Subnormal structure of non-stable unitary groups over rings

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ARTICLE INFO

Article history:
Received 4 February 2009
Received in revised form 24 June 2009
Available online 3 August 2009
Communicated by C.A. Weibel

MSC:
20H25
20G35

ABSTRACT

Let \( R \) be a commutative ring with identity in which 2 is invertible. Let \( H \) denote a subgroup of the unitary group \( U(2n, R, A) \) with \( n \geq 4 \). \( H \) is normalized by \( EU(2n, J, F^0) \) for some form ideal \( (J, F^0) \) of the form ring \( (R, A) \). The purpose of the paper is to prove that \( H \) satisfies a “sandwich” property, i.e. there exists a form ideal \( (I, F^0) \) such that

\[
EU(2n, I, F^0, \Gamma) \subseteq H \subseteq CU(2n, I, F^0).
\]


0. Introduction

Let \( R \) be a commutative ring with identity. For a positive integer \( n \), consider the general linear group \( GL_n(R) \). Let \( E_n(R) \) be the elementary subgroup of \( GL_n(R) \), generated by matrices \( e_{i,j}(a) \), where \( e_{i,j}(a) \) is a matrix with \( a \) in position \( (i, j) \), 1 in the diagonal positions, and zeros elsewhere. For any ideal \( I \) of \( R \), several subgroups of \( GL_n(R) \) are defined and studied as follows. Consider the group homomorphism \( \phi : GL_n(R) \to GL_n(R/I) \) arising from the canonical map \( R \to R/I \). One defines the principal congruence subgroup \( GL_n(R, I) := \ker(\phi) \) and the full congruence subgroup \( C_n(R, I) \) as the pre-image of the center of the group \( GL_n(R/I) \) under the homomorphism \( \phi \). Furthermore, we consider the subgroup \( E_n(I) \) generated by \( e_{i,j}(a) \) where \( a \in I \), and the relative elementary group \( E_n(R, I) \) which is the normal closure of \( E_n(I) \) in \( E_n(R) \).

In a manuscript from 1967 which was published in 1982 [1], Bak studied the subgroups of \( GL_n(R) \) normalized by the relative elementary subgroup \( E_n(R, I) \), for a ring \( R \) with the stable rank condition, and obtained a “sandwich classification” for such subgroups. Since then, the sandwich theorem has been improved several times, due to Wilson [2], Vavilov [3] and Vaserstein [4,5]. Now we have the following theorem (see [5] for a more general form).

Theorem. Let \( R \) be a commutative ring, \( n \geq 3 \) and \( H \) a subgroup of \( GL_n(R) \) normalized by \( E_n(R, J) \) for an ideal \( J \). Then there exists an ideal \( I \) such that

\[
E_n(R, I) \subseteq H \subseteq C_n(R, I : J^4).
\]

Theorems of the above form are essential for classifying the subnormal subgroups of \( GL_n(R) \) (see the proof of Theorem 1 in [5]). Namely, if

\[
H = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_d = GL_n(R)
\]

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doi:10.1016/j.jpaa.2009.07.007
is a subnormal subgroup of $GL_n(R)$ then thanks to the above theorem, there is an ideal $J$ of $R$ such that

$$E_n(R, J^J) \subseteq H \subseteq C_n(R, J).$$

In [1], Bak conjectured that his sandwich classification theorem also holds in the setting of general quadratic groups over rings with stable rank condition [1, Conjecture 1.3]. Although the quadratic setting is much more complicated than the linear one, due mostly to the complexity of its elementary subgroup, it is being gradually established that most results concerning the general linear group and its $K$-theory can be carried over to general quadratic groups and their $K$-theory. For the recent developments on the $K$-theory of general quadratic groups see Bak–Petrov–Tang [6], Hazrat [7], Petrov [8] and Bak–Hazrat–Vavilov [9].

Bak’s conjecture with the stable rank condition on the ring and 2 invertible, was settled positively by Habdank [10]. The conjecture was proven by Zhang [11] in the stable case without any further assumptions on the ring.

The current paper extends the stable relative sandwich classification to the non-stable case. These results build up the full relative sandwich structure of unitary groups on the commutative ring with the invertibility of 2. In the current paper, we follow the notations used by the excellent survey paper [12] by Bak and Vavilov and used by the paper [11].

The rest of the paper is organized as follows. In Section 1, we discuss the elementary elements of unitary groups and some preparatory lemmas concerning elementary computations which are used by the theorems in the subsequent sections. In Section 2, we prove the relative sandwich theorem in the non-stable case. In the final section, we apply the relative sandwich theorem to the subnormal subgroups of non-stable unitary groups over rings and give a classification for them.

1. Elementary computations

In this section we prove several preparatory lemmas concerning the elementary computations of non-stable unitary groups.

From now on, we assume $(R, \Lambda)$ is a commutative form ring. Suppose $H$ is a subgroup of $U(2n, R, \Lambda)$ with $n \geq 4$, and $H$ is normalized by $EU(2n, J, I^J)$, where $(J, I^J)$ is a form ideal of the form ring $(R, \Lambda)$. Let $(I, l^J)$ denote the maximal form ideal of $(R, \Lambda)$ subject to the condition $EU(2n, I, l^J) \subseteq H$. For the uniqueness of $(I, l^J)$, see Lemma 2.7 in [11]. For any form ideal $(J, I^J)$, we use $(I^J)$ to denote the involution invariant ideal generated by the form parameter $I^J$.

Firstly, we shall record a useful lemma whose proof is straightforward.

**Lemma 1.1.** Let $G$ be a group and $S$ be a subgroup of $G$, such that $S$ is normalized by another subgroup $T$ of $G$. Then for any element $t \in T$, the group $gTg^{-1}$ also contains the subgroup $S$.

For convenience, we use $\delta_{i,j}$ to denote the row vector $(0, 0, \ldots, 1, \ldots, 0)$, whose ith entry is 1 and 0 elsewhere, and use $\delta_{i,i}^T$ to denote the column vector $\delta_{i,i}^T$, the transpose of $\delta_{i,i}$. For any element $g \in GL_n(R)$, we use the row vector $g_{i,j}$ to denote the ith row of $g$, and use the column vector $g_{i,j}$ to denote the ith column of $g$.

**Definition 1.2.** For any $g \in U(2n, R, \Lambda)$, we use $g'$ to denote the inverse of $g$, and use $g_{i,j}$ to denote the $(i, j)$th entry of $g$. For $i \neq j \neq \pm k$ and $\alpha \in R$, we use $T(i, j, k, \alpha, g)$ to denote the following element

$$T(i, j, k, \alpha, g) = T_{k, i}(-\alpha g_{i,j})T_{i, j}(\alpha g_{i,i}).$$

It is clear that $T(i, j, k, \alpha, g)$ belongs to $EU(2n, R, \Lambda)$. Furthermore, if $J$ is an involution invariant ideal of $R$ and $\alpha \in J$, then $T(i, j, k, \alpha, g) \in EU(2n, J, I^J_{\text{max}})$.

**Lemma 1.3.** For any $g \in U(2n, R, \Lambda)$ and any $\alpha \in R$, the $(i, i)$th entry of $[g, T(i, j, k, \alpha, g)]$ is zero.

**Proof.** A straightforward computation. □

We denote $[g, T(i, j, k, \alpha, g)] = M(i, j, k, \alpha, g)$ for any $g \in U(2n, R, \Lambda)$, $i \neq j \neq \pm k$ and $\alpha \in R$.

For an involution invariant ideal $I$ and $\alpha \in R$, we denote the involution invariant ideal $l\alpha + l\bar{\alpha}$ by $(l, \alpha)$.

**Lemma 1.4.** Let $H$ be a subgroup of $U(2n, R, \Lambda)$, $n \geq 3$, which is normalized by $EU(2n, J, I^J)$. If $H$ contains a short root elementary element $T_{i,j}(\alpha)$, then

$$EU(2n, J^J, \alpha, I^J_{\text{min}}) \subseteq H.$$

**Proof.** See Lemma 2.6 in [11]. □

**Lemma 1.5.** Let $H$ be a subgroup of $U(2n, R, \Lambda)$, $n \geq 4$, which is normalized by $EU(2n, J, I^J)$. If $H$ contains a long root elementary element $T_{i,-k}(\alpha)$, then

$$EU(2n, J^J, \alpha, I^J_{\text{min}}) \subseteq H.$$

**Proof.** It is well known, see [12, Proposition 5.1], that $EU(2n, J^J, \alpha, I^J_{\text{min}})$ is generated by all root elements

$$T_{i,j}(\delta)T_{i,j}(\xi)T_{i,j}(-\delta)$$

with $\xi \in (J^J, \alpha)$ and $\delta \in R$. 

For a typical short root generator $T_{j,i}(\delta)T_{i,j}(\xi)T_{j,i}(\delta)$ of $EU(2n, (J^4, \alpha), \Gamma_{\min}^{(J^4, \alpha)})$, then $\xi$ can be written as $xyx'y'\alpha + x_1y_1x_1'y_1'$ with $x, y, x', y', x_1, y_1, x_1', y_1' \in J$. Clearly,

$$T_{j,i}(\delta)T_{i,j}(\xi)T_{j,i}(\delta) = (T_{j,i}(\delta)T_{i,j}(xyx'y'\alpha)T_{j,i}(\delta)) \left( T_{j,i}(\delta)T_{i,j}(x_1y_1x_1'y_1')T_{j,i}(\delta) \right).$$

Therefore, it suffices to show that $T_{j,i}(\delta)T_{i,j}(xyx'y'\alpha)T_{j,i}(\delta) \in H$. The same argument works for the $T_{i,j}(x_1y_1x_1'y_1')$. We divide the proof into several cases.

Case 1. The general case, $i \neq \pm j \neq \pm k$.

By the basic rules of the elementary elements, see [12, R1–R6], we have

$$T_{j,i}(\delta)T_{i,j}(xyx'y'\alpha)T_{j,i}(\delta) = T_{j,i}(\delta) \left[ T_{i,k}(x'y'), [T_{k,-\delta}(\alpha), T_{-k}(xy)] \right] T_{j,i}(\delta).$$

Since the subgroup $H$ is normalized by $EU(2n, J, \Gamma')$ and $T_{k,-\delta}(\alpha)$ commutes with $T_{j,i}(\delta)$, it follows that $T_{j,i}(\delta)T_{i,j}(xyx'y'\alpha)T_{j,i}(\delta) \in H$.

Case 2. $i = k, j \neq \pm i$, choose $l \neq \pm k, \pm j$ and $s \neq \pm k, \pm l, \pm j$.

$$T_{j,k}(\delta)T_{k,i}(xyx'y'\alpha)T_{j,i}(\delta) = T_{j,k}(\delta) \left[ T_{k,l}(x'), [T_{l,s}(x'y'), T_{i,j}(y)] \right] T_{j,k}(\delta).$$

By the same argument used in Case 1, it suffices to show that $T_{i,k}(x'y') \in H$. But

$$T_{i,k}(x'y') = \left[ T_{i,k}(x'), [T_{k,-\delta}(\alpha), T_{-k}(y)] \right] \in H.$$

In Case 2, we have shown that $T_{i,k}(x'y') \in H$, therefore $T_{j,k}(\delta)T_{k,i}(xyx'y')T_{j,i}(\delta) \in H$.

In the case of a typical long root generator

$$T_{-l,i}(\delta)T_{i,-l}(\xi)T_{-l,i}(\delta), \quad \xi \in \Gamma_{\min},$$

without loss of generality, we may assume that $i \geq 0$. By definition, it follows that there exist $x, y, x', y', x_1, y_1, x_1', y_1' \in J$ and $\xi \in A$, such that

$$\xi = xyx'y' - \lambda xyx'y' + x_1y_1x_1'y_1' + y_1'\alpha\zeta x_1y_1x_1'y_1'.$$

Hence

$$T_{-l,i}(\delta)T_{i,-l}(\xi)T_{-l,i}(\delta) = T_{-l,i}(\delta)T_{i,-l}(xyx'y' - \lambda xyx'y' + x_1y_1x_1'y_1' + y_1'\alpha\zeta x_1y_1x_1'y_1').$$

It suffices to show that both $T_{-l,i}(\delta)T_{i,-l}(xyx'y' - \lambda xyx'y')T_{-l,i}(\delta)$ and $T_{-l,i}(\delta)T_{i,-l}(x_1y_1x_1'y_1' + y_1'\alpha\zeta x_1y_1x_1'y_1')T_{-l,i}(\delta)$ are contained in $H$.

By Relation (R5) in [12], given a $j \neq \pm i$, we have $T_{i,-l}(xyx'y' - \lambda xyx'y') = [T_{i,j}(xyx'y'), T_{-j}(\delta)]$. Therefore

$$T_{-l,i}(\delta)T_{i,-l}(xyx'y' - \lambda xyx'y')T_{-l,i}(\delta) = T_{-l,i}(\delta)T_{i,-l}(xyx'y')T_{-l,i}(\delta) \left[ T_{-j,i}(\delta)T_{i,j}(xyx'y'), T_{-j,i}(\delta)T_{i,j}(xyx'y') \right].$$

It can be shown that the above element belongs to $H$ by using an argument similar to that used in Cases 1, 2 and 3.

By Relation (R6) in [12], given a $j \neq \pm i$, we have

$$T_{-l,i}(\delta)T_{i,-l}(x_1y_1x_1'y_1' + y_1'\alpha\zeta x_1y_1x_1'y_1').$$

Following the method used in Cases 1, 2 and 3, it is not hard to prove that the above element is contained in $H$.

The above computations show that all typical generators of $EU(2n, (J^4, \alpha), \Gamma_{\min}^{(J^4, \alpha)})$ are contained in $H$. It follows immediately that $EU(2n, (J^4, \alpha), \Gamma_{\min}^{(J^4, \alpha)})$ is a subgroup of $H$, completing the proof.

**Lemma 1.6.** Let $A \in U(2n, R, \Lambda)$ and $B \in EU(2n, J, \Gamma')$. For any elementary matrix $T \in EU(2n, J, \Gamma')$, the commutator $[AB, T] = [A, T]C$ for some $C \in EU(2n, J, \Gamma_{\max}^{\delta})$. 


Proof. Direct computation shows
\[
[AB, T] = ABTB^{-1}A^{-1}T^{-1} \\
= ATA^{-1}T^{-1} (TA (T^{-1}BTB^{-1}) A^{-1}T^{-1}) \\
= [A, T] (TA[T^{-1}, B]A^{-1}T^{-1}).
\]
Let \( C = (TA[T^{-1}, B]A^{-1}T^{-1}). \) It remains to show that \( C \in EU(2n, I, \Gamma_{\text{max}}) \). By the definition of the \( EU \) group, it suffices to show that \( [T^{-1}, B] \in EU(2n, I, \Gamma_{\text{max}}) \), which clearly holds. This finishes the proof.

Lemma 1.7. For any \( g \in H \), if \( g \not\in CU(2n, (I \cup \{j\}), \Gamma_{\text{max}}) \) with \( n \geq 4 \), then there exists an element \( t \in EU(2n, R, \Lambda) \), an element \( \alpha \in J \) and some integers \( i \neq \pm j \), such that the ith row of \( M(i, j, k, \alpha, t^{-1}) \) does not belong to \( CU(2n, I, \Gamma_{\text{max}}) \).

Proof. Let \( g \in H \) satisfying the assumption of the lemma. Since \( g \not\in CU(2n, (I \cup \{j\}), \Gamma_{\text{max}}) \), there exist integers \( i, j \) with \( i \neq j \), such that \( g_{i,j}(2) \notin T \) or \( g_{i,j}(2) \notin T \). For the case that all \( g_{i,j}(2) \notin T \), there exist \( i, j \) with \( i \neq j \) such that \( (g_{i,j}(2)) \notin T \). Now we have to deal with two cases: firstly, if \( i \neq \pm j \), then the non-diagonal entry \( r_{i,j} \) of \( T_{i,j}(1)g_{i,j}(-1) \) generates \( I_{\pm} \), which is clearly not in \( I \). Secondly, if \( g_{i,j}(-1) \notin (I \cup \{j\}) \) and all \( g_{i,j} \notin (I \cup \{j\}) \) for \( i \neq \pm j \), then
\[
(g_{i,j} - g_{-i,-j}) = g_{i,j} + g_{-i,-j}.
\]
The left hand side of the above equation does not belong to \( (I \cup \{j\}) \), but the right hand side does, which is a contradiction. Therefore, for any \( g \notin U(2n, (I \cup \{j\}), \Gamma_{\text{max}}) \) we may find another element \( t \in thH^{-1} \), where \( t \in EU(2n, R, \Lambda) \), which is also denoted by \( g \), such that there is a non-diagonal element \( g_{i,j} \notin (I \cup \{j\}) \).

From now on, we may assume that there is an element \( g \in H^{-1} \), where \( t \in EU(2n, R, \Lambda) \), and the (i, k) entry of \( g \) is not in \( I \cup \{j\} \) with \( i \neq \pm k \). By the assumption \( n \geq 4 \), we may always choose \( i \neq \pm j \neq \pm k \) and any \( \alpha \in J \). Now consider \( M(i, j, k, \alpha, g) \). Since the subgroup \( H \) is normalized by \( EU(2n, J, \Gamma') \), the subgroup \( thH^{-1} \) is normalized by \( EU(2n, J, \Gamma') \). By the previous construction, \( M(i, j, k, \alpha, g) \) belongs \( thH^{-1} \). Lemma 1.3 shows that the (i, j)th entry of \( M(i, j, k, \alpha, g) \) is zero. Furthermore, the ith row of \( M(i, j, k, \alpha, g) \) generates \( 2(i, k, \alpha, g) \), which we denote by \( h \), is
\[
\begin{pmatrix}
(\alpha^{(e(k)-e(-j))}/\alpha^{(e(k)-e(-j))} - \alpha^{(e(k)-e(-j))}/\alpha^{(e(k)-e(-j))}) & \delta_{i,1} \\
(\alpha^{(e(k)-e(-j))} - \alpha^{(e(k)-e(-j))}) & \delta_{i,2} \\
\vdots & \vdots \\
(\alpha^{(e(k)-e(-j))} - \alpha^{(e(k)-e(-j))}) & \delta_{i,n}
\end{pmatrix},
\]
where \( \delta_{i,j} \) is the Kronecker delta. We denote the first summand of the formula (1) by \( v \). Notice that \( g_{i,j} \cdot v = \alpha g_{i,j}g_{j,i} \), and \( g_{i,j} \cdot v = \alpha g_{i,j}g_{j,i} \). It follows that \( v = h_{i,j} \neq \delta_{i,j} \) generates \( \alpha g_{i,j}g_{j,i} \) and \( \alpha g_{j,i}g_{i,j} \). By the construction of the elementary elements \( T(i, j, k, \alpha, g) \), it is routine to check that the entries of \( M(i, j, k, \alpha, g) \) generate \( \alpha g_{i,j}g_{j,i} \) and \( \alpha g_{j,i}g_{i,j} \).

The next step is to show that all the entries of \( M(i, j, k, \alpha, g) \), all \( i \neq \pm j \neq \pm k \), \( \alpha \in J \) and all \( t \in EU(2n, R, \Lambda) \), generate the subgroups \( 2\alpha g_{i,j}g_{j,i} \) and \( \alpha g_{j,i}g_{i,j} \). What is left to show is that the entries of \( M(i, j, k, \alpha, g) \), all \( i \neq \pm j \neq \pm k \), \( \alpha \in J \) and all \( t \in EU(2n, R, \Lambda) \), generate the subgroups \( 2\alpha g_{i,j}g_{j,i} \) and \( \alpha g_{j,i}g_{i,j} \).

Let us go back to the matrix \( g \). Consider \( T_{k,j}(1)g_{k,j}(1), \) whose (i, j) entry is \( g_{i,j} + g_{j,i} \). The entries of \( T_{k,j}(1)g_{k,j}(1) \) generate \( \alpha g_{i,j}g_{j,i} \). By the previous proof, the entries of the ith row of all such \( M(i, j, k, \alpha, T_{k,j}(1)g_{k,j}(1)) \) generate \( \alpha g_{i,j}g_{j,i} \). It follows that all those entries generate the subgroups \( 2\alpha g_{i,j}g_{j,i} \).

Finally, we show that \( 2\alpha g_{i,j}g_{j,i} \) can also be generated by the entries of such vector rows. Let us consider the matrix \( g \). By our assumption \( n \geq 4 \), we may choose \( s \neq \pm j, \pm k, \pm i \). Without loss of generality, we may assume that \( \varepsilon(s) = \varepsilon(j) = \varepsilon(k) = \varepsilon(\pm i) \). Set \( r = T_{j,k}(1)T_{k,j}(1)T_{-k,j}(1)T_{-j,k}(1) \in EU(2n, R, \Lambda) \). The (i, k)th entry and the (i, s)th entry of \( r^{-1}g \) generates \( g_{i,k} \) and \( 2g_{i,k} + g_{j,i} + g_{i,j} \), respectively. Therefore \( 2\alpha g_{i,k} \) can be generated by the entries of
\[
(M(i, j, k, \alpha, r^{-1}g))_{i,k} = \delta_{i,j}.
\]
We have shown that \( 2\alpha g_{i,j}g_{j,i} \) and \( \alpha g_{i,j}g_{j,i} \) can be generated by the entries of \( M(i, j, k, \alpha, r^{-1}g) \). All the entries of \( M(i, j, k, \alpha, r^{-1}g) \) generate \( 2\alpha g_{i,j}g_{j,i} \). It follows immediately that all the entries of \( M(i, j, k, \alpha, r^{-1}g) \) generate \( 2\alpha g_{i,j}g_{j,i} \), which does not belong to \( I \).

By the hypothesis \( g_{i,j} \notin I \), there is an element \( M(i, j, k, \alpha, r^{-1}g) \) for some \( r \in EU(2n, R, \Lambda) \), \( \alpha \in J \) and some integers \( i \neq \pm j, s \), such that \( M(i, j, k, \alpha, r^{-1}g) \) contains a non-trivial ith row modulo \( I \). This finishes the proof.

Lemma 1.8. Suppose that a subgroup \( H \) of \( U(2n, R, \Lambda) \) is normalized by \( EU(2n, J, \Gamma') \), and suppose \( H \) contains \( EU(2n, I, \Gamma_{\text{max}}) \) as a subgroup. If there is an element \( g \in H \) such that some entry of the ith row, say \( g_{i,s} \), satisfies \( f^{(j)}(\Gamma')g_{i,s} \not\in I \) and \( g_{i,s} = 0 \), then there is a non-trivial element \( g \in EU(2n, K, \Gamma_{\text{max}}) \subseteq H \), with \( K \not\subseteq I \).
Proof. Suppose that \( g \) satisfies the assumption of the lemma. Without loss of generality, we may assume that \( i = 1 \). By the assumption of the lemma, we may choose \( \xi \in I^i \) and \( \alpha, \beta, \gamma \in J \) such that \( \alpha \beta \gamma \xi g_{1,k} \notin (I : J^4) \). Consider the commutator

\[
h = [g^{-1}, T_{-1,1}(\xi)] = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ * & * & \ldots & * & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \ldots & * & 1 \end{pmatrix}.
\]

We divide the proof into two cases.

Case 1. There exists an entry \( h_{p,q} \), such that \( \alpha \beta \gamma h_{p,q} \notin (I : J^4) \) when \( p \neq -1 \) and \( q \neq 1 \). Without loss of generality, we may assume that \( q = 2 \). Consider the commutator

\[
h_1 = [h, T_{2,1}(\gamma)] = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ x_2 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{-1} - \lambda x_{-2} & \ldots & -x_2 & 1 \end{pmatrix}.
\]

Since \( \alpha \beta \gamma h_{p,q} \notin (I : J^4) \), there is an entry \( x_k \) such that \( \alpha \beta x_k \notin (I : J^4) \). If \( l \neq -1 \) then consider \( [T_{l,1}(\alpha \beta), h_1] = T_{l,k}(\alpha \beta x_k) \), where \( \alpha \beta x_k \notin (I : J^4) \) and \( T_{l,k}(\alpha \beta x_k) \in H \). By Lemma 1.4, \( EU(2n, J^4 \alpha \beta x_k, I_{\min}) \subseteq H \) where \( J^4 \alpha \beta x_k \subseteq I \). If for all \( l \neq -1, \alpha \beta x_k \in (I : J^4) \), then a direct computation shows that

\[
h_l = T_{-1,1}(\tau) \prod_{k=2}^{n} T_{k,1}(x_k) \prod_{k=-2}^{-n} T_{k,1}(x_k)
\]

with \( \alpha \beta \tau \notin (I : J^4) \). By Lemmas 1.6 and 1.5, we have

\[
EU(2n, J^4 \alpha \beta \tau, I_{\min}) \cdot X \subseteq H
\]

for some \( X \in EU(2n, I, I_{\max}) \). But \( EU(2n, I, I_{\max}) \subseteq H \), so it follows that \( EU(2n, J^4 \alpha \beta \tau, I_{\min}) \subseteq H \), where \( J^4 \alpha \beta \tau \subseteq I \).

Case 2. \( p = -1, q = 1 \) is the only entry satisfying \( \alpha \beta \gamma h_{p,q} \notin (I : J^4) \). We know that \( h \) has the following form:

\[
h = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ h_{2,1} & h_{2,2} & \ldots & h_{2,-2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{-1,1} & h_{-1,2} & \ldots & h_{-1,-2} & 1 \end{pmatrix}.
\]

Denote \( h^{-1} \) by \( h' \). We construct a new matrix \( h_1 \) from \( h^{-1} \):

\[
h_1 = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & h_{2,2} & \ldots & h_{2,-2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & h_{-2,2} & \ldots & h_{-2,-2} & 0 \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}.
\]

By [12, Lemma 2.3], \( h_1 \in U(2n, R, A) \). Furthermore, all the entries of \( h_1 \) belong to the ideal \( (I : J^j) \), so we have \( h_1 \in CU(2n, I : J^j, I_{\max}) \). The key observation in the lemma is

\[
h \cdot h_1 = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ h_{2,1} & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{-2,1} & 0 & \ldots & 0 & 0 \\ h_{-1,1} & h_{-1,2} & \ldots & h_{-1,-2} & 1 \end{pmatrix}.
\]

Denote \( h \cdot h_1 \) by \( T \), then

\[
T = T_{-1,1}(\tau) \prod_{k=2}^{n} T_{k,1}(h_{k,1}) \prod_{k=-2}^{-n} T_{k,1}(h_{k,1})
\]
for some $\tau \not\in (I : J^2)$. Therefore

$$h = T_{-1,1}(\tau) \prod_{k=2}^{n} T_{k,1}(h_{k,1}) \prod_{k=-2}^{-n} T_{k,1}(h_{k,1})^{-1}.$$  

(2)

Since $h \in EU(2n, R, A)$ (see [12, Theorem 1.1]), we have $h_1 \in EU(2n, R, A)$. By the Whitehead–Vaserstein lemma (see e.g. [12, Lemmas 7.3–7.5]), $h_1 \in TU(2n, (I : J^2), I_{\max})$ implies $h_1 \in EU(2n, (I : J^2), I_{\max})$. Finally, by Lemmas 1.6 and 1.5, $H$ contains the elementary subgroup $EU(2n, J^2, I_{\min})$, with $J^2 \not\subseteq I$. This finishes the proof. □

2. Relative sandwich classification

In this section, we prove the main theorem of the current paper, namely the relative sandwich classification theorem. It settles a conjecture of Bak (1967) when $A = A_{\max} = A_{\min}$.

**Theorem 2.1.** Suppose $(R, A)$ is a commutative form ring, in which $2$ is invertible, and $n \geq 4$. Suppose $H$ is a subgroup of the unitary group $U(2n, R, A)$, which is normalized by $EU(2n, J, I^d)$ for some form ideal $(J, I^d)$ of the form ring $(R, A)$. Suppose that $(I, I')$ is the largest form ideal with the property $EU(2n, I, I') \subseteq H$. Then $H$ satisfies a sandwich property

$$EU(2n, I, I') \subseteq H \subseteq CU(2n, I : J^8(I^d), I').$$

**Proof.** The invertibility of $2 \in R$ implies that there is a unique choice of form parameter for any given ideal in $R$ (see [12]), i.e., $I_{\min}^d = I_{\max}^d$ for any ideal $I \subseteq R$. If $H \not\subseteq CU(2n, I : (2)^8(I^d), I')$, then there is an element $g \in H$, such that $g \notin CU(2n, I : (2)^8(I^d), I_{\max})$. Since $2$ is invertible in $R$, we have $(I : (2)^8(I^d)) = (I : J^8(I^d))$. By Lemma 1.7, we may find an element $t \in EU(2n, R, A)$ such that $tgt^{-1} \notin CU(2n, I : J^8(I^d), I')$, and at least one entry in the second diagonal of $tgt^{-1}$ is zero, i.e., there exists an integer $i$, such that the $(i, -i)$th entry of $tgt^{-1}$ is zero. The ideal generated by the entries of $((tgt^{-1})_{i,-i})$ is not in $(I : J^8(I^d))$. Applying Lemma 1.8, we locate an ideal $K \subseteq I$, such that $EU(2n, K, I') \subseteq H$. Lemma 2.7 in [11] implies that $EU(2n, K + I, I') \subseteq H$ which contradicts the maximality of the form ideal $(I, I')$. This completes the proof. □

**Corollary 2.2.** Suppose $(R, A)$ is a commutative form ring, where $2$ is invertible, and suppose $H$ is a subgroup of the unitary group $U(2n, R, A)$, which is normalized by $EU(2n, J, I^d)$ for some form ideal $(J, I^d)$ of $(R, A)$. Suppose that $(I, I')$ is the smallest form ideal with the property $H \subseteq CU(2n, I, I')$. Then $H$ satisfies a sandwich property

$$EU(2n, I^8(I^d), I') \subseteq H \subseteq CU(2n, I, I').$$

**Proof.** Let $(K, I')$ be the largest form ideal such that $EU(2n, K, I') \subseteq H$. By the main theorem

$$H \subseteq CU(2n, K : J^8(I^d), I').$$

By the assumption of the corollary, $I \subseteq (K : J^8(I^d))$, it follows immediately that

$$I^8(I^d) \subseteq K.$$  

By $EU(2n, K, I') \subseteq H$, we have

$$EU(2n, I^8(I^d), I') \subseteq H.$$  

This completes the proof. □

3. The structure of subnormal subgroup

In this section we discuss an application of the sandwich classification, namely analyzing the structure of subnormal subgroups of unitary groups over commutative form rings. We always assume $2$ is invertible in this section. First we recall the definition of a subnormal subgroup.

**Definition 3.1.** A subgroup $H$ of a group $G$ is subnormal if there is a finite chain of subgroups of $G$ such that

$$H = H_0 \triangleleft H_{d-1} \triangleleft \cdots \triangleleft H_0 = G.$$  

In this case, we write $H \triangleleft G$.

Let $H$ be a subgroup of $G \subseteq U(2n, R, A)$. Since we are working on a form ring with a unique form ideal for any given ideal, for convenience, we may skip the form parameter part of any given form ideal. We use $L(H)$ to denote the maximal involution invariant ideal $I$ of the ring $R$ such that $EU(2n, I) \subseteq H$. Similarly, we use $U(H)$ to denote the smallest involution invariant ideal $Q$ of the ring $R$ such that $H \subseteq U(Q)$.

The following theorem is an analog of a theorem of L.N. Vaserstein for the general linear group (see Theorem 1 in [5].)

**Theorem 3.2.** Let $G$ be a subgroup of $U(R)$ containing $EU(2n, R)$ with $n \geq 4$, and let $H$ be a subnormal subgroup of $G$, i.e.,

$$H \triangleleft G \quad \text{for some integer } d.$$
Then
\[ f(U(H), d) \subseteq L(H), \]
where \( f(U(H_0), n) \) is defined recursively, i.e.,
\[ f(U(H_0), 0) = R, \quad f(U(H_0), n) = U(U(H_{n-1})) f(U(H_{n-1}), n - 1) \Lambda (f(U(H_{n-1}), n - 1) \Lambda \Lambda). \]

**Proof.** The proof is by induction.

When \( d = 0 \), then \( H = G \). It follows immediately that
\[ L(H) = R = f(U(H), 0). \]

Suppose we have proved that \( f(U(H_t), t) \subseteq L(H_t) \) for some \( t \). Since \( H_{t+1} \triangleleft H_t \), and \( EU(2n, L(H_t)) \subseteq H_t, H_{t+1} \) is normalized by \( EU(2n, L(H_t)) \). By Corollary 2.2,
\[ U(H_{t+1}) L(H_t) \Lambda (L(H_t) \Lambda \Lambda) \subseteq L(H_{t+1}). \]

Since \( U(H_{t+1}) \subseteq U(H_t) \), we have
\[
\begin{align*}
f(U(H_{t+1}), (t + 1)) &= U(H_{t+1}) f(U(H_t), t) \Lambda (f(U(H_t), t) \Lambda \Lambda) \\
&\subseteq U(H_{t+1}) L(H_t) \Lambda (L(H_t) \Lambda \Lambda) \\
&\subseteq L(H_{t+1}).
\end{align*}
\]

This finishes the proof. \( \square \)

**Acknowledgements**

Thanks are due to Anthony Bak for his hospitality during my stay in Bielefeld. Thanks also go to Anthony Bak, Roozbeh Hazrat, Nikolai Vavilov and Judith Millar for their comments and suggestions.

**References**


**Further reading**


