, thus, Eq. $(*)$ has one positive solution $x^*(t) = H^{\frac{1}{1-\alpha}}h(t)$. □

**Remark 4.1.** For Eq. $(*)$, we can also use the following lemma generalized from [1,2] to prove the results.

**Lemma 4.2.** *Let the norm in Banach space $X$ be monotonic on cone $P$, $A : \mathring{P} \to \mathring{P}$ be positive homogenous of degree $p$, $0 < |p| < 1$* (*i.e.*, $A(tx) = t^p Ax$, $\forall x \in \mathring{P}$, $t > 0$). *In addition, $A$ is increasing* $(0 < p < 1)$ *or decreasing* $(-1 < p < 0)$. *Then $A$ has exactly one positive fixed point in $\mathring{P}$.*

**Proof of Eq.** $(*)$**.** As in the proof above, $A : S \to S$, $AS = S$. For $x_1, x_2 \in S$ with $x_1 \neq x_2$, we will prove $Ax_1 \neq Ax_2$.

In fact, let $x_1 = a_1 h$, $x_2 = a_2 h$, $a_1 \neq a_2$, $a_1, a_2 > 0$. Then $Ax_1 = A(a_1 h) = {a_1}^\alpha Ah$ and $Ax_2 = A(a_2 h) = {a_2}^\alpha Ah$, these together with ${a_1}^\alpha \neq {a_2}^\alpha$ implies the conclusion.

So $A$ is a one-to-one mapping, consequently, $A^{-1} : S \to S$ exists and further

$$A^{-1}(tx) = t^{1/\alpha}A^{-1}x, \quad t \in (0,1).$$

Since $A$ is strictly increasing, we obtain $A^{-1}$ is also increasing. Otherwise, for $y_1, y_2 \in S$, $y_1 \leqslant y_2$, we have $A^{-1}y_1 > A^{-1}y_2$, thus, $AA^{-1}y_1 > AA^{-1}y_2$, i.e., $y_1 > y_2$. This is a contradiction. An application of Lemma 4.2 implies that $A^{-1}$ has exactly one positive fixed point in $S$. Since $Ah = Hh$, $A^{-1}(\lambda x) = \lambda^{1/\alpha}A^{-1}x$, we have $A^{-1}h = H^{-1/\alpha}h$. Let $u_0 = h$, $u_n = A^{-1}u_{n-1}$ $(n = 1, 2, \ldots)$. Then

$$u_1 = A^{-1}u_0 = A^{-1}h = H^{-1/\alpha}h,$$

$$u_2 = A^{-1}u_1 = A^{-1}\bigl(H^{-1/\alpha}h\bigr) = \bigl(H^{-1/\alpha}\bigr)^{(1+\frac{1}{\alpha})}h,$$
$$\ldots$$
$$u_{n+1} = A^{-1}u_n = \bigl(H^{-1/\alpha}\bigr)^{(1+\frac{1}{\alpha}+\cdots+\frac{1}{\alpha^n})}h = H^{\frac{1-\frac{1}{\alpha^n}}{1-\alpha}}h.$$

Thus, we have

$$\bigl\|u_{n+1} - H^{\frac{1}{1-\alpha}}h\bigr\| = \bigl|H^{\frac{1-\frac{1}{\alpha^n}}{1-\alpha}} - H^{\frac{1}{1-\alpha}}\bigr|\|h\| \to 0 \quad (n \to \infty).$$

Consequently, $u_n \to H^{\frac{1}{1-\alpha}}h$ $(n \to \infty)$. In addition,

$$A^{-1}\bigl(H^{\frac{1}{1-\alpha}}h\bigr) = \bigl(H^{\frac{1}{1-\alpha}}\bigr)^{1/\alpha}H^{-1/\alpha}h = H^{\frac{1}{1-\alpha}}h.$$

This implies that $H^{\frac{1}{1-\alpha}}h$ is a fixed point of $A^{-1}$. Hence, $A$ has exactly one positive fixed point $H^{\frac{1}{1-\alpha}}h$ in $S$. □

**Remark 4.3.** For the uniqueness of the solution of Eq. $(*)$, we can also prove it by using the following method.

**Proof.** Let $x_1, x_2$ are the solutions of Eq. $(*)$. Note that $x_1 = a_1h$, $x_2 = a_2h$, then $x_1 = \frac{a_1}{a_2}x_2 = ax_2$, where $a = \frac{a_1}{a_2}$. Evidently, $0 < a < \infty$.

When $a > 1$, we have $Ax_1 = A(ax_2) = a^\alpha Ax_2 = a^\alpha x_2 > ax_2$, i.e., $x_1 > ax_2$. This is a contradiction. When $a < 1$, we have $Ax_1 = A(ax_2) = a^\alpha Ax_2 = a^\alpha x_2 < ax_2$, i.e., $x_1 < ax_2$. This is a contradiction.

So $a = 1$, we obtain $x_1 = x_2$. This completes the proof. □

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