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A two-parameter control for contractive-like multivalued mappings

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ABSTRACT

We propose a general approach to defining a contractive-like multivalued mapping F which avoids any use of the Hausdorff distance between the sets $F(x)$ and $F(y)$. Various fixed point theorems are proved under a two-parameter control of the distance function $d_F(x) = dist(x, F(x))$ between a point $x \in X$ and the value $F(x) \subset X$. Here, both parameters are numerical functions. The first one $\alpha : [0, +\infty) \to [1, +\infty)$ controls the distance between x and some appropriate point $y \in F(x)$ in comparison with $d_F(x)$, whereas the second one $\beta : [0, +\infty) \to [0, 1)$ estimates $d_F(y)$ with respect to $d(x, y)$. It appears that the well harmonized relations between α and β are sufficient for the existence of fixed points of F. Our results generalize several known fixed point theorems.

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0. Introduction

Fixed point theories for singlevalued mappings and multivalued mappings of metric spaces into themself are clearly closely related. As a rule, almost any fixed point theorem for multivalued mappings goes back to some fixed point theorem for singlevalued mappings.

Such a correlation basically deals with the substitution of a given metric, say d, on a space X by the corresponding Hausdorff "metric" H_d on the set of all closed subsets of X. To illustrate this point, if one proves a fixed point theorem for a singlevalued mapping $f : X \to X$ under some contractivity-type restriction as e.g. Browder's condition [1,5,6]:

$$d\big(f(x), f(y)\big) \leqslant \varphi\big(d(x, y)\big) < d(x, y) \tag{$*$}$$

then one can certainly try to verify the existence of fixed points for a multivalued mapping F from X into itself under the analogous assumption:

$$H_d\big(F(x), F(y)\big) \leqslant \varphi\big(d(x, y)\big) < d(x, y). \tag{$**$}$$

Recall that the inequality $H_d(A, B) < \varepsilon$ implies that each of the sets A and B is a subset of an open ε-neighborhood of the other set. The key goal of the present paper is to show that the proximity of $F(x)$ and $F(y)$ with respect to H_d is too restrictive for a successful construction of the Picard sequence of approximations $x_n \to x_*$ which would converge to a fixed point x_* of F, $x_* \in F(x_*)$.

Roughly speaking, there is no need to require that the *entire* set $F(x_n)$ lies in an ε-neighborhood $\mathcal{O}_\varepsilon(F(x_{n+1}))$ of the set $F(x_{n+1})$ and, symmetrically that $F(x_{n+1}) \subset \mathcal{O}_\varepsilon(F(x_n))$. It suffices to find for a chosen $x_n \in X$, a point $x_{n+1} \in F(x_n)$ such that the distance $d_n = d(x_n, x_{n+1})$ is "almost" equal to $dist(x_n, F(x_n))$ and additionally, the distance $dist(x_{n+1}, F(x_{n+1}))$ is "less" than d_n. For the control of "nearness" of d_n to $dist(x_n, F(x_n))$ we propose a numerical function $\alpha : [0, +\infty) \to [1, +\infty)$, whereas the control of $dist(x_{n+1}, F(x_{n+1}))$ with respect to d_n will be provided by a numerical function $\beta : [0, +\infty) \to [0, 1)$. Certain matching behavior of control functions α and β guarantees the convergence of the sequence $\{x_n\}$. A standard verification shows that the limit of the sequence $\{x_n\}$ is a fixed point of the multivalued mapping F.

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1. Preliminaries

For a metric space (X,d), a point $x\in X$ and a subset $A\subset X$ we denote by $\mathcal{O}_\varepsilon(x)$ the open ε-neighborhood of x and by $\mathcal{O}_\varepsilon(A)$ the open ε-neighborhood of A, i.e. $\mathcal{O}_\varepsilon(A)=\bigcup\{\mathcal{O}_\varepsilon(x)\mid x\in A\}$. The distance between x and A is defined by $dist(x,A)=\inf\{d(x,y)\mid y\in A\}$, as usual. For a fixed closed-valued mapping F of a metric space (X,d) into itself we denote by $d_F(\cdot)$ the distance function which is defined by the equality $d_F(x)=dist(x,F(x))$, $x\in X$.

For nonempty subsets $A\subset X$ and $B\subset X$ the *Hausdorff distance* between A and B is defined by $H_d(A,B)=\inf\{\varepsilon>0\mid A\subset \mathcal{O}_\varepsilon(B),\ B\subset\mathcal{O}_\varepsilon(A)\}$, or equivalently, by $H_d(A,B)=\max\{\sup\{dist(x,B)\mid x\in A\},\sup\{dist(y,A)\mid y\in B\}\}$. In order to get the property $(H_d(A,B)=0\Leftrightarrow A=B)$ one needs to use the Hausdorff distance only for closed subsets of X. It is well known that $H_d(\cdot,\cdot)$ is indeed a metric on the family $CB(X)$ of all closed bounded nonempty subsets of X. Moreover, for a complete metric space (X,d) the *compact exponent* of X, i.e. the family of all nonempty subcompacta of X, is a complete metric space with respect to $H_d(\cdot,\cdot)$.

Let $k:[0,\infty)\to[0;1)$ be any numerical function. A multivalued mapping F which associates to each point x of a metric space (X,d) some nonempty closed subset $F(x)\subset X$, is called a *Hausdorff k-contraction* if $H_d(F(x),F(y))\leqslant k(d(x,y))\cdot d(x,y)$, $x,y\in X$. Among others we extract two rather typical restrictions for the contractivity coefficient. A numerical function $k:[0,\infty)\to[0,1)$ is said to have the *Reich property* (R) if $\limsup_{s\to t+0}k(s)<1$ for every $t>0$ (cf. [9,10]), and to have the *Mizoguchi–Takahashi property* (MT) if the same inequality holds for every $t\geqslant 0$ (cf. [8]).

The following is the key notion of the present paper.

Definition 1.1. For numerical functions $\alpha:[0,+\infty)\to[1,+\infty)$ and $\beta:[0,+\infty)\to[0,1)$, a closed-valued mapping F of a metric space (X,d) into itself is said to be an (α,β)-mapping if for each $x\in X$ there exists $y\in F(x)$ such that:

(A) $d(x,y)\leqslant\alpha(d(x,y))\cdot d_F(x)$; and
(B) $d_F(y)\leqslant\beta(d(x,y))\cdot d(x,y)$.

If, in addition, $\alpha(t)\cdot\beta(t)<1$, $t>0$, then F is said to be an (α,β)-contraction.

It easy to see that every Hausdorff k-contraction F is an (α,k)-contraction for every numerical function $\alpha:[0,+\infty)\to(1,+\infty)$ with $\alpha(t)\cdot k(t)<1$, $t>0$. In fact, the assumption (B) is true for an arbitrary $y\in F(x)$ because $d_F(y)\leqslant H_d(F(x),F(y))\leqslant k(d(x,y))\cdot d(x,y)=\beta(d(x,y))\cdot d(x,y)$, whereas the assumption (A) evidently holds for $d_F(x)=0\Leftrightarrow x\in F(x)\Leftrightarrow y=x\Leftrightarrow d(x,y)=0$ and for $d_F(x)>0$ it is true for a suitable $y\in F(x)$ because $d_F(x)<\alpha(d(x,y))\cdot d_F(x)$. Hence, each fixed point theorem for a Hausdorff contraction can be considered as a special case of some fixed point theorem for an (α,β)-contraction.

We shall also need some technical notations.

Definition 1.2.

(1) A numerical function $h:[0,\infty)\to[0,1)$ is said to be essentially positive if $\inf\{h(s)\mid s\geqslant a\}>0$ for every $a>0$;
(2) A nonnegative (and non-identically zero) function $h:X\to[0,1)$ is said to be stably positive if the inequality $h(x)>0$ implies that $\inf\{h(y)\mid y\in\mathcal{O}(x)\}>0$ for some neighborhood $\mathcal{O}(x)$ of the point $x\in X$.

To verify the implication $(x_n\to x_*)\wedge(x_{n+1}\in F(x_n),n\in\mathbb{N})\Rightarrow x_*\in F(x_*)$ one needs some continuity-like restrictions for the mapping F. This implication is definitely true for any Hausdorff contraction F and for any upper semicontinuous closed-valued mapping F. Moreover, it suffices to assume that the distance function d_F is a lower semicontinuous numerical function. So in order to explain the role of the stable positivity of the distance function d_F we shall prove the following simple lemma.

Lemma 1.3. *Let F be a closed-valued mapping such that the distance function d_F is stably positive. Let $\{x_n\}$ be a sequence of points which converges to x_*, where $x_{n+1}\in F(x_n)$, for all $n\in\mathbb{N}$. Then x_* is a fixed point of F.*

Proof. Assume to the contrary, i.e. suppose that x_* does not belong to the set $F(x_*)$. Hence $d_F(x_*)>0$ and $\inf\{d_F(x)\mid x\in\mathcal{O}(x_*)\}=m>0$ for some neighborhood $\mathcal{O}(x_*)$ of the point x_*. For some number $N\in\mathbb{N}$ onwards, all x_n with $n\geqslant N$ lie in $\mathcal{O}(x_*)$. So $0<m\leqslant d_F(x_n)\leqslant d(x_n,x_{n+1})\to 0$, $n\to\infty$. A contradiction. □

The following theorem is the main result of the paper.

Theorem 1.4. *Let numerical functions $\alpha:[0,+\infty)\to[1,+\infty)$ and $\beta:[0,+\infty)\to[0,1)$ be such that*:

(1) *$\beta(\cdot)$ has the property (MT)*;
(2) *$\alpha(t)\leqslant 1+\gamma(1-\beta(t))$, $t\geqslant 0$, where $\gamma:(0,1]\to[0,+\infty)$ has the following properties*:

(i) $\gamma(\cdot)$ *is bounded,* $\lim_{s\to 0}\gamma(s)=0$; *and*
(ii) *the function* $p(s)=s-(1-s)\gamma(s)$, $0\leqslant s<1$ *is essentially positive.*

Then every (α,β)*-mapping* F *of a complete metric space* (X,d) *into itself is an* (α,β)*-contraction and has a fixed point, whenever the distance function* d_F *is stably positive.*

In particular, putting $\gamma(s)=s$, $\alpha(s)=2-\beta(s)$, $p(s)=s^2$ in assumptions (1), (2) and equality instead of inequality in assumption (2) we obtain a recent result of Čirić [2, Theorem 5]. Sometimes there is a possibility for a nonexplicit form of a majorant of the function $\alpha:[0,+\infty)\to[1,+\infty)$.

Theorem 1.5. *Let numerical functions* $\alpha:[0,+\infty)\to[1,+\infty)$ *and* $\beta:[0,+\infty)\to[0,1)$ *be such that*:

(1) $\alpha(\cdot)\beta(\cdot)$ *has the property* (MT); *and*
(2) $\alpha(\cdot)$ *is nonincreasing.*

Then every (α,β)*-contraction* F *of a complete metric space* (X,d) *into itself has a fixed point, whenever the distance function* d_F *is stably positive.*

The referee has noted that Theorem 1.5 follows also by [2, Theorem 6]. One of the key steps in our proofs of Theorems 1.4 and 1.5 is the "boundary" property asserting that $\limsup_{s\to 0}\alpha(s)\beta(s)<1$. As a result, it allows us to use a majorization by a convergent geometric series. However, sometimes it is also possible to work with the equality $\limsup_{s\to 0}\alpha(s)\beta(s)=1$ and hence to use the property (R) instead of (MT). In the following theorem we present a version with a good power-rate upper estimate, say φ for the product $\alpha\beta$. In its proof a majorization is made by generalized harmonic series.

Theorem 1.6. *Let numerical functions* $\alpha:[0,+\infty)\to[1,+\infty)$ *and* $\beta:[0,+\infty)\to[0,1)$ *be such that*:

(1) $\alpha(\cdot)$ *is bounded*; *and*
(2) $\alpha(t)\beta(t)\leqslant\varphi(t)=1-Ct^p$ *for some* $C>0$, $0<p<1$ *and for all* t *in some neighborhood of the zero.*

Then every (α,β)*-contraction* F *of a complete metric space* (X,d) *into itself has a fixed point, whenever the distance function* d_F *is stably positive.*

Finally, in comparison with Theorem 1.4. and Čirić's theorem [2] we have the following:

Example 1.7. There is a finite-valued mapping of the segment $[0,1]$ into itself which:

(1) is not a $(2-\beta,\beta)$-contraction for any $\beta:[0,+\infty)\to[0,1)$;
(2) is not an (a,β)-contraction for any constant $a>1$; and
(3) is an (α,β)-contraction satisfying all assumptions of Theorem 1.4.

2. Proofs

We shall organize the proof of Theorem 1.4 in a sequence of Lemmas 2.1–2.6. The key ingredients are in Lemma 2.5. Its proof preserves the outline of the proof of [2, Theorem 5].

Lemma 2.1. *The product* $\alpha\beta$ *has the property* (MT).

Proof. Pick any $t\geqslant 0$. Due to the property (MT) for β there are numbers $\sigma>0$ and $0\leqslant q<1$ such that $\beta(s)\leqslant q$, $t<s<t+\sigma$. Therefore for all such s we have that $1-\beta(s)\geqslant 1-q>0$ and due to the essential positivity of the function $p(u)=u-(1-u)\gamma(u)$ we see that $\inf\{p(u)\mid u\geqslant 1-q\}=p_q>0$. A simple calculation

$$\alpha(s)\beta(s)\leqslant\big(1+\gamma\big(1-\beta(s)\big)\big)\cdot\beta(s)=1-\big[\big(1-\beta(s)\big)-\beta(s)\cdot\gamma\big(1-\beta(s)\big)\big]=1-p\big(1-\beta(s)\big)$$

shows that $\alpha(s)\beta(s)\leqslant 1-p_q<1$, $t<s<t+\sigma$. Therefore $\limsup_{s\to t+0}\alpha(s)\beta(s)\leqslant 1-p_q<1$. □

Lemma 2.2. *If* $\varepsilon>0$ *then* $\sup\{\alpha(t)\beta(t)\mid t\in\alpha^{-1}([1+\varepsilon,+\infty))\}=Q_\varepsilon<1$.

Proof. Using the equality $\lim_{s\to 0}\gamma(s)=0$ we can pick some $\sigma>0$ such that $\gamma(s)<\varepsilon$ for all $0<s<\sigma$. Then

$$t\in\alpha^{-1}\big([1+\varepsilon,+\infty)\big)\iff\alpha(t)\geqslant 1+\varepsilon\implies 1+\gamma\big(1-\beta(t)\big)\geqslant 1+\varepsilon\implies 1-\beta(t)\geqslant\sigma.$$

Now, exploiting once again the essential positivity of the function $p(\cdot)$, we obtain $\alpha(t)\beta(t) \leqslant 1 - p(1-\beta(t)) \leqslant 1 - \inf\{p(u) \mid u \geqslant \sigma\} = Q_\varepsilon < 1$. □

Lemma 2.3. *For an arbitrary initial point $x_0 \in X$ there exists a sequence $\{x_n\}_{n=0}^{\infty}$ such that for all $n \in \mathbb{N}$ the point x_{n+1} lies in $F(x_n)$ and the following properties hold*:

(A_n) $d_n = d(x_n, x_{n+1}) \leqslant \alpha(d_n) \cdot d_F(x_n)$; *and*
(B_n) $d_F(x_{n+1}) \leqslant \beta(d_n) \cdot d_n$.

Proof. A straightforward induction using Definition 1.1. □

Lemma 2.4. *Let $\{x_n\}_{n=0}^{\infty}$ be a sequence constructed in Lemma* 2.3. *Then $\{d_F(x_n)\}_{n=0}^{\infty}$ is a decreasing numerical sequence, and hence has a nonnegative limit.*

Proof. Applying Lemmas 2.1 and 2.3 we see that

$$d_F(x_{n+1}) \leqslant \beta(d_n) \cdot d_n \leqslant \beta(d_n) \cdot \alpha(d_n) \cdot d_F(x_n) \leqslant \big(1 - p\big(1 - \beta(d_n)\big)\big) \cdot d_F(x_n) < d_F(x_n). \qquad \square$$

Lemma 2.5. *Let $\{x_n\}_{n=0}^{\infty}$ be a sequence constructed in Lemma* 2.3. *Then $\{x_n\}_{n=0}^{\infty}$ is a fundamental sequence in X.*

Proof. Clearly,

$$d_F(x_n) = dist\big(x_n, F(x_n)\big) \leqslant d(x_n, x_{n+1}) = d_n \leqslant C \cdot d_F(x_n),$$

where $C = \sup\{\alpha(t) \mid t \geqslant 0\} < +\infty$.

Denote $\Delta = \lim_{n\to\infty} d_F(x_n)$ and $\nabla = \liminf_{n\to\infty} d_n$. Then $\Delta \leqslant \nabla < +\infty$. There are exactly three possibilities:

(I) $0 < \Delta < \nabla$; or
(II) $0 < \Delta = \nabla$; or
(III) $0 = \Delta = \nabla$.

In the case (I) we divide the segment $[\Delta, \nabla]$ into three equal parts. For all sufficiently large indices n the following inequalities hold:

$$0 < \Delta < d_F(x_n) < \frac{2\Delta + \nabla}{3} < \frac{\Delta + 2\nabla}{3} < d_n.$$

The property (A_n) (cf. Lemma 2.3) implies that

$$\alpha(d_n) \geqslant \frac{d_n}{d_F(x_n)} > \frac{\Delta + 2\nabla}{2\Delta + \nabla} = 1 + \frac{\nabla - \Delta}{2\Delta + \nabla} = 1 + \varepsilon.$$

By Lemma 2.2 this means that for all sufficiently large indices n the inequality $\alpha(d_n)\beta(d_n) \leqslant Q_\varepsilon < 1$ holds. So $d_F(x_{n+1}) \leqslant \alpha(d_n) \cdot \beta(d_n) \cdot d_F(x_n) < Q_\varepsilon \cdot d_F(x_n)$. Hence from some index, say N, onwards the numerical sequence $\{d_F(x_n)\}$ is majorized by a geometrical sequence with the coefficient $Q_\varepsilon < 1$. Therefore $\Delta = 0$. A contradiction.

In the case (II) we have $0 < \nabla = \Delta < d_F(x_n) \leqslant d_n$ and this is why $d_{n_k} \to \nabla + 0$, $k \to \infty$ for some subsequence. Then $d_F(x_{n_k}) \to \nabla + 0$, $k \to \infty$, too. By Lemma 2.1 applied to the product $\alpha\beta$ at the point $\nabla = \Delta$, the right upper limit of this product is less than 1. So there exists a number $0 \leqslant q < 1$ such that for all sufficiently large indices k the following inequalities hold:

$$d_F(x_{n_{k+1}}) \leqslant d_F(x_{n_k+1}) \leqslant \alpha(d_{n_k}) \cdot \beta(d_{n_k}) \cdot d_F(x_{n_k}) \leqslant q \cdot d_F(x_{n_k}).$$

Hence from some index, say K, onwards the numerical sequence $\{d_F(x_{n_k})\}$ is majorized by a geometrical sequence with the coefficient $q < 1$. Therefore $\nabla = 0$. A contradiction.

In the last case (III), recalling that $d_F(x_n) \leqslant d_n \leqslant C \cdot d_F(x_n)$, we conclude that $0 = \Delta = \nabla = \liminf_{n\to\infty} d_n = \lim_{n\to\infty} d_n$. As in the case (II) one can apply Lemma 2.1 to the product $\alpha\beta$ at the point $0 = \Delta = \nabla$. So starting from some index N, we have

$$\sum_{n=N}^{\infty} d(x_n, x_{n+1}) = \sum_{n=N}^{\infty} d_n \leqslant C \cdot \sum_{n=N}^{\infty} d_F(x_n) \leqslant C \cdot \sum_{n=N}^{\infty} q^n < +\infty$$

for some $0 \leqslant q < 1$. By the triangle inequality, the sequence $\{x_n\}$ is fundamental. □

Lemma 2.6 *(End of the proof of Theorem 1.4). Let $\{x_n\}_{n=0}^{\infty}$ be a sequence, constructed in Lemma* 2.3. *Then $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point of the mapping F.*

Proof. Completeness of (X,d) together with Lemma 2.5 guarantees the convergence of $\{x_n\}_{n=0}^{\infty}$ to some point, say x_*. The inclusion $x_* \in F(x_*)$ has already been checked in Lemma 1.3. □

Proof of Theorem 1.5. As above (cf. Lemma 2.3) for an arbitrary initial point $x_0 \in X$, there exists a sequence $\{x_n\}_{n=0}^{\infty}$ such that for all $n \in \mathbb{N}$, the point x_{n+1} lies in $F(x_n)$ and:

(A_n) $d_n = d(x_n, x_{n+1}) \leqslant \alpha(d_n) \cdot d_F(x_n)$; and
(B_n) $d_F(x_{n+1}) \leqslant \beta(d_n) \cdot d_n$.

Also, $d_F(x_{n+1}) \leqslant \alpha(d_n) \cdot \beta(d_n) \cdot d_F(x_n) < d_F(x_n)$, whereas $d_{n+1} \leqslant \alpha(d_{n+1}) \cdot d_F(x_{n+1}) \leqslant \alpha(d_{n+1}) \cdot \beta(d_n) \cdot d_n$. Hence, as above (cf. Lemma 2.4) $\{d_F(x_n)\}_{n=0}^{\infty}$ is a decreasing sequence which converges to some $\Delta \geqslant 0$. It turns out that $\{(d_n)\}_{n=0}^{\infty}$ is a decreasing sequence, too. Indeed, assume to the contrary that $d_{n+1} \geqslant d_n$ for some $n \in \mathbb{N}$. Then $\alpha(d_{n+1}) \leqslant \alpha(d_n)$ and

$$d_n \leqslant d_{n+1} \leqslant \alpha(d_{n+1}) \cdot \beta(d_n) \cdot d_n \leqslant \alpha(d_n) \cdot \beta(d_n) \cdot d_n < d_n.$$

A contradiction. So $d_n \to \nabla + 0$, $n \to \infty$.

The inequality $d_F(x_{n+1}) \leqslant \alpha(d_n) \cdot \beta(d_n) \cdot d_F(x_n)$, together with the assumption (1) and $d_n \to \nabla + 0$, $n \to \infty$, immediately implies that $\Delta = 0$. It now follows from the inequality $d_F(x_n) \leqslant d_n \leqslant \alpha(0) \cdot d_F(x_n)$ that $\nabla = 0$. The rest of the proof is the same as the proof of Theorem 1.4. □

Proof of Theorem 1.6. We preserve the choice of $x_{n+1} \in F(x_n)$ as in the proofs above (cf. Lemma 2.3). So

$$d_F(x_{n+1}) \leqslant \alpha(d_n) \cdot \beta(d_n) \cdot d_F(x_n) \leqslant \varphi(d_n) \cdot d_F(x_n) \leqslant \varphi\big(d_F(x_n)\big) \cdot d_F(x_n),$$

because $d_F(x_n) \leqslant d_n$ and the majorant $\varphi(\cdot)$ is a decreasing function.

Henceforth, in order to check the convergence of the series $\sum_{n=0}^{\infty} d_F(x_n)$ it suffices to show that the series $\sum_{n=0}^{\infty} \varphi_n(t)$ is convergent for all $t > 0$, where

$$\varphi_0(t) = t, \qquad \varphi_{n+1}(t) = \varphi\big(\varphi_n(t)\big) \cdot \varphi_n(t).$$

In other words, by fixing t and omitting t in the brackets, we only need to show that $\sum_{n=0}^{\infty} \varphi_n < +\infty$. We shall complete the proof by checking that

$$\varphi_n \leqslant \frac{1}{(C_1 n + C_2)^{\frac{1}{p}}} \quad \Longleftrightarrow \quad \frac{1}{\varphi_n^p} \geqslant C_1 n + C_2$$

for some constants $C_1 > 0, C_2 > 0$ and for all sufficiently large n. Let us verify that one can take $C_1 = pC$, $C_2 = \varphi_0^{-p}$. To this end we represent

$$\frac{1}{\varphi_n^p} = \left(\frac{1}{\varphi_n^p} - \frac{1}{\varphi_{n-1}^p}\right) + \cdots + \left(\frac{1}{\varphi_1^p} - \frac{1}{\varphi_0^p}\right) + \frac{1}{\varphi_0^p}.$$

Next, for each $k = 0, 1, \ldots, n-1$, by using the Lagrange theorem we see that

$$\begin{aligned}\frac{1}{\varphi_{k+1}^p} - \frac{1}{\varphi_k^p} &= \frac{1}{(\varphi(\varphi_k) \cdot \varphi_k)^p} - \frac{1}{\varphi_k^p} = \frac{1}{\varphi_k^p} \cdot \frac{1 - (1 - C\varphi_k^p)^p}{(1 - C\varphi_k^p)^p} = \big(s = C \cdot \varphi_k^p\big) \\ &= C \cdot \frac{1}{s} \cdot \frac{(-s) \cdot p \cdot (1-\tau)^{p-1} \cdot (-1)}{(1-s)^p} = pC \cdot \left(\frac{1-\tau}{1-s}\right)^p \cdot \frac{1}{1-\tau} > pC = C_1\end{aligned}$$

because $0 < \tau < s$.

Hence the series $\sum_{n=0}^{\infty} d_F(x_n)$ is convergent and the series $\sum_{n=0}^{\infty} d_n$ is also convergent, because $d_F(x_n) \leqslant d_n \leqslant \sup\{\alpha(t)\colon\ t \geqslant 0\} \cdot d_F(x_n)$. Finally, we see that the sequence $\{x_n\}$ is fundamental. The rest of the proof is standard. □

Now we pass to Example 1.7 and we use the idea of [2,7] except that our construction will avoid rather unexpected constants as $\frac{15}{32}, \frac{7}{24}, \frac{17}{96}$, etc., and will be based only on piecewise linear functions. We define an almost singlevalued mapping $F : [0,1] \to [0,1]$ by setting $F(1) = \{\frac{1}{3}, \frac{3}{4}\}$, $F(x) = \{\frac{2x}{3}\}$ if $0 \leqslant x < \frac{3}{4}$, and $F(x) = \{\frac{1}{2}\}$ otherwise.

Proof of (1). Suppose to the contrary that the mapping F is a $(2-\beta, \beta)$-contraction for some numerical function $\beta : [0, +\infty) \to [0, 1)$. Consider the point $x = 1$.

If one chooses the point $y = \frac{3}{4} \in F(1) = F(x)$ then $d(x, y) = \frac{1}{4}$ and by the condition (B) from Definition 1.1 one obtains $d_F(y) = dist(y, F(y)) = d(\frac{3}{4}, \frac{1}{2}) = \frac{1}{4} \leqslant \beta(\frac{1}{4}) \cdot \frac{1}{4}$, or, $\beta(\frac{1}{4}) \geqslant 1$. A contradiction.

For the other choice $y = \frac{1}{3} \in F(1) = F(x)$ and by the condition (A) from Definition 1.1 we conclude that $d(x, y) = \frac{2}{3} \leqslant \alpha(\frac{2}{3}) \cdot d_F(x) = \alpha(\frac{2}{3}) \cdot \frac{1}{4}$, or, $\alpha(\frac{2}{3}) \geqslant \frac{8}{3} > 2$, which contradicts the fact that $\alpha = 2 - \beta < 2$. □

Proof of (2). Suppose to the contrary that the mapping F is an (a,β)-contraction for some constant $a>1$ and for some numerical function $\beta:[0,+\infty)\to[0,1)$. Consider the point $x=\frac{1}{2}$. Then $F(x)=\{\frac{1}{3}\}$ and there is a unique choice of $y\in F(x)$, namely $y=\frac{1}{3}$. Hence, $F(y)=\{\frac{2}{9}\}$ and $d(x,y)=d_F(x)=\frac{1}{6}$ and $d_F(y)=\frac{1}{3}-\frac{2}{9}=\frac{1}{9}$. So the condition (B) looks as follows: $d_F(y)\leqslant\beta(d(x,y))\cdot d(x,y)\Leftrightarrow\beta(\frac{1}{6})\geqslant\frac{2}{3}$.

As in the previous proof, for the point $x=1$ we have $\beta(\frac{1}{4})\geqslant 1$ for the chosen point $y=\frac{3}{4}$, or we have $a=\alpha(\frac{2}{3})\geqslant\frac{8}{3}$ for the chosen point $y=\frac{1}{3}$. However, $a<\frac{3}{2}$ because of the restriction $\alpha(\frac{1}{6})\cdot\beta(\frac{1}{6})=a\cdot\beta(\frac{1}{6})<1$. A contradiction. □

Proof of (3). Define $\alpha(\cdot)$ and $\beta(\cdot)$ by setting $\alpha(x)=\frac{4}{3}$ for $0\leqslant x\leqslant\frac{1}{2}$, $\alpha(x)=\frac{8}{3}$ for $\frac{1}{2}<x$, $\beta(x)=\frac{2}{3}$ for $0\leqslant x\leqslant\frac{1}{3}$, $\beta(x)=\frac{1}{2}$ for $\frac{1}{3}<x\leqslant\frac{1}{2}$, and $\beta(x)=\frac{1}{3}$ for $\frac{1}{2}<x$.

Then the function $\gamma(\cdot)$ with the property that $\alpha(x)=1+\gamma(1-\beta(x))$ can be defined by $\gamma(\frac{1}{3})=\frac{1}{3}=\gamma(\frac{1}{2})$, $\gamma(\frac{2}{3})=\frac{5}{3}$, and $\gamma(t)=0$ otherwise. Therefore for $p(x)=x-(1-x)\gamma(x)$ we have $p(\frac{1}{3})=\frac{1}{9}=p(\frac{2}{3})$, $p(\frac{1}{2})=\frac{1}{3}$, and $p(x)=x$ otherwise and thus all assumptions (1), (2) of Theorem 1.4 are satisfied.

Let us check that F is really an (α,β)-contraction. For the point $x=1$ we choose $y=\frac{1}{3}\in F(x)$. Then $F(y)=\{\frac{2}{9}\}$. The inequality $d(x,y)\leqslant\alpha(d(x,y))\cdot d_F(x)$ from the condition (A) becomes $\frac{2}{3}\leqslant\alpha(\frac{2}{3})\cdot\frac{1}{4}$ and it holds because $\alpha(\frac{2}{3})=\frac{8}{3}$. Also, the inequality $d_F(y)\leqslant\beta(d(x,y))\cdot d(x,y)(x)$ from the condition (B) becomes $\frac{2}{9}\leqslant\beta(\frac{2}{3})\cdot\frac{2}{3}$ and it holds because $\beta(\frac{2}{3})=\frac{1}{3}$.

For the point $0\leqslant x<1$ there are unique $y\in F(x)$, $y=\frac{2x}{3}$ and $z\in F(y)$, $z=\frac{4x}{9}$. In particular, $d(x,y)=d_F(x)$ and the inequality $d(x,y)\leqslant\alpha(d(x,y))\cdot d_F(x)$ from the condition (A) holds because $\alpha(\cdot)>1$. For checking of (B) we calculate

$$d(x,y)=x-\frac{2x}{3}=\frac{x}{3},\quad 0\leqslant x\leqslant\frac{3}{4},\qquad d(x,y)=x-\frac{1}{2},\quad \frac{3}{4}\leqslant x\leqslant 1,$$
$$d_F(y)=\frac{2x}{3}-\frac{4x}{9}=\frac{2x}{9},\quad 0\leqslant x\leqslant\frac{3}{4},\qquad d_F(y)=\frac{1}{6},\quad \frac{3}{4}\leqslant x\leqslant 1.$$

So for $0\leqslant x\leqslant\frac{3}{4}$, the inequality $d_F(y)\leqslant\beta(d(x,y))\cdot d(x,y)$ from (B) becomes $\frac{2x}{9}\leqslant\beta(\frac{x}{3})\cdot\frac{x}{3}=\frac{2}{3}\cdot\frac{x}{3}$. For points $\frac{3}{4}<x\leqslant\frac{5}{6}$ we have $\frac{1}{4}<d(x,y)=x-\frac{1}{2}\leqslant\frac{1}{3}$ and the desired inequality becomes $\frac{1}{6}=\frac{2}{3}\cdot\frac{1}{4}<\frac{2}{3}\cdot(x-\frac{1}{2})=\beta(x-\frac{1}{2})\cdot(x-\frac{1}{2})$. Finally, if $\frac{5}{6}<x<1$ then $\frac{1}{3}<d(x,y)=x-\frac{1}{2}<\frac{1}{2}$ and the desired inequality becomes $\frac{1}{6}=\frac{1}{2}\cdot\frac{1}{3}<\frac{1}{2}\cdot(x-\frac{1}{2})=\beta(x-\frac{1}{2})\cdot(x-\frac{1}{2})$. □

3. Concluding remarks

Remark 3.1. Lower semicontinuity of a nonnegative numerical function implies its stable positivity, but not vice versa. For example, let F be a Hausdorff k-contraction of (X,d) and x_0 a non-fixed point of F. Define a new mapping, say G, by setting $G(x)=F(x)$, $x\neq x_0$ and by letting $G(x_0)$ be an arbitrary closed subset of X with $dist(x_0,F(x_0))<dist(x_0,G(x_0))$. Then the distance function $d_G(\cdot)$ is evidently stably positive over X, but $d_G(\cdot)$ is not lower semicontinuous at the point x_0. Observe also that the upper semicontinuity of a closed-valued mapping F implies the lower semicontinuity of the distance function d_F which, in turn, implies its stable positivity, due to Lemma 1.3.

Remark 3.2. As it was pointed above, the special case of Theorem 1.4 with $\gamma(s)=s$, $\alpha(s)=2-\beta(s)$, and $p(s)=s^2$ in fact coincides with a recent result of Čirić [2, Theorem 5]. For the mappings with proximinal values it suffices to take $\gamma(s)=0$, $\alpha(s)=1$, and $p(s)=s$ to obtain results of [4,7] and [2, Theorem 7]. In particular, we obtain results on fixed points for singlevalued contractions and for compact-valued Hausdorff contractions. The Mizoguchi–Takahashi theorem [8] is also a special case of Theorem 1.4, as was discussed above (after Definition 1.1).

Remark 3.3. The essential positivity of a numerical function $h:[0,\infty)\to[0,1)$ is equivalent to the property that h admits an increasing positive minorant. Hence the assumption (2)(ii) of Theorem 1.4, after dividing by $1-s$, can be expressed in the following form

$$\gamma(s)=\frac{s}{1-s}-\mu(s)\quad\Longleftrightarrow\quad\gamma(s)=\bigl(s+s^2+s^3+\cdots\bigr)-\mu(s)$$

for some positive increasing function $\mu(\cdot)$.

In particular, if one takes an arbitrary function $\gamma:(0,1]\to[0,+\infty)$ such that $0<\gamma(s)\leqslant s+s^2+\cdots+s^m$, $m\in\mathbb{N}$, then the assumption (2)(i) from Theorem 1.4 is evident, whereas (2)(ii) holds because

$$p(s)=s-(1-s)\gamma(s)\geqslant s-(1-s)\bigl(s+s^2+\cdots+s^m\bigr)=s^{m+1}.$$

Hence, these cases are sources of new fixed point theorems.

Remark 3.4. Recall that Reich [9] showed that the property (R) guarantees existence of fixed points for an arbitrary compact-valued $k(\cdot)$-contraction F of a complete metric space. In 1974 he also proposed [10] the still unresolved problem on possibility of removing the compactness condition. In 1989 Mizoguchi and Takahashi [8] obtained the result for any closed-valued $k(\cdot)$-contractions but under the condition (MT) stronger than (R). So we still have the following interesting open problem.

Question 3.5. Is Theorem 1.4 valid for compact-valued (α, β)-contractions but with (MT) replaced by (R) in the assumption (1)?

Roughly speaking, the difficulty is that Reich proved his theorem by passing to the compact exponent of (X, d) endowed with the Hausdorff distance H_d and by using an appropriate fixed point theorem for singlevalued mapping of such complete metric space into itself. However, in our case a compact-valued (α, β)-contraction does not generate a singlevalued mapping of the compact exponent into itself. Moreover, in the absence of Hausdorff distance one needs to find another kind of "metric" in the compact exponent which agrees with the notion of an (α, β)-contraction.

Remark 3.6. Formally, Theorem 1.6 admits the following abstract form.

Theorem 3.7. *Let numerical functions $\alpha : [0, +\infty) \to [1, +\infty)$ and $\beta : [0, +\infty) \to [0, 1)$ be such that*:

(1) *$\alpha(\cdot)$ is bounded*; *and*
(2) *the product $\alpha(t)\beta(t)$ has a nonincreasing majorant $\varphi(t) < 1$ with $\sum_{n=0}^{\infty} \varphi_n(t) < +\infty$, where $\varphi_0(t) = t$, $\varphi_{n+1}(t) = \varphi(\varphi_n(t)) \cdot \varphi_n(t)$.*

Then every (α, β)-contraction F of a complete metric space (X, d) into itself has a fixed point, whenever the distance function d_F is stably positive.

A special case of Theorem 1.6 for Hausdorff $k(\cdot)$-contractions with power-rate majorants for $k(\cdot)$ yields the main result of [3]. See also [11] for examples of (in this sense) "summable" functions $\varphi(\cdot)$ which have no power-rate upper estimates.

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