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Quadratic Eigenvalue Problems

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1. INTRODUCTION

We consider differential equations of the form

$$Lu - \mu^2 Mu - \mu u = 0, \tag{1.1}$$

$$Lu + \frac{1}{\mu} Mu - \mu u = 0, \tag{1.2}$$

where the operators L and M are linear ordinary differential operators and L is of higher order. Chandrasekhar [1] studied *Problem I* (p. 386), which is of the type (1.1), and *Problem II* (p. 430), which is of the type (1.2). In each L and M are positive definite operators and the order of L is at least twice that of M. In this case (1.1) and (1.2) reduce to quadratic eigenvalue problems in which operators are compact: Class I (Section 2) and Class II (Sections 4-6). We show that each class has an equivalent linear form

$$Kw = \kappa w, \tag{1.3}$$

where K is compact, symmetric and one to one. It follows that the eigenvalues may be determined by one of the classical variational principles and that the eigenfunctions are complete. The reformulation of the problems into classes I-II make the results applicable to partial differential equations, matrix equations or to any eigenvalue problem which falls into either of the two classes.

Problem I, which arises in the theory of hydrodynamic stability, also appears in the theory of wave propagation and in that context, it was studied by Cohen [2]. Cohen reduced the problem to a linear (but not self-adjoint) form. He was able to show that the eigenfunctions are complete but the method makes use of the fact that L is of the second order. Another treatment, using the equivalence with a compact, symmetric linear problem, is given in [3]. Chandrasekhar obtained a variational formulation by solving for μ

in $(Lu, u) - \mu^2(Mu, u) - \mu(u, u) = 0$, thus obtaining a "nonlinear variational principle." He obtained a similar result for Problem II.

Problem II has been considered also by Shinbrot [4] and Turner [5]. The former showed that the eigenfunctions are "nearly" complete while the latter, with the aid of a "pseudo-inner product," attempted the construction of a nonlinear variational principle which corresponds say, to the Rayleigh-Ritz method.

The phenomenon of a quadratic eigenvalue problem having a symmetric linear equivalent has also been observed by Duffin[1] [6] and Muller [7].

Applications to (1.1) and (1.2) are discussed in Sections 3 and 7.

Added 7 March 1968. An earlier version of this paper was submitted to this Journal in May 1967. Since that time a short proof of the completeness of the eigenvectors of (4.1) (which is slightly less general than our class II problems) has been obtained independently by H. F. Weinberger [13]. Weinberger's method is different from ours. He does not show that the eigenvalue problem is equivalent to one of the form (1.3) but he does obtain that the eigenvectors which correspond only to the positive eigenvalues are complete with respect to the norm $(Ax, x)^{1/2}$. Using Weinberger's technique one can show that the eigenvectors corresponding to the positive eigenvalues of our class I problems are complete with respect to the original norm.

2. Class I Problems

Let L and M be ordinary differential operators on $L_2[a, b]$, the order of L exceeds that of M, and the boundary conditions are such that L and M are positive definite on their respective domains. If the domain of $L^{1/2}$ is contained in the domain of M,

$$D(L^{1/2}) \subset D(M) \tag{2.1}$$

then (1.1) has the equivalent form

$$x - \lambda Ax - \lambda^2 Bx = 0. \tag{2.2}$$

Here $\lambda = \mu$, $x = L^{1/2}u$ and

$$A = L^{-1}, \qquad B = L^{-1/2}ML^{-1/2} \tag{2.3}$$

are compact, positive definite operators.

[1] Duffin shows that $\lambda^2 Ax + \lambda Bx + Cx = 0$ can be reduced to symmetric linear form. His method leads to further generality since it does not require that the operators be positive definite. See also [10]–[12].

It will be shown in the next section that the condition (2.1) may be replaced by one which is easier to apply; that the order of L is at least twice the order of M. In the event that (2.1) is not satisfied then Turner shows that if B is now defined as the completion of $L^{-1/2}ML^{-1/2}$ then B continues to be compact and positive definite (in fact, it can be shown that B is Hilbert-Schmidt) and the eigenvalue problem (1.1) is "inbedded" in (2.2) i.e., every solution of (2.1) corresponds to a solution of (2.2).

More generally, consider the problem of finding a number λ such that for a nonzero vector x, in a Hilbert space $\mathscr{H}$

$$x - \lambda Ax - \lambda^2 Bx = 0 \qquad \text{(class I)} \tag{2.4}$$

where A and B are compact and symmetric on $\mathscr{H} \to \mathscr{H}$ and B is positive definite. By introducing

$$y = \lambda B^{1/2} x, \tag{2.5}$$

(2.4) takes the form

$$Kw = \kappa w, \tag{2.6}$$

where $\kappa = \lambda^{-1}$, $w = \binom{x}{y}$ is in $\mathscr{H} \times \mathscr{H}$, and

$$K = \begin{pmatrix} A & B^{1/2} \\ B^{1/2} & 0 \end{pmatrix} \tag{2.7}$$

is compact, symmetric and one to one on $\mathscr{H} \times \mathscr{H} \to \mathscr{H} \times \mathscr{H}$. It follows that K has at most a denumerable number of solutions

$$Kw_n = \kappa_n w_n, \qquad n = 1, 2, \ldots . \tag{2.8}$$

These eigenvectors may be selected so that

$$((w_n, w_m)) \equiv (x_n, x_m) + (y_n, y_m) = \delta_{nm} . \tag{2.9}$$

If $\mathscr{H}$ is separable then the eigenvectors $\{w_n\}$ form a basis for $\mathscr{H} \times \mathscr{H}$. For a detailed account of compact symmetric operators on a separable Hilbert space see Riesz and Sz. Nagy, Chapter VI.

THEOREM 2.1. *Problem* (2.4) *has a denumerable number of solutions:*

$$x_n - \lambda_n A x_n - \lambda_n^2 B x_n = 0, \qquad n = 1, 2, \ldots, \tag{2.10}$$

which may be selected so that

$$(x_n, x_m) + \lambda_n \lambda_m (Bx_n, x_m) = \delta_{nm} . \tag{2.11}$$

If $\mathscr{H}$ *is separable then one has the expansion*

$$f = \sum (f, x_n)\, x_n, \qquad f \in \mathscr{H}. \tag{2.12}$$

PROOF. Let $y_n = \lambda_n B^{1/2} x_n$ and $w_n = \binom{x_n}{y_n}$ then (2.10) is satisfied if and only if (2.8) is satisfied. If $\mathscr{H}$ is separable, $\{w_n\}$ is complete in $\mathscr{H} \times \mathscr{H}$ and we have, in view of (2.9),

$$F = \sum ((F, w_n))\, w_n\,, \qquad F \in \mathscr{H} \times \mathscr{H}.$$

Taking $F = \binom{f}{0}$ this reduces to (2.12).

3. Eigenfunction Expansions for (1.1)

Let L and M be positive definite, self-adjoint, ordinary differential operators defined on $L_2[a, b]$. We assume (2.1) so that (1.1) and (2.2) are completely equivalent. A useful characterization of $D(L^{1/2})$ is given by Krein (cf. [14], p. 387) as follows. Let the order of L be 2ℓ and let $\sum$ denote a set of 2ℓ homogeneous boundary conditions associated with L, then $D(L^{1/2})$ consists of those functions in $L_2[a, b]$ which have ℓ strong derivatives in $L_2[a, b]$, the first $\ell - 1$ being continuous, and which satisfy only those boundary conditions involving derivatives of order at most $\ell - 1$. It follows from this characterization that if M is another differential operator of order $2m$ such that $D(L) \subset D(M)$ and $2m \leqslant \ell$ then (2.1) is satisfied. To see this observe that $D(L) \subset D(M)$ implies that every boundary condition associated with M is also associated with L. Since $2m \leqslant \ell$ such a boundary condition would involve not more than $\ell - 1$ derivatives and thus it must also be associated with $L^{1/2}$.

Krein also shows that the elements in $D(L^{1/2})$ form a Hilbert space $\mathscr{H}_L$ with respect to the inner product

$$(\cdot, \cdot)_L = (L(\cdot), \cdot), \tag{3.1}$$

and that a sequence $f_j \to f$ in $\mathscr{H}_L$ means that the differentiated sequences $f_j^{(s)} \to f^{(s)}$ in $L_2[a, b]$ for $k = 0, 1,..., \ell$, and that the convergence is pointwise, uniformly and absolutely, in $[a, b]$, for $s = 0, 1,..., \ell - 1$. That is, $\mathscr{H}_L$ is a subspace of the Sobolev space of order ℓ.

It is not difficult to see, in view of the above remarks, that the eigenfunctions of (1.1) are not only complete in $L_2[a, b]$ but the expansion converges pointwise, uniformly and absolutely, in $[a, b]$ and that term by term differentiation of the series is permissible.

THEOREM 3.1. *Let L and M be positive definite, ordinary differential operators on $L_2[a, b]$ satisfying* (2.1), *then* (1.1) *has a denumerable number of solutions:*

$$Lu_n - \mu_n{}^2 M u_n - \mu_n u_n = 0, \qquad n = 1, 2,..., \tag{3.2}$$

which may be selected so that

$$(Lu_n, u_m) + \mu_n\mu_m(Mu_n, u_m) = \delta_{nm}, \tag{3.3}$$

and one has the expansion

$$f^{(s)} = \sum (f, u_n)\, u_n^{(s)}, \qquad f \in D(L^{1/2}), \tag{3.4}$$

where the convergence is in $L_2[a, b]$ *for* $s = 0, 1,..., \ell$ *and pointwise in* $[a, b]$, *uniformly and absolutely, for* $s = 0, 1,..., \ell - 1$. *Here* 2ℓ *is the order of* L.

PROOF. Let $g = L^{1/2}f$. Then by Theorem 2.1

$$g = \sum (g, x_n)\, x_n,$$

where $x_n = L^{1/2}u_n$. That is, if $g_j = \sum_{n=1}^{j} (g, x_n)\, x_n$ is the partial sum then $\| g_j - g \| \to$ as $j \to \infty$. If $f_j = \sum_{n=1}^{j} (f, u_n)\, u_n$ then $L^{1/2}f_j = g_j$ and hence $\| f_j - f \|_L = \| g_j - g \| \to 0$ as $j \to \infty$.

4. CLASS II PROBLEMS

Consider the eigenvalue problem (1.2) where L and M are as in Section 2. This corresponds to the "compact eigenvalue problem"

$$Ax - \lambda^2 Bx - \lambda x = 0, \tag{4.1}$$

where $\lambda = \mu^{-1}$, $x = L^{1/2}u$ and A and B are given by (2.3). The above remarks concerning the equivalence of (1.1) and (2.2) apply to that of (1.2) and (4.1). In particular, (1.2) and (4.1) are completely equivalent in the event that (2.1) is satisfied.

We will study the slightly more general problem

$$Ax - \lambda^2 Bx - \lambda Cx = 0 \qquad \text{(class II)}, \tag{4.2}$$

where A, B, and C are symmetric operators on a separable Hilbert space $\mathscr{H}$ and A and B are compact and positive definite. Another condition on C is given below but this will be such as to include the case: $C = I$.

If we apply the trick of introducing the new variable $y = \lambda B^{1/2}x$ then (4.2) reduces to the symmetric, linear form

$$T_1 w = \lambda T_2 w, \tag{4.3}$$

where $w = \binom{x}{y}$ is in $\mathscr{H} \times \mathscr{H}$ and

$$T_1 = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \qquad T_2 = \begin{pmatrix} C & B^{1/2} \\ B^{1/2} & 0 \end{pmatrix} \tag{4.4}$$

are symmetric operators on $\mathscr{H} \times \mathscr{H} \to \mathscr{H} \times \mathscr{H}$. In view of the symmetry, if we define

$$((\cdot, \cdot))_{T_1} \equiv ((T_1(\cdot), \cdot)) \tag{4.5}$$

on $\mathscr{H} \times \mathscr{H}$ (cf. (2.9) for the definition of the product on the right), then the eigenvectors of (4.3) may be selected so as to satisfy the orthogonality relationship

$$((w_\alpha , w_\beta))_{T_1} = \delta_{\alpha\beta} . \tag{4.6}$$

Since $\mathscr{H}$ is separable this means that there are at most a denumerable number of mutually orthogonal vectors $\{T^{1/2}w_\alpha\}$ in $\mathscr{H} \times \mathscr{H}$ and therefore, at most a denumerable number of distinct solutions

$$T_1 w_n = \lambda_n T_2 w_n , \qquad n = 1, 2, \ldots . \tag{4.7}$$

The form (4.3) is useful but it is not as desireable as the form (1.3) in which one has a single operator K which is symmetric, compact and one to one. For such problems, not only are the eigenvectors complete but also there are numerical procedures for calculating the eigenvalues. The remainder of this section will be devoted to providing such an equivalent form for (4.2). One may point out that simply multiplying by T_1^{-1} will not suffice for the operator $T_1^{-1}T_2$ is unbounded and has an unbounded inverse. This is true no matter what norm we choose for (4.2) has a sequence of eigenvalues tending to zero as well as a sequence tending to infinity. This will be established in Section 6 and it was observed in the special case $C = I$ by Turner. One sees therefore, that any transformation which produces the desired form (1.3) must be nonlinear.

Let

$$E_\lambda \equiv A - \lambda^2 B - \lambda C \tag{4.8}$$

and let

$$\rho = \{\lambda : E_\lambda \text{ has a bounded inverse}\}.$$

In addition to the conditions on the operators cited above we assume that C is such that ρ contains at least one real number. For example, if C has a bounded inverse then E_λ is a Fredholm operator, i.e., either $\lambda \in \rho$ or λ is an eigenvalue. However, we have observed above that there are at most a denumerable number of eigenvalues. Therefore, the condition is satisfied when C is invertable. On the other hand, the condition is not satisfied when C is compact and in fact, when $A = B = C$ the equivalence we seek is impossible.

5. An Equivalent Compact Problem

Since A is positive definite on $\mathscr{H}$, T_1 has a positive definite square root:

$$T_1^{1/2} = \begin{pmatrix} A^{1/2} & 0 \\ 0 & I \end{pmatrix}.$$

Let $D(T)$ denote the set of vectors v in Range $(T_1^{1/2})$ such that $T_2 T_1^{-1/2} v$ is in Range $(T_1^{1/2})$ and let

$$Tv = T_1^{-1/2} T_2 T_1^{-1/2} v, \qquad v \in D(T).$$

Then (4.3) is satisfied for $\lambda = \mu^{-1}$ and $w = T_1^{1/2} v$ if

$$Tv = \mu v, \qquad v \in D(T_1). \tag{5.1}$$

If for each $f \in \mathscr{H} \times \mathscr{H}$ the equation

$$Tv - \mu v = f \tag{5.2}$$

has a unique solution $v \in D(T)$ then we write

$$v = R_\mu f.$$

If the linear operator R_μ is bounded then it is called the resolvent and the set ρ^*, of numbers μ, for which the resolvent is defined, is called the resolvent set. The resolvent is discussed in most textbooks on functional analysis (e.g., cf. [9]).

Lemma 5.1. *Let $\mu, \nu \in \rho^*$, $w \in \mathscr{H} \times \mathscr{H}$, then the following statements are true:*

(i) *R_μ is one to one mapping $\mathscr{H} \times \mathscr{H}$ onto $D(T)$.*

(ii) $TR_\mu w = w + \mu T_\mu w$

(iii) $TR_\mu w = R_\mu T w$, $w \in D(T)$

(iv) $(R_\mu - R_\nu)\, w = (\mu - \nu)\, R_\mu R_\nu w$.

The above lemma is immediate from the definition of the resolvent. The next lemma describes how to construct the resolvent.

Lemma 5.2. *If $\lambda \in \rho$ then $\mu = \lambda^{-1} \in \rho^*$ then*

$$R_\mu = -\lambda \begin{pmatrix} A^{1/2} E_\lambda^{-1} A^{1/2} & \lambda A^{1/2} E_\lambda^{-1} B^{1/2} \\ \lambda B^{1/2} E_\lambda^{-1} A^{1/2} & \lambda^2 B^{1/2} E_\lambda^{-1} B^{1/2} + I \end{pmatrix}. \tag{5.3}$$

PROOF. Observe first that $T_1^{1/2}$ is a left factor of R_μ so that Range $(R_\mu) \subset$ Range $(T_1^{1/2})$. Using matrix multiplication and the identity:

$$\lambda C E_\lambda^{-1} + \lambda^2 B E_\lambda^{-1} = A E_\lambda^{-1} - I$$

we see that Range $(T_2 T_1^{-1/2} R_\mu) \subset$ Range $(T_1^{1/2})$ and that

$$T R_\mu f - \mu R_\mu f = f, \qquad f \in \mathscr{H} \times \mathscr{H}.$$

Finally, the solution $v = R_\mu f$ is unique for suppose that (5.2) has a second solution then by substracting we see that μ is an eigenvalue of (5.1) and hence, λ is an eigenvalue of (4.3). This contradicts the assumption that $\lambda \in \rho$.

Observe from (5.3) that

$$R_\mu = \mu^{-1}(K_\mu + P_2), \tag{5.4}$$

where P_2 is the projection $P_2\binom{x}{y} \equiv \binom{0}{y}$ and K_μ is compact on $\mathscr{H} \times \mathscr{H}$. Moreover, K_μ is symmetric when μ is real.

LEMMA 5.3. *Let σ^{-1} and τ^{-1} be real numbers in ρ (not necessarily distinct) then*

$$K(\sigma, \tau) \equiv R_\sigma + \tau R_\sigma R_\tau \tag{5.5}$$

is compact, symmetric and one to one.

PROOF. The symmetry follows from the symmetry of R_μ, $\mu^{-1} \in \rho$ (cf. (5.3)) and the commutivity of R_σ and R_τ (cf. Lemma 5.1, (iv)). The compactness follows from the identity

$$K(\sigma, \tau) = \mu^{-1}(K_\mu K_\nu + K_\mu P_2 + P_2 K_\nu - K_\mu),$$

which is easily obtained with the aid of (5.4). In view of Lemma 5.1, (ii), (5.5) reduces to

$$K(\sigma, \tau) = T_\sigma T R_\tau\,. \tag{5.6}$$

Since R_σ, R_τ, T_1 and T_2 are all one to one the same is true of $K(\sigma, \tau)$.

Observe from (5.6) that K is symmetric in σ and τ and that (5.1) implies that

$$Kv = \kappa v \qquad (K = K(\sigma, \tau)), \tag{5.7}$$

where by (5.6)

$$\kappa = \frac{\mu}{[(\mu - \sigma)(\mu - \tau)]}\,. \tag{5.8}$$

In order to establish the equivalence of the eigenvalues problems (5.1) and (5.7) one must show that the converse is true. One may select a complete sequence of eigenvectors of K $\{v_n\}$ such that

$$((v_n, v_m)) = \delta_{nm}\,. \tag{5.9}$$

If one of the eigenvalues corresponds to more than one eigenvector of K, then there is more than one sequence of eigenvectors satisfying (5.9). The following theorem shows that a sequence may be selected so that each member is also an eigenvector of T.

THEOREM 5.3. *Let σ^{-1} and τ^{-1} be (not necessarily distinct) real numbers in ρ, then $K = K(\sigma, \tau)$ and T share a complete orthonormal sequence of eigenvectors of both.*

PROOF. Let κ be an eigenvalue of K and let V_κ be the linear space of eigenvectors of K corresponding to κ. Since K is compact V_κ is finite dimensional. Since T commutes with K on $D(T)$ (cf. Lemma 5.1, (iii)) and since $V_\kappa \subset D(T)$, T maps V_κ into itself. Therefore, T may be considered as a one to one, symmetric matrix mapping $V_\kappa \to V_\kappa$, and thus, one can select an orthonormal basis for V_κ of eigenvectors of T. The union of these basis vectors for each eigenvalue κ of K forms the desired sequence.

By using the completeness of the eigenvectors of T, the next theorem shows that (4.3) and (5.1) are equivalent. The existence of at least one $K = K(\sigma, \tau)$ follows from the restriction on C given above.

THEOREM 5.4. *The eigenvalue problem* (4.3) *has a solution if and only if* (5.1) *has a solution with $w = T_1^{-1/2}v$ and $\lambda = \mu^{-1}$.*

PROOF. It is easy to see that every solution of (5.1) corresponds to a solution of (4.3). It remains to show the converse. Let $\{v_n\}$ denote a complete orthonormal sequence of eigenvectors of T. If $w_n = T_1^{-1/2}v_n$, $n = 1, 2, \ldots$, then $\{w_n\}$ is a sequence of eigenvectors of (4.3) such that $\{T_1^{1/2}w_n\}$ is complete. If $T_1 w = \lambda T_2 w$, $w \neq 0$, is a solution of (4.3) such that λ^{-1} is not an eigenvalue of T then, in view of (4.6) and the completeness of $\{T_1^{1/2}w_n\}$, $T_1^{1/2}w = 0$ which implies that $w = 0$. If $\mu = \lambda^{-1}$ is an eigenvalue of T then the completeness of $\{T_1^{1/2}w_n\}$ and the orthogonality (4.6) imply that $T_1^{1/2}w$ is in the space V spanned by the eigenvectors of T corresponding to the eigenvalue μ. Since the vectors in V are also eigenvectors of some $K(\sigma, \tau)$ corresponding to the eigenvalue κ given by (5.9), and since $K(\sigma, \tau)$ is compact, V is finite dimensional. Therefore, $T_1^{1/2}w = v$ where v is an eigenvector of T corresponding to μ.

REMARK. It was necessary to show that the subspace of eigenvectors of T corresponding to a single eigenvalue is finite dimensional to insure that $T_1^{1/2}w$ is in $D(T)$. It is worth noting that the eigenvalues of (4.2) have finite multiplicity.

As a consequence of Theorems 5.3-5.4, if σ^{-1} and τ^{-1} are any real numbers in ρ then the eigenvalues of (4.2) and those of the compact, symmetric operator $K(\sigma, \tau)$ are related by

$$\kappa_n = \frac{\lambda_n}{[1 - \lambda_n \sigma]\,[1 - \lambda_n \tau]}, \qquad n = 1, 2, \ldots. \tag{5.10}$$

THEOREM 5.5. *Let $\{w_n\}$ be the eigenvectors of* (4.3) *normalized by* (4.6), *then for each $F \in \mathscr{H} \times \mathscr{H}$.*

$$T_1^{1/2} F = \sum ((F, w_n))_{T_1} T_1^{1/2} w_n . \tag{5.11}$$

Let $\{x_n\}$ be the eigenvectors of (4.2) *normalized by*

$$(Ax_n, x_m) + \lambda_n \lambda_m (Bx_n, x_m) = \delta_{nm}, \tag{5.12}$$

where $\{\lambda_n\}$ are the corresponding eigenvalues, then for each $f \in \mathscr{H}$

$$A^{1/2} f = \sum (Af, x_n) A^{1/2} x_n \tag{5.13}$$

PROOF. The expansion (5.11) follows from the completeness of the eigenvectors of T, $v_n = T_1^{1/2} w_n$, $n = 1, 2, \ldots$ and the orthogonality relation (4.6). Recalling that $w_n = \binom{x_n}{y_n}$ where $y_n = \lambda_n B^{1/2} x_n$, x_n, $n = 1, 2, \ldots$, being the eigenvectors of (4.2) we obtain (5.12) from (4.6). Letting $F = \binom{f}{0}$ in (5.11) yields (5.13).

The next theorem gives conditions under which the operator $T_1^{1/2}$ can be "removed" from both sides of (5.11).

THEOREM 5.6. *Under the hypothesis of Theorem 5.5*

$$T_1^{-1/2} F = \sum ((T_1^{-1/2} F, w_n))_{T_1} w_n, \qquad F \in D(T). \tag{5.14}$$

PROOF. Let $\chi \in D(T)$, $\sigma^{-1} \in \rho$ be real, and $\emptyset = T\chi - \sigma\chi$, then

$$\chi = R_\sigma \emptyset. \tag{5.15}$$

In view of Theorem 5.3

$$\emptyset = \sum ((\emptyset, v_n))\, v_n .$$

Making use of (5.3) and (5.15), the first component

$$(\chi)_1 = A^{1/2} \sum ((\emptyset, v_n))\, [B_1 (v_n)_1 + B_2 (v_n)_2],$$

where B_1 and B_2 are bounded operators on $\mathscr{H}$. It follows that

$$\| A^{-1/2} (\chi)_1 \| \leqslant \beta_0 \, ||| \emptyset ||| , \tag{5.16}$$

where β_0 is a positive constant. Now let

$$F_s = \sum_{n=1}^{s} ((F, v_n))\, v_n ,$$

$\chi_s = F - F_s$ and $\phi_s = R_\mu \chi_s$, then (5.16) yields

$$\| A^{-1/2}(F - F_s)_1 \| \leqslant \beta_0 \||\, \phi_s \|| \to 0$$

as $s \to \infty$. This is equivalent to the first component expansion of (5.14); the second component expansion follows from Theorem 5.5.

6. Distribution of Eigenvalues

Theorem 6.1. *The eigenvalues of* (4.2) *are:*

(i) *real*

(ii) *discrete*

(iii) *have precisely two accumulation points: zero and infinity.*

Proof. The first two properties and the fact that the eigenvalues have at most two accumulation points: zero and infinity follow from formula (5.10) and corresponding facts about the eigenvalues of a compact, symmetric operator. Moreover, since the sequence $\{\kappa_n\}$ accumulates at zero we know that one of the points zero or infinity must be an accumulation point of $\{\lambda_n\}$. Let us suppose that infinity is the only accumulation point and that if $\lambda_n = \mu_n^{-1}$, $n = 1, 2,...$ then

$$| \mu_n | < \beta, \qquad n = 1, 2,... \tag{6.1}$$

for some positive constant β. It follows from Theorems 5.3-5.4 that

$$\||\, Tw \|| < \beta \||\, w \||, \qquad w \in D(T).$$

The operator $A^{-1/2}CA^{-1/2}$, which is the first row-first column entry of the matrix T, therefore satisfies

$$\| A^{-1/2}CA^{-1/2}x \| \leqslant \beta \| x \|, \qquad x \in D(A^{-1/2}CA^{-1/2}).$$

Since the eigenvectors of (5.1) are complete, $D(T)$ is dense in $\mathscr{H} \times \mathscr{H}$ and hence, $D(A^{-1/2}CA^{-1/2})$ is dense in $\mathscr{H}$. It follows that $A^{-1/2}CA^{-1/2}$ has a bounded completion F and that

$$C = A^{1/2}FA^{1/2}.$$

If this is true, C is compact, E_λ is compact (cf. (4.8)), and ρ is empty, contradicting the assumption that ρ contains at least one real number.

We conclude that zero is always an accumulation point of (4.2). Since (4.2) may also be written as

$$Bx - \mu^2 Ax - \mu Cx = 0$$

where $\mu = -(1/\lambda)$, it follows that infinity must also be an accumulation point.

The above theorem generalizes a result by Turner in the case $C = I$. However, Turner shows also that the positive eigenvalues lie in $(0, \| A \|)$ and that the negative eigenvalues lie in $(-\infty, -\| B \|^{-1})$ which we are not able to deduce in the more general case.

7. Eigenfunction Expansion for (1.2)

Since the convergence of the expansions in Theorem 5.5 are weaker then the corresponding expansions in Theorem 2.1 the next theorem is not as strong as its counterpart, Theorem 3.1. However, we are able to obtain results concerning term by term differentiation and uniform, absolute convergence of the eigenfunction expansion.

THEOREM 7.1. *Let L and M be positive definite, ordinary differential operators on $L_2[a, b]$ satisfying (2,1), then (1.2) has a denumerable number of solutions:*

$$Lu_n + \frac{1}{\mu_n} Mu_n - \mu_n u_n = 0, \qquad n = 1, 2, \ldots,$$

which may be selected so that

$$(u_n , u_m) = \frac{1}{\mu_n \mu_m} (Mu_n , u_m) = \delta_{nm}$$

and one has the expansion

$$f^{(s)} = \sum (f, u_n) u_n^{(s)}, \qquad f \in D(L), \tag{7.1}$$

where the convergence is in $L_2[a, b]$ for $s = 0, 1, \ldots, \ell$ and pointwise in $[a, b]$, uniformly and absolutely, for $s = 0, 1, \ldots, \ell - 1$. Here 2ℓ is the order of L.

PROOF. Let $F = \binom{f}{0} \in \mathscr{H} \times \mathscr{H}$ where here $\mathscr{H} = L_2[a, b]$, $C = I$ and A and B are given by (2.3). It is not difficult to see that $F \in D(T)$. Theorem 5.6 yields

$$L^{1/2} f = \sum (f, x_n) L^{1/2} x_n ,$$

where the convergence is in $L_2[a, b]$. Letting $f_s = \sum_{n=1}^{s} (f, x_n) x_n$ we see from the proof of Theorem 3.1 that $f_s \to f$ in $\mathscr{H}_L$.

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