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EMBEDDING FINITE POSETS IN CUBES

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In this paper we define the n-cube Q_n as the poset obtained by taking the cartesian product of n chains each consisting of two points. For a finite poset X, we then define $\dim_2 X$ as the smallest positive integer n such that X can be embedded as a subposet of Q_n. For any poset X we then have $\log_2 |X| \leqslant \dim_2 X \leqslant |X|$. For the distributive lattice $L = \underline{2}^X$, $\dim_2 L = |X|$ and for the crown S_n^k, $\dim_2(S_n^k) = n + k$. For each $k \geqslant 2$, there exist positive constants c_1 and c_2 so that for the poset X consisting of all one element and k-element subsets of an n-element set, the inequality $c_1 \log_2 n < \dim_2(X) < c_2 \log_2 n$ holds for all n with $k < n$. A poset is called Q-critical if $\dim_2(X - x) < \dim_2(X)$ for every $x \in X$. We define a join operation $\oplus$ on posets under which the collection $\mathcal{Q}$ of all Q-critical posets which are not chains forms a semigroup in which unique factorization holds. We then completely determine the subcollection $\mathcal{M} \subseteq \mathcal{Q}$ consisting of all posets X for which $\dim_2(X) = |X|$.

1. Introduction

A partially ordered set or poset is a pair (X, P) where X is a set and P is a reflexive, antisymmetric, and transitive relation on X. The notations $(x, y) \in P$, $x \leqslant y$ in P, and $x P y$ are used interchangeably. If neither (x, y) nor (y, x) is in P, we say x and y are incomparable and write $x I y$. For convenience we frequently denote a poset by a single symbol; we also use $X = Y$ and $X \subseteq Y$ for X is isomorphic to Y and X is isomorphic to a subposet of Y.

The n-cube Q_n is the set of all 0–1 sequences of length n. We consider Q_n as a poset with the natural partial ordering P defined by $f P g$ iff $f(i) \leqslant g(i)$ for all $i \leqslant n$. Q_n is then isomorphic to the poset consisting of all subsets of an n-element set ordered by inclusion. Equivalently, Q_n is the poset obtained by taking the cartesian product of n copies of the two element chain $0 < 1$.

In this paper we denote an n-element chain by $\underline{n}$ and label the points of $\underline{n}$ so that $0 < 1 < 2 < \ldots < n-1$ in $\underline{n}$. With this notation, $Q_n = \underline{2}^n$.

We also denote an n-element antichain (a poset in which distinct points are always incomparable) by $\bar{n}$.

For a poset (X, P), Dushnik and Miller [4] defined the dimension of (X, P), denoted $\dim(X, P)$, as the smallest positive integer n for which there exist linear orders $L_1, L_2, ..., L_n$ on X such that $x \leq y$ in P iff $x \leq y$ in each L_i. Equivalently, Ore [8] defined $\dim(X, P)$ as the smallest positive integer n for which $(X, P) \subseteq C_1 \times C_2 \times ... \times C_n$ where each C_i is a chain. For a finite poset X and an integer $k \geq 2$, we define* the k-dimension of X, $\dim_k X$, as the smallest positive integer n for which $X \subseteq \underline{k}^n$. In this paper we are concerned primarily with the case $k = 2$, i.e., the embedding of finite posets in cubes. We refer the reader to [12, 13] for theorems when $k \geq 3$. We also note that the problem of embedding graphs in cubes is discussed in [5, 6].

For each $n \geq 1$, the length of the longest chain in Q_n is easily seen to be $n + 1$. If we take $\dim_2(\underline{1})$ to be zero by convention, then we have $\dim_2 \underline{n} = n - 1$ for all $n \geq 1$. On the other hand, it follows immediately from Sperner's theorem [10], that $\dim_2(\bar{n})$ is the smallest positive integer t for which $\binom{t}{[t/2]} \geq n$.

If $X \subseteq Q_n$ and $f: X \to Q_n$ is an embedding, then the map $g: \hat{X} \to Q_n$ defined by $g(x)(i) = 1 - f(x)(i)$ is an embedding of the dual $\hat{X}$ of X in Q_n and thus $\dim_2(X) = \dim_2(\hat{X})$.

For any poset X we have the trivial lower bound $\dim_2(X) \geq \log_2|X|$ since $|Q_n| = 2^n$. If $X = \{x_1, x_2, ..., x_p\}$, then the map $f: X \to Q_p$ defined by $f(x_i)(j) = 0$ if $x_j \leq x_i$ in X and $f(x_i)(j) = 1$ otherwise is an embedding of $\hat{X}$ in Q_p and thus we have the upper bound $\dim_2(X) \leq |X|$.

A poset X for which $\dim_2(X-x) < \dim_2(X)$ for every $x \in X$ is called a Q-critical poset. Every chain of two or more points is Q-critical. We denote the set of all Q-critical posets which are not chains by $\mathbf{Q}$. A poset X for which $\dim_2(X) = |X|$ is called an MQ poset and we denote the set of all MQ posets by $\mathcal{M}$. Clearly every MQ poset is also Q-critical.

For arbitrary posets X and Y we define the join (or ordinal sum) of X and Y, denoted $X \oplus Y$, as the poset obtained by placing all elements of X under all elements of Y. This operation is analogous to the join operation $G_1 + G_2$ defined for graphs by Zykov for if $G(X)$ is the comparability graph of X, then $G(X \oplus Y) = G(X) + G(Y)$. However, in this paper, we will use the symbol + to denote the free sum or cardinal sum of posets as defined by Birkhoff [1, p. 55]. In subsequent sections of this paper, we will show that both $(\mathcal{M}, \oplus)$ and $(\mathbf{Q}, \oplus)$ are semigroups

* This concept has been studied by Novak [7] who used the terminology k-pseudo-dimension.

with no prime posets which are composite in the semigroup of all posets under $\oplus$. We will then completely determine the set of all prime MQ posets.

For any pair of posets we have $\dim_2(X \times Y) \leqslant \dim_2(X) + \dim_2(Y)$ and $\dim(X \times Y) \leqslant \dim X + \dim Y$. If X and Y have universal bounds, then $\dim_2(X \times Y) = \dim_2(X) + \dim_2(Y)$ and $\dim(X \times Y) = \dim X + \dim Y$ (in particular, $\dim Q_n = n$). In subsequent sections, we will discuss the analogy between $\dim_2(X)$ and $\dim X$ in more detail.

2. Embedding the join of posets in cubes

In this section, we produce a formula for computing $\dim_2(X)$ in terms of its prime join factors.

Lemma 2.1. $\dim_2(X \oplus Y) \geqslant \dim_2(X) + \dim_2(Y)$ *for every* X, Y.

Proof. Let $f : X \oplus Y \to Q_n$ be an embedding. Define $A \subseteq \{1, 2, ..., n\}$ by $A = \{i$: there exist $x, x' \in X$ such that $f(x)(i) = 0$ and $f(x')(i) = 1\}$. We observe that for each $y \in Y$ and for each $i \in A$, $f(y)(i) = 1$. Now define $B = \{i$: there exist $y, y' \in Y$ such that $f(y)(i) = 0$ and $f(y')(i) = 1\}$. Then it is easy to see that A and B are disjoint and that $|A| \geqslant \dim_2(X)$ and $|B| \geqslant \dim_2(Y)$.

Lemma 2.2. *If X has a greatest element and Y a least element, then* $\dim_2(X \oplus Y) \geqslant 1 + \dim_2(X) + \dim_2(Y)$.

Proof. Let $f : X \oplus Y \to Q_n$ be any embedding and let A and B be defined as in the preceding lemma. Let x be the greatest element of X and let y be the least element of Y. Then define $C = \{i: f(x)(i) = 0$ and $f(y)(i) = 1\}$. It follows that A, B and C are mutually disjoint and that C is nonempty.

Theorem 2.3. $\dim_2(X \oplus Y) = \dim_2(X) + \dim_2(Y)$ *unless X has a greatest element and Y has a least element. In that case* $\dim_2(X \oplus Y) = 1 + \dim_2(X) + \dim_2(Y)$.

Proof. Let $f : X \to Q_n$ and $g : Y \to Q_m$ be embeddings. Define $h : X \oplus Y \to Q_{m+n}$ by $h(x)(i) = 0$ for $1 \leqslant i \leqslant m$; $h(x)(i) = f(x)(i)$ for $m + 1 \leqslant i \leqslant n+m$; $h(y)(i) = g(y)(i)$ for $1 \leqslant i \leqslant m$; and $h(y)(i) = 1$ for $m+1 \leqslant i \leqslant m + n$.

Then h is an embedding of $X \oplus Y$ unless $h(x) = h(y)$ for some $x \in X$, $y \in Y$. It is easy to see that this may occur only if X has a greatest element and Y has a least element. In this case, it suffices to add one additional term to these sequences: a zero for each point in X and a one for each point in Y.

It should be noted that the order of factors in the join operation is important.

Corollary 2.4. *If $X = P_1 \oplus P_2 \oplus P_3 \oplus \ldots \oplus P_t$ is the decomposition of X into prime join factors, then* $\dim_2(X) = s + \Sigma \dim_2(P_i)$, *where s is the number of subscripts $i \leqslant t - 1$ such that $P_i = P_{i+1} = \underline{1}$.*

3. The structure of Q-critical posets

It follows immediately from the formula for $\dim_2(X)$ given in the preceding section that the chains are the only Q-critical posets which have $\underline{1}$ as a join factor. Also we see that $X \oplus Y$ is Q-critical and does not have $\underline{1}$ as a join factor iff both X and Y are Q-critical and neither has $\underline{1}$ as a join factor. Similarly $X \oplus Y$ is MQ iff both X and Y are MQ.

Example 3.1. For each $n \geqslant 2$, let $L_n = \underline{n-1} + \underline{1}$. Since every chain in Q_{n-1} of length $n - 1$ contains at least one of the two universal bounds and these points compare with every other point of Q_{n-1}, it follows that L_n cannot be embedded in Q_{n-1} and thus $Q(L_n) = n$. Clearly each L_n is a prime MQ poset.

Example 3.2. For each $n \geqslant 4$, let N_n denote the poset consisting of two disjoint chains $a_1 < a_2$ and $b_1 < b_2 < \ldots < b_{n-2}$ with a_2 also covering b_{n-3}. Then $Q(N_n) = n - 1$.

Example 3.3. The only MQ poset on two points is $L_2 = \overline{2}$. The only MQ posets on three points are L_3 and $\overline{3}$. The only prime MQ posets on four points are L_4, $L_3 + \underline{1}$ and $\overline{4}$. The only prime MQ poset on five points is L_5.

Lemma 3.4. *If a is a maximal element of a finite poset X and $X - a$ does not have a greatest element, then* $\dim_2 X \leqslant 1 + \dim_2(X - a)$.

Proof. Let $f : (X - a) \to Q_t$ be an embedding. Define $g : X \to Q_{t+1}$ by $g(x)(i) = f(x)(i)$ for every $x \in X - a$ and every $i \leq t$; $g(a)(i) = 1$ for every $i \leq t$; $g(x)(t+1) = 0$ if $x \leq a$ and $g(x)(t+1) = 1$ if $x \leq a$ for every $x \in X$. It follows easily that g is an embedding of X in Q_{t+1}.

Theorem 3.5. *For $n \geq 5$, the only prime MQ poset is L_n.*

Proof. Assume validity for $n \leq k$, where $k \geq 5$. Now suppose that X is a prime MQ poset on $k+1$ points. Since X is prime, it has two or more maximal elements. Suppose that X has only two maximal elements a and b. If a is the greatest element of $X - b$ and b is the greatest element of $X - a$, then X has L_2 as a join factor. Now suppose that a is the greatest element of $X - b$ but that b is not the greatest element of $X - a$. Choose $c \in X$ such that a covers c but b and c are incomparable. By Lemma 3.4, $X - a$ is an MQ poset and if $X - a$ is composite so is X. If $X - a$ is L_k, then X is either L_{k+1} or N_{k+1} and since N_{k+1} is not MQ, X must be L_{k+1}.

Now suppose that a is not the greatest element of $X - b$ and that b is not the greatest element of $X - a$. Choose elements c, d such that a covers c, b covers d, but a is incomparable with d and b is incomparable with c. Now $X - a$ and $X - b$ are both MQ posets and if either has a join factor, so does X. Hence we may assume that $X - a = X - b = L_k$. But it is easy to see that no such poset exists. The contradiction shows that X must have at least three maximal elements.

Choose any three maximal elements a, b and c. Then by Lemma 3.4, we conclude that each of the posets $X - a$, $X - b$ and $X - c$, must be L_k. Clearly this is not possible.

4. Embedding distributive lattices in cubes

In this section, we develop a formula for $\dim_2(L)$ when L is a distributive lattice. We employ the concept of exponentiation (cardinal power) of posets and define X^Y as the collection of all order reversing functions from Y to X with $f \leq g$ in X^Y iff $f(y) \leq g(y)$ in X for every $y \in Y$. We refer the reader to [1, p. 57] for elementary properties of X^Y. In particular, we note that for each distributive lattice L there is a unique poset X for which $L = \underline{2}^X$.

Theorem 4.1. *If $L = \underline{2}^X$ is a distributive lattice, then* $\dim_2(L) = |X|$.

Proof. Let $|X| = n$. Then $\underline{2}^X \subseteq \underline{2}^{\bar{n}} = \underline{2}^n = Q_n$ and thus $\dim_2(L) \leq n$. On the other hand, if we let Y be a linear extension of X, then $\underline{n+1} = \underline{2}^{\underline{n}} = \underline{2}^Y \subseteq \underline{2}^X$ and thus $n = \dim_2(\underline{n+1}) \leq \dim_2(L)$.

Theorem 4.1 is a special case of a result for embedding distributive lattices in chains of bounded lengths. We state this result and refer the reader to [12] for the proof.

Theorem 4.2. *Let $L = \underline{2}^X$ be a distributive lattice and let $k \geq 2$ be a positive integer. Then the smallest positive integer t for which L can be embedded in $\underline{k}^t$ is equal to the smallest positive integer s for which there exists a decomposition $X = C_1 \cup C_2 \cup \ldots \cup C_s$, where each C_i is a chain containing at most $k-1$ points.*

We note that Theorem 4.2 includes Dilworth's elegant result [2] for the dimension of a distributive lattice, $\dim \underline{2}^X = \text{width } X$.

5. Embedding crowns in cubes

For $n \geq 3$, $k \geq 0$, the crown S_n^k is defined in [11] as a poset with $n+k$ maximal elements $a_1, a_2, \ldots, a_{n+k}$ and $n+k$ minimal elements $b_1, b_2, \ldots, b_{n+k}$. Each b_i is incomparable with $a_i, a_{i+1}, \ldots, a_{i+k}$ (cyclically) and less than the remaining $n-1$ maximal elements. In [11], it is shown that $\dim S_n^k = \{2(n+k)/(k+2)\}$. To determine $\dim_2(S_n^k)$, we note that $\dim_2(S_n^k)$ is the smallest integer t for which there exists an order preserving map $f: S_n^k \to Q_t$ such that for every incomparable max-min pair $a, b \in S_n^k$, there exists $i \leq t$ with $f(b)(i) = 1$ and $f(a)(i) = 0$.

Theorem 5.1. $\dim_2(S_n^k) = n + k$ *for every* $n \geq 3$, $k \geq 0$.

Proof. The map $f: S_n^k \to Q_{n+k}$, defined by $f(b_j)(i) = 1$ if $i = j$, 0 otherwise, and $f(a)(i) = 0$ if $a \, I \, b_i$, 1 otherwise, shows that $\dim_2(S_n^k) \leq n + k$. Now suppose that $\dim_2(S_n^k) = t$. Choose an embedding $g: S_n^k \to Q_t$ with

$$M = \sum_{i=1}^{t} \sum_{x \in S_n^k} g(x)(i)$$

as small as possible. For each $i \leq t$, let B_i denote the set of minimal elements b for which $g(b)(i) = 1$. It is clear that each $B_i \neq \phi$ and that $g(a)(i) = 0$ iff a is incomparable with each $b \in B_i$. For each i, choose a maximal element a^i such that $g(a^i)(i) = 0$ and let A_i be the set of all maximal elements a such that $g(a)(i) = 0$. Then B_i is a subset of the set D_i consisting of all $k + 1$ minimal elements which are incomparable with a^i. Subscripts interpreted cyclically impose a linear order on each D_i. Then for each i, let b^i be the largest element in B_i as determined by this linear order on D_i. Suppose that there exist distinct integers $i, j \leqslant t$ with $b^i = b^j$. It follows that either $A_i \subseteq A_j$ or $A_j \subseteq A_i$; we assume without loss of generality that $A_i \subseteq A_j$. Then define $h : S_n^k \to Q_t$ by $h(b^i)(i) = 0$ and $h(x) = g(x)$ otherwise. It is easy to see that h is an embedding but M has been reduced by 1. The contradiction shows that $b^i \neq b^j$ for every distinct pair i,j and thus $\dim_2 S_n^k = t \geqslant n + k$.

6. Embedding collections of sets in cubes

Dushnik [3] and Spencer [9] use the notation $N(n, k)$ for the dimension of the poset X consisting of all one element and $(k - 1)$-element subsets of an n-element set ($n \geqslant k \geqslant 3$) ordered by inclusion. We will denote $\dim_2(X)$ by $Q(n, k)$. It is easy to see that the following alternate definition of $Q(n, k)$ is valid.

Lemma 6.1. *$Q(n, k)$ is the smallest integer t for which there exists a collection $A_1, A_2, ..., A_t$ of subsets of $\{1, 2, ..., n\}$ so that for each k-element subset $F \subseteq \{1, 2, ..., n\}$ and each $a \in F$, there exists $i \leqslant t$ such that $F \cap A_i = \{a\}$.*

Since $|X| = n + \binom{n}{k-1}$, we see that for each $k \geqslant 3$, there exists a positive constant c_1 so that $Q(n, k) \geqslant c_1 \log_2 n$ for all $n \geqslant k$. We can modify Spencer's probabilistic argument [9] to produce the following upper bound

Theorem 6.2. *For each $k \geqslant 3$, there exists a positive constant c_2 so that $Q(n, k) \leqslant c_2 \log_2 n$ for all $n \geqslant k$.*

Proof. Let s be a positive integer. Then there are 2^{ns} s-tuples of subsets of $\{1, 2, ..., n\}$. For each k-element subset $F \subseteq \{1, 2, ..., n\}$ and each $a \in F$, $(2^k - 1)^s 2^{ns-ks}$ of these s-tuples fail to satisfy the requirements of Lemma 6.1. There are $\binom{n}{k}k < n^{k+1}$ ways to choose F and a. In order to insure the existence of an s-tuple of subsets of $\{1, 2, ..., n\}$ satisfying the requirements of the lemma, it is sufficient to choose s so that

$n^{k+1}(2^k-1)^s 2^{ns-ks} < 2^{ns}$. But it is easy to see that this inequality holds if

$$s > \{(k+1)/[k - \log_2(2^k - 1)]\}\log_2 n,$$

and the theorem is proved.

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