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# A new version of Bishop frame and an application to spherical images

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ABSTRACT

In this work, we introduce a new version of Bishop frame using a common vector field as binormal vector field of a regular curve and call this frame as "Type-2 Bishop Frame". Thereafter, by translating type-2 Bishop frame vectors to the center of unit sphere of three-dimensional Euclidean space, we introduce new spherical images and call them as type-2 Bishop spherical images. Frenet–Serret apparatus of these new spherical images are obtained in terms of base curve's type-2 Bishop invariants. Additionally, we express some interesting relations and illustrate two examples of our main results.



## 1. Introduction

The local theory of space curves are mainly developed by the Frenet–Serret theorem which expresses the derivative of a geometrically chosen basis of $\mathbb{R}^3$ by the aid of itself is proved. Then it is observed that by the solution of some of special ordinary differential equations, further classical topics, for instance spherical curves, Bertrand curves, involutes and evolutes are investigated. One of the mentioned works is spherical images of a regular curve in the Euclidean space. It is a well-known concept in the local differential geometry of curves. Such curves are obtained in terms of the Frenet–Serret vector fields (for details, see [11]).

Bishop frame, which is also called alternative or parallel frame of the curves, was introduced by L.R. Bishop in 1975 by means of parallel vector fields. Recently, many research papers related to this concept have been treated in the Euclidean space, see [7,8]; in Minkowski space, see [5,6,9,16,21]; and in dual space, see [17]. And, recently, this special frame is extended to study of canal and tubular surfaces, we refer to [13–15].

Bishop and Frenet–Serret frames have a common vector field, namely the tangent vector field of the Frenet–Serret frame. In this work, using common vector field as the binormal vector of Frenet–Serret frame, we introduce a new version of the Bishop frame. We call it as "Type-2 Bishop Frame" of regular curves. Thereafter, translating new frame's vector fields to the center of unit sphere, we obtain new spherical images. We call them as "Type-2 Bishop Spherical Images" of regular curves. We also distinguish them by the names, $\zeta_1$, $\zeta_2$ and Binormal Bishop spherical images. Besides, we investigate their Frenet–Serret apparatus according to type-2 Bishop invariants. We establish some relations on general helix and slant helix of spherical images and illustrate two examples of main results.

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## 2. Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space $\mathbb{E}^3$ are briefly presented (a more complete elementary treatment can be found in [11]).

The Euclidean 3-space $\mathbb{E}^3$ provided with the standard flat metric given by

$$\langle,\rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where $(x_1, x_2, x_3)$ is a rectangular coordinate system of $\mathbb{E}^3$. Recall that, the norm of an arbitrary vector $a \in \mathbb{E}^3$ is given by $\|a\| = \sqrt{\langle a, a\rangle}$. $\varphi$ is called a unit speed curve if velocity vector $v$ of $\varphi$ satisfies $\|v\| = 1$. For vectors $v, w \in \mathbb{E}^3$ it is said to be orthogonal if and only if $\langle v, w\rangle = 0$. Let $\vartheta = \vartheta(s)$ be a regular curve in $\mathbb{E}^3$. If the tangent vector of this curve forms a constant angle with a fixed constant vector $U$, then this curve is called a general helix or an inclined curve. The sphere of radius $r > 0$ and with center in the origin in the space $\mathbb{E}^3$ is defined by

$$S^2 = \{p = (p_1, p_2, p_3) \in \mathbb{E}^3\colon\ \langle p, p\rangle = r^2\}.$$

Denote by $\{T, N, B\}$ the moving Frenet–Serret frame along the curve $\varphi$ in the space $\mathbb{E}^3$. For an arbitrary curve $\varphi$ with first and second curvature, $\kappa$ and $\tau$ in the space $\mathbb{E}^3$, the following Frenet–Serret formulae is given in [11]

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{1}$$

where

$$\langle T, T\rangle = \langle N, N\rangle = \langle B, B\rangle = 1,$$
$$\langle T, N\rangle = \langle T, B\rangle = \langle T, N\rangle = \langle N, B\rangle = 0.$$

Here, curvature functions are defined by $\kappa = \kappa(s) = \|T'(s)\|$ and $\tau(s) = -\langle N, B'\rangle$.

Let $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ be vectors in $\mathbb{E}^3$ and $e_1, e_2, e_3$ be positive oriented natural basis of $\mathbb{E}^3$. Cross product of $u$ and $v$ is defined by

$$u \times v = \begin{vmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Mixed product of $u$, $v$ and $w$ is defined by the determinant

$$[u, v, w] = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Torsion of the curve $\varphi$ is given by the aid of the mixed product

$$\tau = \frac{[\varphi', \varphi'', \varphi''']}{\kappa^2}.$$

In the rest of the paper, we suppose everywhere $\kappa \neq 0$ and $\tau \neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. One can express parallel transport of an orthonormal frame along a curve simply by parallel transporting each component of the frame [8]. The tangent vector and any convenient arbitrary basis for the remainder of the frame are used (for details, see [10]). The Bishop frame is expressed as [8,10]

$$\begin{bmatrix} T' \\ M_1' \\ M_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \end{bmatrix}.$$

Here, we shall call the set $\{T, M_1, M_2\}$ as *Bishop trihedra* and $k_1$ and $k_2$ as *Bishop curvatures*. The relation matrix may be expressed as

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta(s) & \sin\theta(s) \\ 0 & -\sin\theta(s) & \cos\theta(s) \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \end{bmatrix},$$

where $\theta(s) = \arctan\frac{k_2}{k_1}$, $\tau(s) = \theta'(s)$ and $\kappa(s) = \sqrt{k_1^2 + k_2^2}$. Here, Bishop curvatures are defined by

$$\begin{cases} k_1 = \kappa\cos\theta(s), \\ k_2 = \kappa\sin\theta(s). \end{cases}$$

Izumiya and Takeuchi [12] have introduced the concept of slant helix in the Euclidean 3-space $\mathbb{E}^3$ saying that the normal lines makes a constant angle with a fixed direction [12]. They characterize a slant helix by the condition that the function

$$\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)'$$

is constant. In further researches, spherical images, the tangent and the binormal indicatrix and some characterizations of such curves in Euclidean space and Lorentz–Minkowski spaces are presented (see [1–4,18,19]).

It is well known that for a unit speed curve with non-vanishing curvatures the following propositions hold [11,12].

**Proposition 2.1.** *Let $\varphi = \varphi(s)$ be a regular curve with curvatures $\kappa$ and $\tau$. The curve $\varphi$ lies on the surface of a sphere if and only if*

$$\frac{\tau}{\kappa} + \left[\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right]' = 0.$$

**Proposition 2.2.** *Let $\varphi = \varphi(s)$ be a regular curve with curvatures $\kappa$ and $\tau$. $\varphi$ is a general helix if and only if*

$$\frac{\kappa}{\tau} = constant.$$

**Proposition 2.3.** *Let $\varphi = \varphi(s)$ be a regular curve with curvatures $\kappa$ and $\tau$. $\varphi$ is a slant helix if and only if*

$$\sigma(s) = \left[\frac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\left(\frac{\tau}{\kappa}\right)'\right] = constant.$$

## 3. Type-2 Bishop frame of a regular curve

Let $\alpha = \alpha(s)$ be a unit speed regular curve in $\mathbb{E}^3$ and (1) be its Frenet–Serret frame. Let us express a relatively parallel adapted frame, denoting derivatives with respect to arc length by a dash

$$\begin{bmatrix}\zeta_1'\\ \zeta_2'\\ B'\end{bmatrix} = \begin{bmatrix}0 & 0 & -\epsilon_1\\ 0 & 0 & -\epsilon_2\\ \epsilon_1 & \epsilon_2 & 0\end{bmatrix}\begin{bmatrix}\zeta_1\\ \zeta_2\\ B\end{bmatrix}. \tag{2}$$

We shall call this frame as “Type-2 Bishop Frame”. In order to investigate this new frame’s relation with Frenet–Serret frame, first we write

$$B' = -\tau N = \epsilon_1\zeta_1 + \epsilon_2\zeta_2.$$

Taking the norm of both sides, we have

$$\tau = \sqrt{\epsilon_1^2 + \epsilon_2^2}. \tag{3}$$

We write the tangent vector according to frame $\{\zeta_1, \zeta_2, B\}$ as

$$T = \sin\theta(s)\zeta_1 - \cos\theta(s)\zeta_2,$$

and differentiate with respect to $s$

$$T' = \kappa N = \theta'(s)\big(\cos\theta(s)\zeta_1 + \sin\theta(s)\zeta_2\big) + \sin\theta(s)\zeta_1' - \cos\theta(s)\zeta_2'. \tag{4}$$

Substituting $\zeta_1' = -\epsilon_1 B$ and $\zeta_2' = -\epsilon_2 B$ to Eq. (4), we have

$$\kappa N = \theta'(s)\big(\cos\theta(s)\zeta_1 + \sin\theta(s)\zeta_2\big).$$

In the above equation let us take $\theta'(s) = \kappa(s)$. So we immediately arrive at

$$N = \cos\theta(s)\zeta_1 + \sin\theta(s)\zeta_2.$$

Considering the obtained equations, the relation matrix between Frenet–Serret and type-2 Bishop frames can be expressed

$$\begin{bmatrix}T\\ N\\ B\end{bmatrix} = \begin{bmatrix}\sin\theta(s) & -\cos\theta(s) & 0\\ \cos\theta(s) & \sin\theta(s) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\zeta_1\\ \zeta_2\\ B\end{bmatrix}. \tag{5}$$

Besides, Eq. (3) can also be written as

$$1=\sqrt{\frac{\epsilon_1^2}{\tau^2}+\frac{\epsilon_2^2}{\tau^2}},\tag{6}$$

and so by (6), we may express

$$\begin{cases}\epsilon_1(s)=-\tau\cos\theta(s),\\ \epsilon_2(s)=-\tau\sin\theta(s).\end{cases}$$

By this way, we conclude

$$\theta(s)=\arctan\left(\frac{\epsilon_2}{\epsilon_1}\right).$$

The frame $\{\zeta_1,\zeta_2,B\}$ is properly oriented, and $\tau$ and $\theta(s)=\int_0^s\kappa(s)\,ds$ are polar coordinates for the curve $\alpha=\alpha(s)$. We shall call the set $\{\zeta_1,\zeta_2,B,\epsilon_1,\epsilon_2\}$ as type-2 Bishop invariants of the curve $\alpha=\alpha(s)$. In Section 5, we shall give examples how to find a regular curve's type-2 Bishop trihedra.

## 4. New spherical images of a regular curve

### 4.1. $\zeta_1$ Bishop spherical image

**Definition 4.1.** Let $\alpha=\alpha(s)$ be a regular curve in $\mathbb{E}^3$. If we translate of the first vector field of type-2 Bishop frame to the center $O$ of the unit sphere $S^2$, we obtain a spherical image $\varphi=\varphi(s_\varphi)$. This curve is called $\zeta_1$ *Bishop spherical image or indicatrix of the curve* $\alpha=\alpha(s)$.

Let $\varphi=\varphi(s_\varphi)$ be $\zeta_1$ Bishop spherical image of a regular curve $\alpha=\alpha(s)$. We shall investigate relations among type-2 Bishop and Frenet–Serret invariants. First, we differentiate

$$\varphi'=\frac{d\varphi}{ds_\varphi}\frac{ds_\varphi}{ds}=-\epsilon_1 B.$$

Here, we shall denote differentiation according to $s$ by a dash, and differentiation according to $s_\varphi$ by a dot. Taking the norm of both sides the equation above, we have

$$T_\varphi=B\tag{7}$$

and

$$\frac{ds_\varphi}{ds}=\epsilon_1.\tag{8}$$

We differentiate (7) as

$$T'_\varphi=\dot{T}_\varphi\frac{ds_\varphi}{ds}=\epsilon_1\zeta_1+\epsilon_2\zeta_2.$$

So, we have

$$\dot{T}_\varphi=-\zeta_1-\frac{\epsilon_2}{\epsilon_1}\zeta_2.$$

Since, we have the first curvature and the principal normal of $\varphi$

$$\kappa_\varphi=\|\dot{T}_\varphi\|=\sqrt{1+\left(\frac{\epsilon_2}{\epsilon_1}\right)^2}\tag{9}$$

and

$$N_\varphi=-\frac{\zeta_1}{\kappa_\varphi}-\frac{\epsilon_2}{\epsilon_1\kappa_\varphi}\zeta_2.$$

Cross product of $T_\varphi\times N_\varphi$ gives us the binormal vector field of $\zeta_1$ Bishop spherical image of $\alpha=\alpha(s)$

$$B_\varphi=\frac{\epsilon_2}{\epsilon_1\kappa_\varphi}\zeta_1-\frac{1}{\kappa_\varphi}\zeta_2.$$

Using the formula of the torsion, we write a relation

$$\tau_\varphi = -\frac{\epsilon_1 (\frac{\epsilon_2}{\epsilon_1})'}{\epsilon_1^2 + \epsilon_2^2}. \tag{10}$$

Considering Eqs. (9) and (10), we give:

**Corollary 4.2.** *Let $\varphi = \varphi(s_\varphi)$ be $\zeta_1$ Bishop spherical image of the curve $\alpha = \alpha(s)$. If the ratio of type-2 Bishop curvatures of $\alpha = \alpha(s)$ is constant ($\frac{\epsilon_2}{\epsilon_1} = constant$, i.e.), then, the $\zeta_1$ Bishop spherical indicatrix $\varphi(s_\varphi)$ is a circle in the osculating plane.*

**Proof.** Let $\varphi = \varphi(s_\varphi)$ be $\zeta_1$ Bishop spherical image of a regular curve $\alpha = \alpha(s)$. If the ratio of type-2 Bishop curvatures of $\alpha = \alpha(s)$ is constant, in terms of Eqs. (9) and (10), we have $\kappa_\varphi = constant$ and $\tau_\varphi = 0$, respectively. Therefore, $\varphi$ is a circle in the osculating plane. □

**Theorem 4.3.** *Let $\varphi = \varphi(s_\varphi)$ be $\zeta_1$ Bishop spherical image of a regular curve $\alpha = \alpha(s)$. There exists a relation among Frenet–Serret invariants $\varphi(s_\varphi)$ and type-2 Bishop curvatures of $\alpha = \alpha(s)$ as follows*

$$\frac{\epsilon_2}{\epsilon_1} = \int_0^{s_\varphi} \kappa_\varphi^2 \tau_\varphi \, ds_\varphi. \tag{11}$$

**Proof.** Let $\varphi = \varphi(s_\varphi)$ be $\zeta_1$ Bishop spherical image of a regular curve $\alpha = \alpha(s)$. Then, Eqs. (8) and (10) hold. Using (8) in (10) and by the chain rule, we have

$$\tau_\varphi = -\frac{\epsilon_1 \frac{d}{ds_\varphi}(\frac{\epsilon_2}{\epsilon_1}) \frac{ds_\varphi}{ds}}{\epsilon_1^2 + \epsilon_2^2}. \tag{12}$$

Substituting (9) to (12) and integrating both sides, we have (11) as desired. □

In the light of Propositions 2.2 and 2.3, we express the following theorems without proofs:

**Theorem 4.4.** *Let $\varphi = \varphi(s_\varphi)$ be $\zeta_1$ Bishop spherical image of a regular curve $\alpha = \alpha(s)$. If $\varphi$ is a **general helix**, then, type-2 Bishop curvatures of $\alpha$ satisfy*

$$\frac{\epsilon_1^2 (\frac{\epsilon_2}{\epsilon_1})'}{(\epsilon_1^2 + \epsilon_2^2)^{\frac{3}{2}}} = constant.$$

**Theorem 4.5.** *Let $\varphi = \varphi(s_\varphi)$ be $\zeta_1$ Bishop spherical image of a regular curve $\alpha = \alpha(s)$. If $\varphi$ is a **slant helix**, then, type-2 Bishop curvatures of $\alpha$ satisfy*

$$\left[\frac{\epsilon_1^2 (\frac{\epsilon_2}{\epsilon_1})'}{(\epsilon_1^2 + \epsilon_2^2)^{\frac{3}{2}}}\right]' \frac{(\epsilon_1^2 + \epsilon_2^2)^4}{\epsilon_1^3 [(\frac{\epsilon_2}{\epsilon_1})'^2 + (\epsilon_1^2 + \epsilon_2^2)^3]^{\frac{3}{2}}} = constant.$$

We know that $\varphi$ is a spherical curve, so, by Proposition 2.1 one can prove:

**Theorem 4.6.** *Let $\varphi$ be $\zeta_1$ Bishop spherical image of a regular curve $\alpha = \alpha(s)$. Type-2 Bishop curvatures of the regular curve $\alpha = \alpha(s)$ satisfy the following differential equation*

$$\frac{\epsilon_1^2 (\frac{\epsilon_2}{\epsilon_1})'}{(\epsilon_1^2 + \epsilon_2^2)^{\frac{3}{2}}} - \left[\frac{\epsilon_1 \epsilon_2}{\sqrt{\epsilon_1^2 + \epsilon_2^2}}\right]' = constant.$$

**Remark 4.7.** Considering $\theta_\varphi = \int_0^{s_\varphi} \kappa_\varphi \, ds_\varphi$ and using the transformation matrix, one can obtain the type-2 Bishop trihedra $\{\zeta_{1\varphi}, \zeta_{2\varphi}, B_\varphi\}$ of the curve $\varphi = \varphi(s_\varphi)$.

*4.2. $\zeta_2$ Bishop spherical image*

**Definition 4.8.** Let $\alpha = \alpha(s)$ be a regular curve in $\mathbb{E}^3$. If we translate of the second vector field of type-2 Bishop frame to the center $O$ of the unit sphere $S^2$, we obtain a spherical image $\beta = \beta(s_\beta)$. This curve is called $\zeta_2$ *Bishop spherical image or indicatrix of the curve* $\alpha = \alpha(s)$.

Let $\beta = \beta(s_\beta)$ be $\zeta_2$ spherical image of the regular curve $\alpha = \alpha(s)$. We can write that

$$\beta' = \frac{d\beta}{ds_\beta}\frac{ds_\beta}{ds} = -\epsilon_2 B.$$

Similar to $\zeta_1$ Bishop spherical image, one can have

$$T_\beta = B \tag{13}$$

and

$$\frac{ds_\beta}{ds} = \epsilon_2. \tag{14}$$

So, by differentiating of the formula (13), we get

$$T'_\beta = \dot{T}_\beta \frac{ds_\beta}{ds} = \epsilon_1\zeta_1 + \epsilon_2\zeta_2$$

or in other words

$$\dot{T}_\beta = -\frac{\epsilon_1}{\epsilon_2}\zeta_1 - \zeta_2.$$

Since, we express

$$\kappa_\beta = \|\dot{T}_\beta\| = \sqrt{1 + \left(\frac{\epsilon_1}{\epsilon_2}\right)^2} \tag{15}$$

and

$$N_\beta = -\frac{\epsilon_1}{\epsilon_2\kappa_\beta}\zeta_1 - \frac{\zeta_2}{\kappa_\beta}.$$

Cross product $T_\beta \times N_\beta$ gives us

$$B_\beta = \frac{1}{\kappa_\beta}\zeta_1 - \frac{\epsilon_1}{\epsilon_2\kappa_\beta}\zeta_2.$$

By the formula of the torsion, we have

$$\tau_\beta = \frac{\epsilon_2(\frac{\epsilon_1}{\epsilon_2})'}{\epsilon_1^2 + \epsilon_2^2}. \tag{16}$$

In terms of Eqs. (15) and (16), we may give:

**Corollary 4.9.** *Let $\beta = \beta(s_\beta)$ be $\zeta_2$ spherical image of a regular curve $\alpha = \alpha(s)$. If the ratio of type-2 Bishop curvatures of $\alpha = \alpha(s)$ is constant ($\frac{\epsilon_1}{\epsilon_2}$ = constant, i.e.), then, the $\zeta_2$ Bishop spherical indicatrix $\beta(s_\beta)$ is a circle in the osculating plane.*

**Theorem 4.10.** *Let $\beta = \beta(s_\beta)$ be $\zeta_2$ spherical image of a regular curve $\alpha = \alpha(s)$. Then, there exists a relation among Frenet–Serret invariants of $\beta(s_\beta)$ and type-2 Bishop curvatures of $\alpha = \alpha(s)$ as follows*

$$\frac{\epsilon_1}{\epsilon_2} + \int_0^{s_\beta} \kappa_\beta^2 \tau_\beta \, ds_\beta = 0.$$

**Proof.** Similar to proof of Theorem 4.3, above equation can be obtained by Eqs. (14), (15) and (16). □

In the light of Propositions 2.2 and 2.3, we also give the following theorems for the curve $\beta = \beta(s_\beta)$:

**Theorem 4.11.** *Let $\beta = \beta(s_\beta)$ be $\zeta_2$ Bishop spherical image of a regular curve $\alpha = \alpha(s)$. If $\beta$ is a* ***general helix****, then, type-2 Bishop curvatures of $\alpha$ satisfy*

$$\frac{\epsilon_2^2(\frac{\epsilon_1}{\epsilon_2})'}{(\epsilon_1^2+\epsilon_2^2)^{\frac{3}{2}}} = constant.$$

**Theorem 4.12.** *Let $\beta = \beta(s_\beta)$ be $\zeta_2$ Bishop spherical image of a regular curve $\alpha = \alpha(s)$. If $\beta$ is a* ***slant helix****, then, type-2 Bishop curvatures of $\alpha$ satisfy*

$$\left[\frac{\epsilon_2^2(\frac{\epsilon_1}{\epsilon_2})'}{(\epsilon_1^2+\epsilon_2^2)^{\frac{3}{2}}}\right]' \frac{(\epsilon_1^2+\epsilon_2^2)^4}{\epsilon_2^3[(\frac{\epsilon_1}{\epsilon_2})'^2+(\epsilon_1^2\epsilon_2^2)^3]^{\frac{3}{2}}} = constant.$$

We also know that $\beta$ is a spherical curve. By Proposition 1, it is safe to report that:

**Theorem 4.13.** *Let $\beta = \beta(s_\beta)$ be $\zeta_2$ Bishop spherical image of a regular curve $\alpha = \alpha(s)$. Type-2 Bishop curvatures of the regular curve $\alpha = \alpha(s)$ satisfy the following differential equation*

$$\frac{\epsilon_2^2(\frac{\epsilon_1}{\epsilon_2})'}{(\epsilon_1^2+\epsilon_2^2)^{\frac{3}{2}}} + \left[\frac{\epsilon_1\epsilon_2}{\sqrt{\epsilon_1^2+\epsilon_2^2}}\right]' = constant.$$

**Remark 4.14.** Considering $\theta_\beta = \int_0^{s_\beta} \kappa_\beta \, ds_\beta$ and using the transformation matrix, one can obtain the type-2 Bishop trihedra $\{\zeta_{1\beta}, \zeta_{2\beta}, B_\beta\}$ of the curve $\beta = \beta(s_\beta)$.

### *4.3. Binormal Bishop spherical image*

**Definition 4.15.** Let $\alpha = \alpha(s)$ be a regular curve in $\mathbb{E}^3$. If we translate of the third vector field of type-2 Bishop frame to the center $O$ of the unit sphere $S^2$, we obtain a spherical image $\phi = \phi(s_\phi)$. This curve is called *Binormal Bishop spherical image or indicatrix of the curve $\alpha = \alpha(s)$.*

Here, one question may come to mind about binormal spherical image, since, Frenet–Serret and type-2 Bishop frames have a common binormal vector field. Images of such binormal images are the same as we shall demonstrate in Section 5. But, here we are concerned with the binormal Bishop spherical image's Frenet–Serret apparatus according to type-2 Bishop invariants.

Let $\phi = \phi(s_\phi)$ be binormal Bishop spherical image of a regular curve $\alpha = \alpha(s)$. One can differentiate of $\phi$ with respect to $s$:

$$\phi' = \frac{d\phi}{ds_\phi}\frac{ds_\phi}{ds} = \epsilon_1\zeta_1 + \epsilon_2\zeta_2.$$

In terms of type-2 Bishop frame vector fields (2), we have the tangent vector of the spherical image as follows

$$T_\phi = \frac{\epsilon_1\zeta_1+\epsilon_2\zeta_2}{\sqrt{\epsilon_1^2+\epsilon_2^2}},$$

where

$$\frac{ds_\phi}{ds} = \sqrt{\epsilon_1^2+\epsilon_2^2} = \kappa(s).$$

In order to determine first curvature of $\phi$, we write

$$\dot{T}_\phi = \frac{\epsilon_2^3}{(\epsilon_1^2+\epsilon_2^2)^2}\left(\frac{\epsilon_1}{\epsilon_2}\right)'\zeta_1 + \frac{\epsilon_1^3}{(\epsilon_1^2+\epsilon_2^2)^2}\left(\frac{\epsilon_2}{\epsilon_1}\right)'\zeta_2 - B.$$

Since, we immediately arrive at

$$\kappa_\phi = \|\dot{T}_\phi\| = \sqrt{\left[\frac{\epsilon_2^3}{(\epsilon_1^2+\epsilon_2^2)^2}\left(\frac{\epsilon_1}{\epsilon_2}\right)'\right]^2 + \left[\frac{\epsilon_1^3}{(\epsilon_1^2+\epsilon_2^2)^2}\left(\frac{\epsilon_2}{\epsilon_1}\right)'\right]^2 + 1}. \tag{17}$$

Therefore, we have the principal normal

$$N_\phi = \frac{1}{\kappa_\phi}\left\{\frac{\epsilon_2^3}{(\epsilon_1^2+\epsilon_2^2)^2}\left(\frac{\epsilon_1}{\epsilon_2}\right)'\zeta_1 + \frac{\epsilon_1^3}{(\epsilon_1^2+\epsilon_2^2)^2}\left(\frac{\epsilon_2}{\epsilon_1}\right)'\zeta_2 - B\right\}.$$

By the cross product of $T_\phi \times N_\phi$, we obtain the binormal vector field

$$B_\phi = \frac{1}{\kappa_\phi}\left\{\begin{array}{c} [\frac{\epsilon_1^4}{(\epsilon_1^2+\epsilon_2^2)^{\frac{5}{2}}}(\frac{\epsilon_2}{\epsilon_1})' - \frac{\epsilon_2^4}{(\epsilon_1^2+\epsilon_2^2)^{\frac{5}{2}}}(\frac{\epsilon_1}{\epsilon_2})']B \\ -[\frac{\epsilon_2}{\sqrt{\epsilon_1^2+\epsilon_2^2}}]\zeta_1 + [\frac{\epsilon_1}{\sqrt{\epsilon_1^2+\epsilon_2^2}}]\zeta_2 \end{array}\right\}.$$

By means of obtained equations, we express the torsion of the binormal Bishop spherical image

$$\tau_\phi = \frac{\begin{pmatrix} -\epsilon_1\{3\epsilon_2'(\epsilon_1\epsilon_1'+\epsilon_2\epsilon_2') - (\epsilon_1^2+\epsilon_2^2)[\epsilon_2'' - \epsilon_2(\epsilon_1^2+\epsilon_2^2)]\} \\ +\epsilon_2\{3\epsilon_1'(\epsilon_1\epsilon_1'+\epsilon_2\epsilon_2') - (\epsilon_1^2+\epsilon_2^2)[\epsilon_1'' - \epsilon_1(\epsilon_1^2+\epsilon_2^2)]\} \end{pmatrix}}{[\epsilon_1^2(\frac{\epsilon_2}{\epsilon_1})']^2 + (\epsilon_1^2+\epsilon_2^2)^3}. \tag{18}$$

Consequently, we determined Frenet–Serret invariants of the binormal Bishop spherical indicatrix according to type-2 Bishop invariants. In terms of Eqs. (17) and (18), we give:

**Corollary 4.16.** *Let $\phi = \phi(s_\phi)$ be binormal Bishop spherical image of a regular curve $\alpha = \alpha(s)$. If the ratio of type-2 Bishop curvatures of $\alpha = \alpha(s)$ is constant ($\frac{\epsilon_1}{\epsilon_2}$ = constant, i.e.), then, binormal Bishop spherical indicatrix $\beta(s_\beta)$ is a circle in the osculating plane.*

**Remark 4.17.** Considering $\theta_\phi = \int_0^{s_\phi}\kappa_\phi\, ds_\phi$ and using the transformation matrix, one can obtain type-2 Bishop trihedra $\{\zeta_{1\phi}, \zeta_{2\phi}, B_\phi\}$ of the curve $\phi = \phi(s_\phi)$.

## 5. Examples

In this section, first we show how to find a regular curve's type-2 Bishop trihedra and thereafter illustrate two examples of new spherical images.

**Example 5.1.** First, let us consider a unit speed circular helix of $\mathbb{E}^3$ by

$$\xi = \xi(s) = \left(24\cos\frac{s}{25}, 24\sin\frac{s}{25}, \frac{7s}{25}\right). \tag{19}$$

See the curve $\xi = \xi(s)$ in Fig. 1.

One can calculate its Frenet–Serret apparatus as the following

$$\begin{cases} \kappa = \dfrac{24}{625}, \\ \tau = \dfrac{7}{625}, \\ T = \dfrac{1}{25}\left(-24\sin\dfrac{s}{25}, 24\cos\dfrac{s}{25}, 7\right), \\ N = \left(-\cos\dfrac{s}{25}, -\sin\dfrac{s}{25}, 0\right), \\ B = \dfrac{1}{25}\left(7\sin\dfrac{s}{25}, -7\cos\dfrac{s}{25}, 24\right). \end{cases}$$

In order to compare our main results with spherical images according to Frenet–Serret frame, we first plot classical spherical images of $\xi$ in Fig. 2.

Now we focus on the type-2 Bishop trihedra. In order to form the transformation matrix (5), let us express

$$\theta(s) = \int_0^s \frac{24}{625}\,ds = \frac{24s}{625}.$$

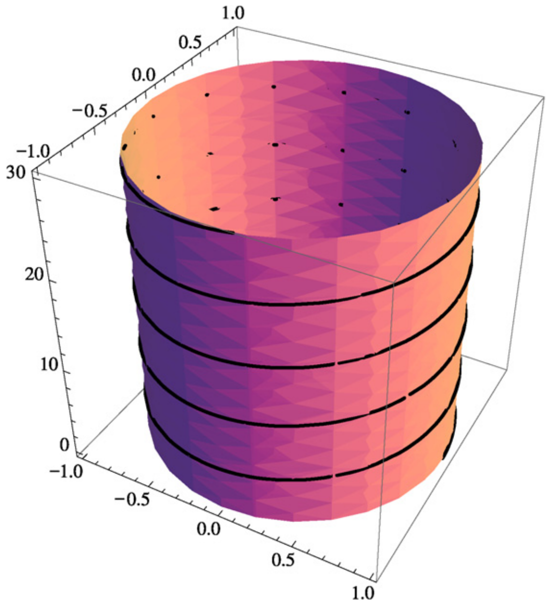

**Fig. 1.** Circular helix $\xi = \xi(s)$.

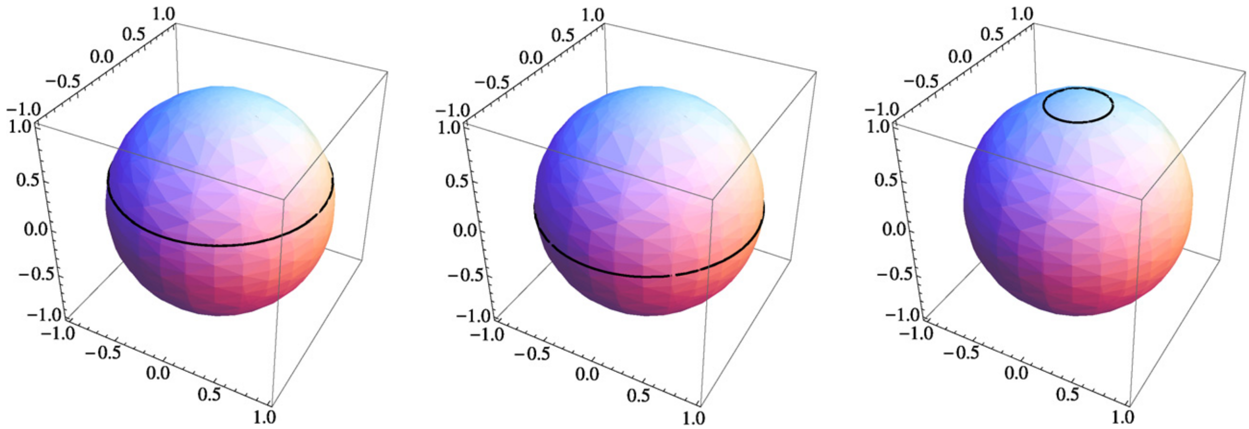

**Fig. 2.** Spherical images of $\xi = \xi(s)$ with respect to Frenet–Serret frame.

Since, we can write the transformation matrix

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \sin\frac{24s}{625} & -\cos\frac{24s}{625} & 0 \\ \cos\frac{24s}{625} & \sin\frac{24s}{625} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ B \end{bmatrix}.$$

By the method of Cramer, one can obtain type-2 Bishop trihedra of $\xi$ as follows:

The $\zeta_1$:

$$\zeta_1 = \left(-\cos\frac{24s}{625}\cos\frac{s}{25} - \frac{24}{25}\sin\frac{24s}{625}\sin\frac{s}{25}, \frac{24}{25}\cos\frac{s}{25}\sin\frac{24s}{625} - \cos\frac{24s}{625}\sin\frac{s}{25}, \frac{7}{25}\sin\frac{24s}{625}\right).$$

The $\zeta_2$:

$$\zeta_2 = \left(-\cos\frac{s}{25}\sin\frac{24s}{625} + \frac{24}{25}\cos\frac{24s}{625}\sin\frac{s}{25}, -\frac{24}{25}\cos\frac{24s}{625}\cos\frac{s}{25} - \sin\frac{24s}{625}\sin\frac{s}{25}, -\frac{7}{25}\cos\frac{24s}{625}\right).$$

The $B$:

$$B = \frac{1}{25}\left(7\sin\frac{s}{25}, -7\cos\frac{s}{25}, 24\right).$$

So, we can illustrate new spherical images, see Fig. 3.

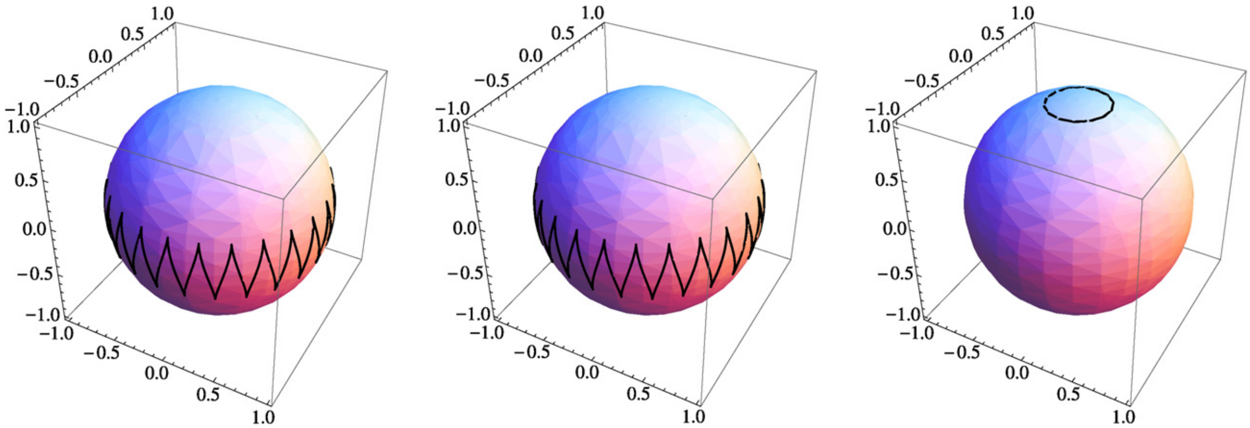

**Fig. 3.** $\zeta_1$, $\zeta_2$ and binormal Bishop spherical images of $\xi = \xi(s)$.

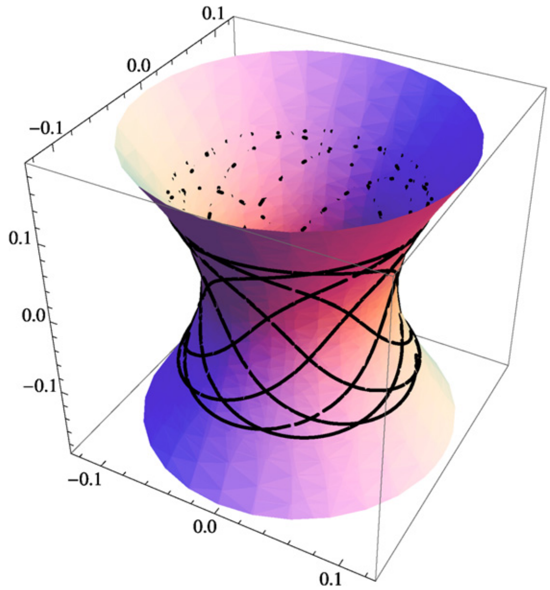

**Fig. 4.** The curve $\omega = \omega(s)$.

**Example 5.2.** Next, let us consider the following unit speed curve $\omega(s) = (\omega_1, \omega_2, \omega_3)$ of $\mathbb{E}^3$:

$$\begin{cases} \omega_1 = \dfrac{25}{306}\sin 9s - \dfrac{9}{850}\sin 25s, \\ \omega_2 = -\dfrac{25}{306}\cos 9s + \dfrac{9}{850}\cos 25s, \\ \omega_3 = \dfrac{15}{136}\sin 8s. \end{cases}$$

It is rendered in Fig. 4.

And, this curve's curvature functions are expressed as in [20]:

$$\begin{cases} \kappa(s) = -15\sin 8s, \\ \tau(s) = 15\cos 8s. \end{cases}$$

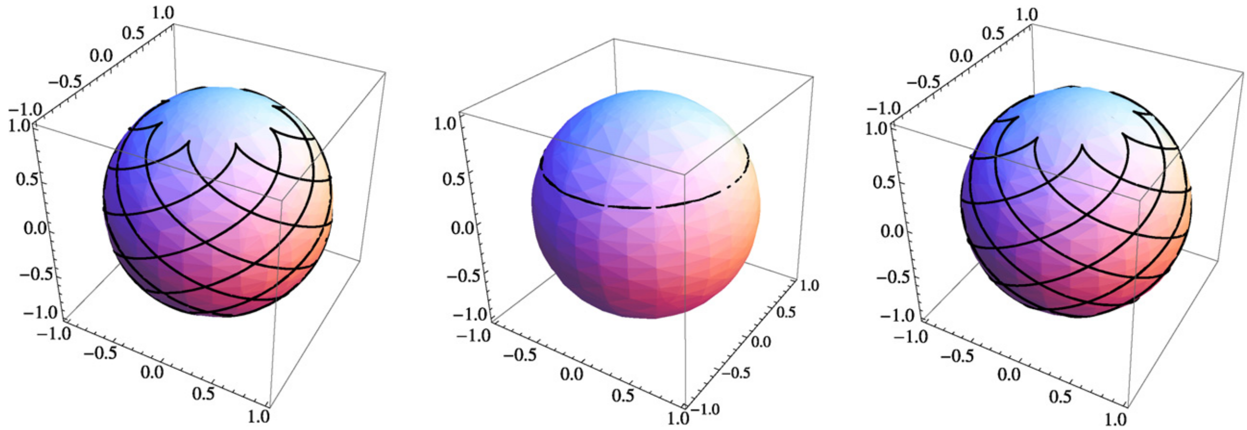


**Fig. 5.** Spherical images of $\omega = \omega(s)$ with respect to Frenet–Serret frame.

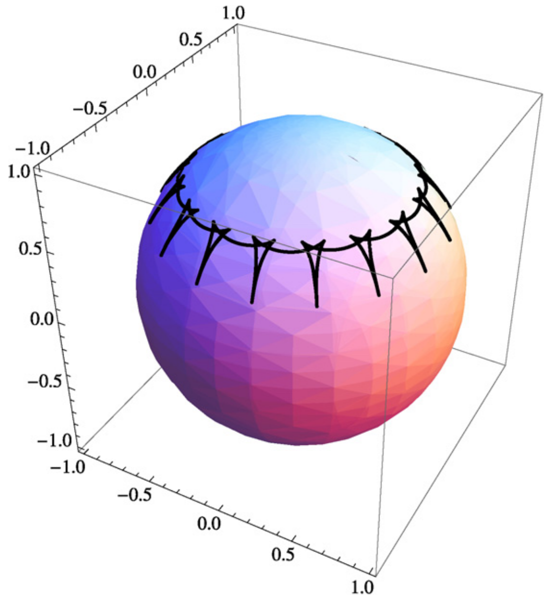


**Fig. 6.** $\zeta_1$ Bishop spherical image of $\omega = \omega(s)$.

The Frenet–Serret frame of the $\omega = \omega(s)$ may be written by the aid Mathematica program as follows

$$\begin{cases} T = \left(\frac{25}{34}\cos 9s - \frac{9}{34}\cos 25s, \frac{25}{34}\sin 9s - \frac{9}{34}\sin 25s, \frac{15}{17}\cos 8s\right), \\ N = \left(\frac{15}{34}\csc 8s(\sin 9s - \sin 25s), -\frac{15}{34}\csc 8s(\cos 9s - \cos 25s), \frac{8}{17}\right), \\ B = \left(\frac{1}{34}(25\sin 9s + 9\sin 25s), \frac{1}{34}(-25\cos 9s - 9\cos 25s), -\frac{15}{17}\sin 8s\right). \end{cases}$$

Similar to above case, we first plot spherical images of $\omega = \omega(s)$ with respect to Frenet–Serret frame, see Fig. 5.

In order to form the transformation matrix, we also need

$$\theta(s) = -\int_0^s 15\sin(8s)\,ds = \frac{15}{8}\cos(8s).$$

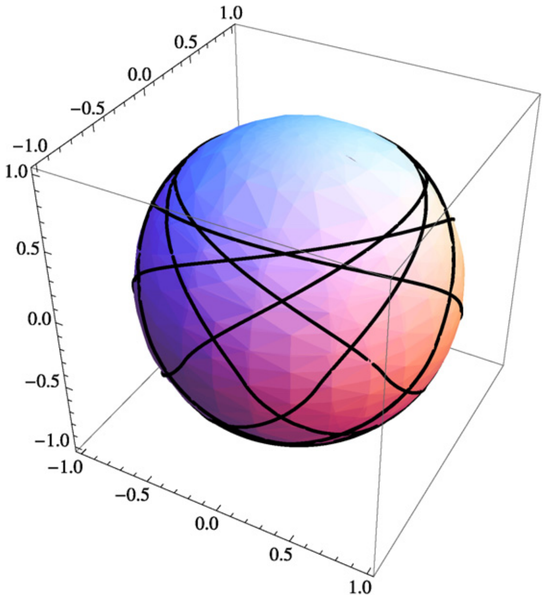


**Fig. 7.** $\zeta_2$ Bishop spherical image of $\omega = \omega(s)$.

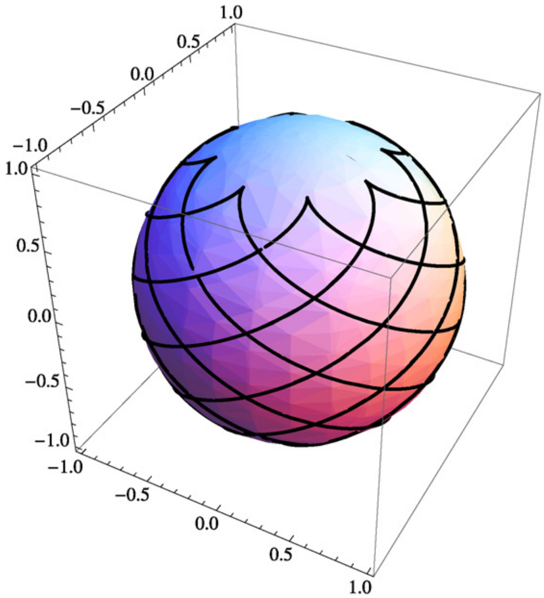


**Fig. 8.** Binormal Bishop spherical image of $\omega = \omega(s)$.

The transformation matrix for the curve $\omega = \omega(s)$ has the form

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \sin(\frac{15}{8}\cos 8s) & -\cos(\frac{15}{8}\cos 8s) & 0 \\ \cos(\frac{15}{8}\cos 8s) & \sin(\frac{15}{8}\cos 8s) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ B \end{bmatrix}.$$

By the solution of the system above, we have type-2 Bishop spherical images of the unit speed curve $\omega = \omega(s)$, see Figs. 6, 7 and 8.

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