A remark about the interpolation of spaces of continuous, vector-valued functions

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Abstract

If $A$ is a sectorial operator on a Banach space $X$, then the space $C([0, 1]; (X, D(A))_\theta, \infty)$ is a subspace of the interpolation space $(C([0, 1]; X), C([0, 1]; D(A)))_\theta, \infty$. The inclusion is strict in general.

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Let $X$ be a Banach space, and let $A$ be a closed, sectorial operator on $X$, i.e., $(-\infty, 0) \subset \rho(A)$ and $\sup_{\lambda > 0} \|\lambda(\lambda + A)^{-1}\| < \infty$.

Consider the abstract Cauchy problem

\[
\begin{cases}
\dot{u} + Au = f, & t \in [0, 1], \\
u(0) = 0.
\end{cases}
\] (1)

Under the assumption that $-A$ is the generator of an analytic $C_0$-semigroup and that a space $\mathcal{F}([0, 1]; X) \subset L^1(0, 1; X)$ of $X$-valued functions on the interval $[0, 1]$ is given, the problem of maximal regularity is to find out when $Au \in \mathcal{F}([0, 1]; X)$ for every $f \in \mathcal{F}([0, 1]; X)$.

Important results in this direction go back to the pioneering work of Da Prato and Grisvard [4], and articles of Sinestrari [8], Dore and Venni [5], or Acquistapace and Terreni [1]; see also the monograph of Lunardi [6] for a recent account and more references or the recent article of Clément et al. [3]. In this context, the theory of interpolation spaces (or

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intermediate spaces) is important. It is the aim of this note to study some particular interpolation spaces of spaces of continuous, vector-valued functions.

For every $0 < \theta < 1$, we define

$$
D_A(\theta, \infty) := \left\{ x \in X : \sup_{\lambda > 0} \| \lambda^\theta A(\lambda + A)^{-1} x \| < \infty \right\}.
$$

Together with the norm $\| x \|_{D_A(\theta, \infty)} := \sup_{\lambda > 0} \| \lambda^\theta A(\lambda + A)^{-1} x \|$ the space $D_A(\theta, \infty)$ is a Banach space. It is known that $D_A(\theta, \infty)$ coincides with the real interpolation space $(X, D(A))_{\theta, \infty}$, where $D(A)$ is equipped with the graph norm so that it becomes a Banach space [6, Propositions 2.2.2, 2.2.6].

Let the operator $\tilde{A}$ be defined by

$$
D(\tilde{A}) := C([0, 1]; D(A)),
$$

$$(\tilde{A} f)(s) := A f(s), \quad f \in D(\tilde{A}), \; s \in [0, 1]. \tag{3}$$

Then $\tilde{A}$ is a sectorial operator on $C([0, 1]; X)$, and

$$
D(\tilde{A}) = \left\{ f \in C([0, 1]; X) : \sup_{\lambda > 0} \| \lambda^\theta \tilde{A}(\lambda + \tilde{A})^{-1} f \|_{C([0,1];X)} < \infty \right\} = \left\{ f \in C([0, 1]; X) : \sup_{\lambda > 0} \sup_{s \in [0, 1]} \| \lambda^\theta A(\lambda + A)^{-1} f(s) \| < \infty \right\} = C([0, 1]; X) \cap B([0, 1]; D_A(\theta, \infty)),
$$

where $B([0, 1]; D_A(\theta, \infty))$ denotes the space of all bounded functions from $[0, 1]$ with values in $D_A(\theta, \infty)$. This shows in particular that

$$
C([0, 1]; D_A(\theta, \infty)) \subset (C([0, 1]; X), C([0, 1]; D(A)))_{\theta, \infty}.
$$

The following two examples on $X = c_0$ and $X = l^2$ show that the converse inclusion is not true.

Example 1. Let $X := c_0(\mathbb{N})$, and define

$$
D(A) := \left\{ (x_n)_{n \in \mathbb{N}} \in X : (x_n)_{n \in \mathbb{N}} \in X \right\},
\quad A(x_n)_{n \in \mathbb{N}} := (nx_n)_{n \in \mathbb{N}}. \tag{4}
$$

Then $A$ is a sectorial operator on $c_0$, and

$$
D(\tilde{A}) = \left\{ f = (f_n)_{n \geq 1} \in C([0, 1]; c_0) : \sup_{n \geq 1} \sup_{s \in [0, 1]} \| \lambda^\theta n(\lambda + n)^{-1} f_n(s) \| < \infty \right\} = \left\{ f = (f_n)_{n \geq 1} \in C([0, 1]; c_0) : \sup_{n \geq 1} \sup_{s \in [0, 1]} \| \theta^\theta (1 - \theta)^{1-\theta} n^\theta f_n(s) \| < \infty \right\}.
$$
In the last equality we have used the fact that the maximum of the function \( \lambda \mapsto \lambda^0 / (\lambda + n) \) on \((0, \infty)\) is \( \theta^0 (1 - \theta)^{1-\theta} n^{-\theta - 1} \).

Define the function \( f : [0, 1] \rightarrow X \) by

\[
  f(s) = \left( f_n(s) \right)_{n \in \mathbb{N}} := \left( \sin(n s) \right)_{n \in \mathbb{N}}, \quad s \in [0, 1].
\]

From the above equality it is clear that \( f \in (C([0, 1]; c_0), C([0, 1]; D(A)))_{\theta, \infty} \). We will see that the bounded function \( f : [0, 1] \rightarrow D_A(\theta, \infty) \) is not continuous in \( s = 0 \). Indeed,

\[
  \| f(s) - f(0) \|_{D_A(\theta, \infty)} = \sup_{n \geq 1} \| \theta^0 (1 - \theta)^{1-\theta} \sin(n s) \|.
\]

This shows that there exists a sequence \( s_k \downarrow 0 \) such that \( \| f(s_k) - f(0) \|_{D_A(\theta, \infty)} \geq \theta^0 (1 - \theta)^{1-\theta} \). Hence, the function \( f \) cannot be continuous in \( s = 0 \), i.e., \( f \notin C([0, 1]; D_A(\theta, \infty)) \).

**Example 2.** Now let us consider the case \( X := l^2(\mathbb{N}) \). We define the operator \( A \) on \( l^2 \) as in Eq. (4). The operator \( A \) is sectorial, and

\[
  D_A(\theta, \infty) = \left( C([0, 1]; l^2), C([0, 1]; D(A)) \right)_{\theta, \infty} = \left\{ f = (f_n)_{n \geq 1} \in C([0, 1]; l^2) : \sup_{\lambda > 0} \sup_{s \in [0, 1]} \lambda^2 n^2 (\lambda + n)^{-2} f_n(s)^2 < \infty \right\}.
\]

If \( \theta = 1/2 \) and \( f_n(s) := \sin(n s) / n \), then we have \( f = (f_n)_{n \geq 1} \in C([0, 1]; l^2) \), and

\[
  \sup_{\lambda > 0} \sup_{s \in [0, 1]} \lambda^2 n^2 (\lambda + n)^{-2} f_n(s)^2 \leq \frac{\lambda}{(\lambda + n)^2} \leq \frac{\lambda}{\lambda + n} \int_0^\infty \frac{ds}{(\lambda + s)^2} = 1.
\]

Thus, \( f = (f_n)_{n \geq 1} \in (C([0, 1]; l^2), C([0, 1]; D(A)))_{1/2, \infty} = D_A(1/2, \infty) \).

We claim that \( f = (f_n)_{n \geq 1} \notin C([0, 1]; D_A(1/2, \infty)) \). Indeed, let \( k \in \mathbb{N} \cup \{0\} \), and for \( s \in [0, 1] \) let \( [s] \in \mathbb{Z} \) be such that \( 0 \leq s - [s] < 1 \). Then, when \( s \in [0, 1] \) is small enough, we have for every \( \lambda > 0 \),

\[
  \sum_{[k \pi + 3\pi/4] / s}^{[k \pi + 3\pi/4] / s + 1} \lambda^2 n^2 (\lambda + n)^{-2} f_n(s)^2 = \sum_{[k \pi + 3\pi/4] / s}^{[k \pi + 3\pi/4] / s + 1} \lambda (\lambda + n)^{-2} \sin^2(n s)
\]

\[
  \geq \frac{\lambda}{2} \sum_{[k \pi + 3\pi/4] / s}^{[k \pi + 3\pi/4] / s + 1} \frac{1}{(\lambda + n)^2} \geq \frac{\lambda}{2} \sum_{[k \pi + 3\pi/4] / s}^{[k \pi + 3\pi/4] / s + 1} \frac{1}{(\lambda + n)^2}
\]

\[
  \geq \frac{\lambda}{2} \int_{[k \pi + 3\pi/4] / s}^{[k \pi + 3\pi/4] / s + 1} \frac{dx}{(\lambda + x)^2} \geq \frac{\lambda}{2} \int_{[k \pi + 3\pi/4] / s}^{[k \pi + 3\pi/4] / s + 1} \frac{\pi/4 s}{(\lambda + x)^2} \geq \frac{\pi}{8(k \pi + 2\pi/3 + \lambda s)^2}.
\]
Hence, when \( s \in (0, 1] \) is small enough, we have

\[
\| f(s) - f(0) \|^2_{DA(1/2, \infty)} = \sup_{\lambda > 0} \sum_{n \geq 1} \lambda(\lambda + n)^{-2} \sin^2(ns)
\]

\[
\geq \sup_{\lambda > 0} \sum_{k \geq 0} \frac{1}{8} \lambda(\lambda + n)^{-2} \sin^2(ns)
\]

\[
\geq \sup_{\lambda > 0} \frac{\pi \lambda s}{8} \int_1^\infty \frac{dx}{(\pi + 2\pi/3 + \lambda s)^2}
\]

\[
= \sup_{\lambda > 0} \frac{\lambda s}{8} \frac{1}{\pi + 2\pi/3 + \lambda s} = \frac{1}{8}.
\]

This shows that \( f \) is not continuous in \( s = 0 \), and therefore \( f \notin C([0, 1]; DA(1/2, \infty)) \).

**Remarks 3.** (a) Let

\[
D_A(\theta) := \left\{ x \in X : \lim_{\lambda \to \infty} \| \lambda^n A(\lambda + A)^{-1} x \| = 0 \right\}.
\]  

This space is a closed subspace of \( DA(\theta, \infty) \). In fact, it is the closure of the domain \( D(A) \) in \( DA(\theta, \infty) \) [8, Proposition 1.8]. By Arzela–Ascoli type arguments, one can show the following proposition (see, e.g., [2, Theorem 6.5], [3, Lemma 9]).

**Proposition 4.**

\[
D_{\tilde{A}}(\theta) = C([0, 1]; DA(\theta)).
\]  

If one remarks that [4, Théorème 3.11] remains true if the space \( DB(\theta, \infty) \) is replaced by \( D_B(\theta) \), then Proposition 4 implies the following result (see also [8, Theorem 5.1], [6, Corollary 4.3.10]).

**Proposition 5.** Assume that \( -A \) is the generator of an analytic \( C_0 \)-semigroup. For every \( f \in C([0, 1]; DA(\theta)) \) the solution \( u \) of problem (1) satisfies \( \dot{u}, Au \in C([0, 1]; DA(\theta)) \).

(b) By [4, Théorème 3.11], Proposition 5 remains true if the space \( C([0, 1]; DA(\theta)) \) is replaced by the space \( D_{\tilde{A}}(\theta, \infty) = C([0, 1]; X) \cap B([0, 1]; DA(\theta, \infty)) \) (see again also [8, Theorem 5.1], [6, Corollary 4.3.10]). In [4, Théorème 4.7(ii')], it is claimed that one may also replace it by the space \( C([0, 1]; DA(\theta, \infty)) \). However, since this space is in general a proper subspace of \( D_{\tilde{A}}(\theta, \infty) \) by Examples 1 and 2, this statement does not follow from [4, Théorème 3.11]. This was already remarked in [7, Remark 3], but no explicit counterexample was given. In [8, Remark 3, p. 57] it is even noted that [4, Théorème 4.7(ii')] is not true as it is stated. The same remarks hold for [4, Théorèmes 7.23, 8.8, 8.9].
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References