Arithmetical Properties of a Certain Power Series

KUMIKO NISHIOKA

Department of Mathematics, Nara Women's University,
Kita-Uoya Nishimachi, Nara, 630 Japan

IEKATA SHIKAWA

Department of Mathematics, Keio University,
Hiyoshi, Yokohama, 223 Japan

AND

JUN-ICHI TAMURA

Faculty of General Education, International Junior College,
Ekoda 4-15-1, Nakano-ku, Tokyo, 165 Japan

Communicated by Hans Zassenhaus

Received June 18, 1990; revised March 6, 1991

The function

\[ S(\theta, \phi; x, y) = \sum_{k=1}^{\infty} \sum_{m=k\theta + \phi}^{\infty} x^m y^m, \]

where \( \theta > 0 \) is irrational and \( \phi \) is real, satisfies Mahler-type functional equations which enable us to represent it by a gap-like series and then by a continued fraction. Using these representations, we describe the sequence \( \{ [k+1]0 + d - [k0 + ] \} \) by a chain of substitutions and give algebraic independence results for the values of \( f(\theta, \phi, x, y) \) at some algebraic points when the partial quotients of the continued fraction of \( \theta \) are unbounded, and irrationality measures for the values at some rational points.

\[ \frac{\theta + \phi}{1 - \phi} = \sum_{k=1}^{\infty} ([k+1] \theta + \phi - [k\theta + \phi]) x^k, \]

\[ \frac{[k+1]0 + d - [k0 + ]}{1 - \phi} = \sum_{k=1}^{\infty} [k\theta + \phi] x^k. \]

INTRODUCTION

One of the motivations of the present paper is to investigate the sequence \( \{ [k+1] \theta + \phi - [k\theta + \phi] \}_{k=1}^{\infty} \) in connection with a billiard problem in the unit square stated in Section 2, where \( \theta > 0 \) is irrational and \( \phi \) is real and \( [t] \) is the integral part of a real number \( t \). For this purpose it is natural to consider the power series

\[ \frac{1 - \theta}{\phi} = \sum_{k=1}^{\infty} [k\theta + \phi] x^k. \]
The last series can be obtained from the function

\[ f(\theta, \phi; x, y) = \sum_{k=1}^{\infty} \sum_{1 \leq m < k\theta + \phi} x^k y^m \]

by specialising to \( y = 1 \); namely, we have

\[ f(\theta, \phi; x, 1) = \sum_{k=1}^{\infty} [k\theta + \phi] x^k. \]

Throughout this paper it should be understood that, if \( k\theta + \phi < 0 \), then the term corresponding to \( k \) must be omitted from the sum. \( f(\theta, \phi; x, y) \) is considered as a perturbation of the Mahler function \( \sum_{k=1}^{\infty} \sum_{1 \leq m < k\theta} x^k y^m = f(\theta, 0; x, y) \) which has been treated by many authors. Historical surveys and references on the transcendence of the special values of \( f(\theta, 0; x, y) \) can be found in [9, 10, 12] and some other aspects of the function \( \sum_{k=1}^{\infty} [k\theta] x^k = f(\theta, 0; x, 1) \) are discussed in, e.g., [14, 8].

\( f(\theta, \phi; x, y) \) satisfies certain functional equations ((2), (3) below) which with a suitable expansion of \( \phi \) related to the continued fraction of \( \theta \) enable us to represent the function in a series and then in a continued fraction whose practical quotients are rational functions of \( x \) and \( y \) (see Theorem 1 and Corollary 1 in Section 1). With these representations everything proceeds more or less as for the case \( \phi = 0 \). In Section 2, the sequence \( \{ [k + 1 + \phi] - [k\theta + \phi] \}_{k=1}^{\infty} \) is described by a chain of substitutions. In particular, if \( \theta \) is quadratic irrational and \( \phi \) is in the field \( \mathbb{Q}(\theta) \), it can be written by a fixed point of a substitution. In Section 3, irrationality measures for \( f(\theta, \phi; 1/a, 1/b) \), where \( a, b \) are integers with \( |a| \geq 2, b \neq 0 \), are given. Finally Section 4, we prove the algebraic independence of the values \( f(\theta, \phi; \alpha_i, \beta_i) \) \( (i = 1, \ldots, n) \) with some algebraic numbers \( \alpha \) and \( \beta \), when the partial quotients of the continued fraction of \( \theta \) are unbounded, by using Evertse’s theorem on S-unit equations and non-vanishing of the regulator of an algebraic number field.

1. Functional Equations

Throughout this paper \( \theta > 0 \) is irrational. Then \( f(\theta, \phi; x, y) \) is a transcendental function over the field of rational functions for any given real \( \phi \) (see, e.g., [11]). Assuming for the moment \( |x| < 1 \) and \( |y| < 1 \), we have

\[ f(\theta, \phi; x, y) = -f(1/\theta, -\phi/\theta; y, x) + \frac{x}{1-x} \frac{y}{1-y} + \delta(\theta, \phi; x, y), \]  

(2)

where \( \delta(\theta, \phi; x, y) = x^{k_0} y^{m_0} \), if \( k_0 \theta + \phi = m_0 \) for some positive integers \( k_0 \).
and $m_0$; it equals 0, otherwise. Since $\theta$ is irrational, such a pair $(k_0, m_0)$ is unique, if it exists. The series (1) converges in $|x| < 1$, $|x||y]|^q < 1$, while in the right-hand side of (2), $f(1/\theta, -\phi/\theta; y, x)$ converges in $|y| < 1$, $|x||y]|^q < 1$. So the function $f(\theta, \phi; x, y)$ defined by (1) can be continued analytically in the region.

$$|x||y]|^q < 1.$$ 

It can be shown that, if $a \geq 0$ and $b$ are integers,

$$f(\theta, b + \phi; x, y) - \frac{x}{1-x} \frac{y}{1-y} = y^b \left( f(\theta, \phi; x, y) - \frac{x}{1-x} \frac{y}{1-y} \right),$$

$$f(a + \theta, \phi; x, y) - \frac{x}{1-x} \frac{y}{1-y} = f(\theta, \phi; xy^a, y) - \frac{x y^a}{1-x y^a} \frac{y}{1-y},$$

so that

$$f(a + \theta, b + \phi; x, y) = \frac{x}{1-x} \frac{y}{1-y} + y^b \left( f(\theta, \phi; xy^a, y) - \frac{x y^a}{1-x y^a} \frac{y}{1-y} \right).$$

(3)

Let $\theta = [a_0; a_1, a_2, ...]$ denote the simple continued fraction of $\theta$, where

$$\theta = a_0 + \theta_0, \quad a_0 = [\theta], \quad 1/\theta_{n-1} = a_n + \theta_n,$$

$$a_n = [1/\theta_{n-1}] (n \geq 1).$$

(4)

The $n$th convergent $p_n/q_n = [a_0; a_1, ..., a_n]$ of $\theta$ is given by the recurrence relations

$$p_n = a_n p_{n-1} + p_{n-2} (n \geq 1), \quad p_{-2} = 0, p_{-1} = 1,$$

$$q_n = a_n q_{n-1} + q_{n-2} (n \geq 1), \quad q_{-2} = 1, q_{-1} = 0.$$

We note that, if $n \geq 0$,

$$p_n + p_{n-1} = \sum_{v=1}^{n} a_v p_{v-1} + p_0 + p_{-1}, \quad q_n + q_{n-1} = \sum_{v=1}^{n} a_v q_{v-1} + q_0 + q_{-1},$$

(5)

$$\theta_0 \theta_1 \cdots \theta_n = (-1)^n (q_n \theta - p_n).$$

(6)

We expand $\phi$ in terms of the sequence $\{\theta_0, \theta_1, \ldots\}$. Put

$$\phi = b_0 - \phi_0, \quad b_0 = -[-\phi]$$

$$\phi_{n-1} = b_n - \phi_n, \quad b_n = -[-\phi_{n-1}/\theta_{n-1}] (n \geq 1).$$

(7)
Then $b_n$ is an integer with $0 \leq b_n \leq a_n + 1$ $(n \geq 1)$, $0 \leq \phi_n < 1$ $(n \geq 0)$, and
\[
\phi = b_0 + \sum_{n=1}^{N} (-1)^n \theta_0 \theta_1 \cdots \theta_{n-1} b_n + (-1)^{N-1} \theta_0 \theta_1 \cdots \theta_{N-1} \phi_N \quad (N \geq 0).
\] (8)

Since $\theta_{n-1} \theta_n < 1/2$ and $\theta_{n-1} b_n < 2$, the series
\[
\phi = b_0 + \sum_{n=1}^{\infty} (-1)^n \theta_0 \theta_1 \cdots \theta_{n-1} b_n = b_0 \cdot b_1 b_2 \ldots, \quad \text{say} \quad (9)
\]
is convergent. The series terminates if and only if $\phi_n = 0$ for some $n \geq 0$, or what amounts to the same thing, $b_{n+1} = 0$ for some $n \geq 0$. In this case, if $N$ is the smallest integer $n$ for which $\phi_n = 0$, then $b_n = 0$ for all $n > N$ and
\[
\phi = b_0 \cdot b_1 b_2 \cdots b_N.
\]

Otherwise, $b_n \geq 1$ and $0 < \phi_n < 1$ for all $n \geq 1$.

Suppose that $\delta(\theta_n, -\phi_n; x, y) \neq 0$ for some $n \geq 0$, or equivalently, $k_n \theta_n - \phi_n = m_n$ for some $n \geq 0$ and positive integers $k_n$ and $m_n$. Then it follows from (6) and (8) that
\[
\phi = b_0 + \sum_{v=1}^{n} b_v p_{v-1} + m_n p_{n-1} + k_n p_n - \left( \sum_{v=1}^{n} b_v q_{v-1} + m_n q_{n-1} + k_n q_n \right) \theta.
\]

Therefore, we have the following

**Lemma 1.** Let $\theta > 0$ be irrational and $\phi$ be real. If
\[
k\theta + \phi \neq m \quad \text{for all integers } k \geq 1 \text{ and } m,
\] (10)

then we have
\[
\delta(\theta_n, -\phi_n; x, y) = 0 \quad \text{for all } n \geq 0.
\]

**Remark 1.** Let $\theta$ and $\phi$ be as in Lemma 1. If
\[
k_0 \theta + \phi = m_0
\]
for some integers $k_0 \geq 1$ and $m_0$, then by (1)
\[
f(\theta, \phi; x, y) = p(x, y)/(1 - x) + x^k y^m f(\theta, 0; x, y),
\]
where $p(x, y)$ is a polynomial with integral coefficients, and in the last term $\phi = 0$ satisfies the condition (10).
EXAMPLE 1. Assuming (10) and 0 < \theta < 1, -1 < \phi \leq 0, we have by (1) and (2)

\[
\sum_{k=0}^{\infty} \left[(k+1) \theta + \phi\right] x^k - \frac{1-x}{x} \sum_{k=1}^{\infty} \left[k\theta + \phi\right] x^k = \sum_{k=1}^{\infty} x^{[k\theta - \phi/\theta]}, \quad |x| < 1.
\]

In what follows, we assume (10) unless mentioned otherwise. It follows from (2), (3), and Lemma 1 that

\[
f(\theta, \phi; x, y) = \frac{x}{1-x-1-y} = \frac{x^{b_0}}{1-x - y} \frac{x}{1-y} + y^{b_0}f(\theta_0, -\phi_0; x y^{a_0}, y)
\]

and

\[
f(\theta_n, -\phi_n; x, y) = x^{b_{n+1}} \frac{x^{a_{n+1}} y}{1-x^{a_{n+1}}} \frac{x}{1-y} - x^{b_{n+1}} f(\theta_{n+1}, -\phi_{n+1}; x^{a_{n+1}} y, x)
\]

for \(n \geq 0\). Hence we get

\[
f(\theta, \phi; x, y) = \sum_{n=1}^{N} (-1)^{n-1} x^{n y x^n} \frac{x^{q_n y^{p_n}}}{1-x^{q_n y^{p_n}}} \frac{x^{q_{n-1} y^{p_{n-1}}}}{1-x^{q_{n-1} y^{p_{n-1}}}} + (-1)^N x^{q_N y^{p_N}} f(\theta_N, -\phi_N; x^{q_N y^{p_N}}, x^{q_N y^{p_N}}),
\]

where

\[
s_n = \sum_{v=0}^{n} b_v p_{v-1} (n \geq 0), \quad s_n = 0 (n < 0),
\]

\[
t_n = \sum_{v=0}^{n} b_v q_{v-1} (n \geq 0), \quad t_n = 0 (n < 0).
\]

and \(b_v\) is defined by (7). Therefore we have the following.

THEOREM 1. Let \(\theta > 0\) be an irrational number and \(\phi\) be a real number satisfying (10). Then we have

\[
f(\theta, \phi; x, y) = \sum_{n=1}^{\infty} (-1)^{n-1} x^{n y x^n} \frac{x^{q_n y^{p_n}}}{1-x^{q_n y^{p_n}}} \frac{x^{q_{n-1} y^{p_{n-1}}}}{1-x^{q_{n-1} y^{p_{n-1}}}} + (-1)^N x^{q_N y^{p_N}} f(\theta_N, -\phi_N; x^{q_N y^{p_N}}, x^{q_N y^{p_N}}),
\]
where \( p_n/q_n \) is the \( n \)th convergent of \( \theta \) and \( t_n, s_n \) are defined by (12). The series converges in the region \(|x|, |y| < 1\) except at points \((x, y)\) such that \(x^{s_n} y^{t_n} = 1\) for some \(n \geq 0\).

It follows from Theorem 1 that

\[
\left( \frac{1 - xy^{q_0}}{xy^{q_0} + b_0} \right)^2 f(\theta, \phi; x, y) = A_0 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{Q_n Q_{n-1}},
\]

where

\[
A_0 = \left( \frac{1 - xy^{q_0}}{xy^{q_0} + b_0} \right)^2 \left( \frac{x}{1 - x} \frac{y}{1 - y} - \frac{y^{q_0}}{1 - y^{q_0}} \right),
\]

\[
Q_n = \frac{1 - x^{q_n} y^{t_n}}{x^{q_n} + y^{t_n} + \sigma_n} \frac{xy^{q_0} + b_0}{1 - xy^{q_0}}
\]

and

\[
\sigma_n = b_n p_{n-1} + b_{n-2} p_{n-3} + \cdots (n \geq 0),
\]

\[
\tau_n = b_n q_{n-1} + b_{n-2} q_{n-3} + \cdots (n \geq 0)
\]

with \(b_n = 0\) \((n < 0)\). Noticing that \(Q_0 = 1\) and putting

\[
A_n = (Q_n - Q_{n-2})/Q_{n-1} \quad (n \geq 1)
\]

we have the following

**COROLLARY 1.** Let \( \theta \) and \( \phi \) be as in Theorem 1. Then we have

\[
\left( \frac{1 - xy^{q_0}}{xy^{q_0} + b_0} \right)^2 f(\theta, \phi; x, y) = [A_0; A_1, A_2, \ldots],
\]

where the \( A_n \) \((n \geq 0)\) are defined above. The \(n\)th convergent \([A_0; A_1, \ldots, A_n]\) is given by \(P_n/Q_n\), where

\[
P_{-1} = 1, \quad P_0 = A_0, \quad P_n = A_n P_{n-1} + P_{n-2} \quad (n \geq 1),
\]

\[
Q_{-1} = 0, \quad Q_0 = 1, \quad Q_n = A_n Q_{n-1} + Q_{n-2} \quad (n \geq 1).
\]

**EXAMPLE 2.** Let \( \theta \) and \( \phi \) be as in Theorem 1. If \(0 < \theta < 1\) and \(-1 < \phi \leq 0\), then we have

\[
\left( \frac{1-x}{x} \right)^2 f(\theta, \phi; x, y) = [0; A_1, A_2, \ldots],
\]
where

\[ Q_n = \frac{1 - x^{q_n} y^{p_n} x}{x^{q_n + \tau_n} y^{p_n + \sigma_n} 1 - x} \quad (n \geq 0) \]

and

\[ A_n = x^{-q_n + q_n - 1 - \tau_n + \tau_n + s_n - s_n + s_n - s_n - 1} \times \frac{1 - x^{q_n} y^{p_n} - (x^{q_n - 1} y^{p_n - 1})^{(a_n + b_n)} (1 - x^{q_n - 2} y^{p_n - 2})}{1 - x^{q_n - 1} y^{p_n - 1}} \quad (n \geq 1). \]

2. The sequence \( \{(k + 1) \theta + \phi] - [k \theta + \phi] \}_{k=1}^{\infty} \)

To investigate the sequence in the title of this section we may assume the condition (10), since if \( k_0 \theta + \phi = m_0 \) for some integers \( k_0 \geq 1 \) and \( m_0, [(k + 1) \theta + \phi] - [k \theta + \phi] = [(k - k_0 + 1) \theta] - [(k - k_0) \theta] \). Furthermore, we may assume \( 0 < \theta < 1 \) and \( -1 < \phi \leq 0 \). Then it follows from Example 1 that

\[ \sum_{k=1}^{\infty} [(k + 1) \theta + \phi] - [k \theta + \phi] x^k = \frac{1 - x}{x} f(\theta, \phi; x, 1), \quad |x| < 1. \]  

(13)

We shall study the coefficients of the Taylor expansion of the function \( (1 - x)/x f(\theta, \phi; x) \) at \( x = 0 \). By Example 2 we have

\[ \left( \frac{1 - x}{x} \right)^2 f(\theta, \phi; x, 1) = \lim_{n \to \infty} P_n(x) \frac{1}{Q_n(x)} = \lim_{n \to \infty} P_n^*(x), \]  

(14)

where

\[ P_n^*(x) = x^{q_n + \tau_n} p_n, \quad Q_n^*(x) = (1 - x^{q_n})/(1 - x) \quad (n \geq 1). \]

Here it follows from Corollary 1 and Example 2 that

\[ P_n^*(x) = A_n^*(x) P_{n-1}^*(x) + x^{(a_n + b_n) q_n - 1} P_{n-2}^*(x) \quad (n \geq 1) \]  

(15)

with \( P_{-1}^*(x) = 1, P_0^*(x) = 0 \), where

\[ A_n^*(x) = \frac{1 - x^{q_n} - x^{(a_n + b_n) q_n - 1}(1 - x^{q_n - 2})}{1 - x^{q_n - 1}} \quad (n \geq 1). \]  

(16)

We first give the proof of our theorem when \( \phi = 0 \). Although the result (Theorem 2) is known in this special case (see, e.g., [4]), it provides a
natural introduction to the case of $\phi = 0$. Since $\phi = 0$, (10) is satisfied. It follows from (15) and (16) that

$$P_n^*(x) = P_{n-1}^*(x) + x^{q_{n-1}} P_{n-1}^*(x) + \cdots + x^{(u_n - 1)q_{n-1}} P_{n-2}^*(x)$$

for $n \geq 1$, where $P_{n-1}^*(x) = 1$, $P_0^*(x) = 0$. Here it can be shown by induction on $n$ that

$$\text{ord}_{x=0} P_n^*(x) = a_1, \quad \text{deg} P_n^*(x) = q_n - \frac{1 + (-1)^n}{2} (n \geq 1).$$

and so

$$\text{ord}_{x=0} x^{q_{n-1}} P_{n-1}^*(x) - \text{deg} P_{n-1}^*(x) = \text{ord}_{x=0} x^{(u_n - 1)q_{n-1}} P_{n-2}^*(x) - \text{deg} x^{(u_n - 1)q_{n-1}} P_{n-1}^*(x)$$

$$= a_1 + 1 + (-1)^{n-1} \geq 1 \quad (n \geq 3).$$

Hence $P_n^*(x)$ converges in $|x| < 1$ to a power series as $n \to \infty$, and hence (13) and (14) yield

$$\sum_{k=1}^{\infty} \left(\left[(k+1) \theta\right] - \left[k \theta\right]\right) x^k = \lim_{n \to \infty} P_n^*(x), \quad |x| < 1.$$

Therefore we obtain the following

**Theorem 2.** Let $\theta = [0; a_1, a_2, \ldots]$ be irrational and let $p_n/q_n = [0; a_1, \ldots, a_n]$. Then we have

$$\left\{\left[(k+1) \theta\right] - \left[k \theta\right]\right\}_{k=1}^{\infty} = \lim_{n \to \infty} w_n,$$

where $w_n$ is the sequence of length $q_n$ whose terms are 0 or 1 defined by

$$w_0 = 0, \quad w_1 = 0 \cdots 01, \quad w_n = \underbrace{w_{n-1} \cdots w_{n-1}}_{a_n \text{ times}} w_{n-2} \quad (n \geq 2)$$

and $\lim_{n \to \infty} w_n$ is the infinite sequence of 0, 1 whose first $q_n$ terms coincide with those of $w_n$.

The construction of $w_n$ has a natural interpretation into the language of substitutions. Let $\Sigma = \{a, b, \ldots, d\}$ be the set of finite symbols $a, b, \ldots, d$ and $\Sigma^*$ be the free monoid generated by $a, b, \ldots, d$; namely, $\Sigma^*$ is the set of all finite words on $a, b, \ldots, d$ including the empty word $\epsilon$, in which the product of two words $u, v \in \Sigma^*$ is defined by the concatenation $uv$. A substitution $\sigma$
on $\Sigma^*$ is an endomorphism on $\Sigma^*$. Let $\Sigma^\infty$ be the set of all infinite words on $a, b, ..., d$. A substitution $\sigma$ on $\Sigma^*$ can be extended to $\Sigma^\infty$ by defining, for a given $w = w_1 w_2 \cdots \Sigma^\infty$ ($w_i \in \Sigma$), $\sigma(w) = \sigma(w_1) \sigma(w_2) \cdots$ ($\in \Sigma^\infty$). A fixed point of a substitution $\sigma$ is an infinite word $w \in \Sigma^\infty$ such that $\sigma(w) = w$. We remark that any substitution $\sigma$ on $\Sigma^\infty$ of the form

$$\sigma(a) = au (u \neq \varepsilon), \quad \sigma(b) \neq \varepsilon, ..., \sigma(d) \neq \varepsilon$$

has the unique fixed point $w$ prefixed by $a$, namely,

$$w = a(u) \sigma(u) \cdots \Sigma^\infty.$$ 

Here the product $\sigma \circ \tau$ of two substitutions $\sigma$ and $\tau$ on $\Sigma^*$ is defined by $\sigma \circ \tau(u) = \sigma(\tau(u)) (u \in \Sigma^*)$, and $\sigma^a = \sigma \circ \sigma^{a-1}$.

**COROLLARY 2.** Let $\theta = [0; a_1, a_2, ...]$ be irrational. We define the substitution $\sigma_i$ on $\{a, b\}^*$ by

$$\begin{align*}
\sigma_i: & \\
& \begin{cases} 
a \rightarrow a \cdots \overbrace{ab}^{i \text{ times}}, \\
b \rightarrow a,
\end{cases}
\end{align*}$$

and put

$$w_1 = \sigma_{a_1-1}(a) \quad \text{with} \quad a = 0, b = 1, w_0 = 0,$$

$$w_n = \sigma_{a_n}(a) \quad \text{with} \quad a = w_{n-1}, b = w_{n-2} \quad (n \geq 2).$$

Then we have

$$\left\{ \left[ (k+1) \theta \right] - \left[ k \theta \right] \right\}_{k=1}^\infty = \lim_{n \to \infty} w_n.$$ 

Corollary 2 is illustrated in Figs. 1 and 2.

If $\theta$ is a quadratic irrational, the sequence $\{ [ (k+1) \theta ] - [k \theta] \}_{k=1}^\infty$ can be given as a fixed point of a certain substitution. Let $\theta, 0 < \theta < 1$, be a quadratic irrational. Then $\theta$ can be written as a periodic continued fraction

$$\theta = [0; a_1, ..., a_j, a_{j+1}, ..., a_{j+1}].$$

![Diagram](image_url)

$w_n = w_{n-1} w_{n-1} \cdots w_{n-1} w_{n-2}$

Fig. 1. The word $w_n (n \geq 2)$. 
Let $\sigma$ be the substitution on \{0, 1\} defined by
\[
\sigma = \sigma_{a_1} \circ \sigma_{a_2} \circ \cdots \circ \sigma_{a_j},
\]
where $\sigma$ denotes the identity map when $j = 0$. Put $a = \sigma(0), \ b = \sigma(1) \in \{0, 1\}$, and define the substitution $\tau$ on \{a, b\} by
\[
\tau = \sigma_{a_{j+1}} \circ \cdots \circ \sigma_{a_{j+1}},
\]
which is of the form (17).

**Corollary 3.** Let $\theta$ be a quadratic irrational with $0 < \theta < 1$. Then with the notations as above
\[
\{(k + 1) \theta - [k\theta]\}_{k=1}^{\infty} = \omega(a, b; \tau),
\]
where $\omega(a, b; \tau)$ is the fixed point of $\tau$ prefixed by $a$.

Now we consider the sequence (12) with $\phi \neq 0$. We put for brevity $A_n^* = A_n^*(x), \ \pi_n = x^{-a_1-b_1}P_n^*(x), \ X_n = x^{a_n}(n \geq 1)$.

Then it follows from (15) and (16) that
\[
\pi_n = A_n^* \pi_{n-1} + X_n^{a_{n-1}+b_n} \pi_{n-2} (n \geq 2), \ \pi_0 = 0, \ \pi_1 = 1, \quad (18)
\]
where
\[ A_n^* = \frac{1 - X_n - X_{n-1}^{\alpha_n + \beta_n}(1 - X_{n-2})}{1 - X_{n-1}} \]
\[ = (1 - X_n)(1 + X_{n-1} + \cdots + X_{n-1}^{\beta_n-1}) + X_{n-1}^{\beta_n} + \cdots + X_{n-1}^{\alpha_n + \beta_n - 1}. \]  

**Lemma 2.** We have
\[ \pi_n = \alpha_n + \beta_n - X_n \alpha_n \quad (n \geq 1), \]
(20)
where \( \beta_0 = 0, \alpha_1 = 0, \beta_1 = 1, \) and if \( n \geq 2 \)
\[ \alpha_n = \alpha_{n-1} + (1 + X_{n-1} + \cdots + X_{n-1}^{\beta_n-1}) \beta_{n-1}, \]
\[ \beta_n = (X_{n-1}^{\beta_n} + X_{n-1}^{\beta_n+1} + \cdots + X_{n-1}^{\alpha_n + \beta_n-1}) \beta_{n-1} + X_{n-1}^{\alpha_n + \beta_n} X_{n-1}^{\beta_n} \beta_{n-2}. \]
(21)

**Proof:** Equation (20) is true if \( n = 1. \) Let \( n \geq 2 \) and assume that it is true up to \( n - 1. \) Then it follows from (18), (19), and (20) with \( n - 1 \) that
\[ \pi_n = A_n^*(1 - X_{n-1}) \alpha_{n-1} + A_n^* \beta_{n-1} + X_{n-1}^{\alpha_n + \beta_n} \pi_{n-2} \]
\[ = (1 - X_n) \{ (\alpha_{n-1} + (1 + X_{n-1} + \cdots + X_{n-1}^{\beta_n-1}) \beta_{n-1} \}
\[ + (X_{n-1}^{\beta_n} + \cdots + X_{n-1}^{\alpha_n + \beta_n-1}) \beta_{n-1} + X_{n-1}^{\alpha_n + \beta_n} \{ \pi_{n-2} - (1 - X_{n-2}) \alpha_{n-1} \}. \]

Here using (20) with \( n - 2, \) we have
\[ \pi_{n-2} - (1 - X_{n-2}) \alpha_{n-1} = X_{n-2}^{\beta_n-1} \beta_{n-2} \]
and hence (20) follows.

The following lemma gives the order at \( x = 0 \) and the degree of \( \alpha_n \) and \( \beta_n \) as polynomials in \( x. \)

**Lemma 3.** We have for \( n \geq 2 \)
\[ \text{ord}_{x=0} \alpha_n = 0, \quad \text{deg} \alpha_n = (b_n - 1)q_{n-1} + \text{deg} \beta_{n-1}, \]
\[ \text{ord}_{x=0} \beta_n = \sum_{v=2}^{n} b_v q_{v-1}, \quad \text{deg} \beta_1 = 0, \]
\[ \text{deg} \beta_n = \text{ord}_{x=0} \beta_n + q_n - a_1 - \frac{1 + (-1)^n}{2}. \]

**Proof.** We prove only the last equality. It is true for \( n = 2, 3. \) Let \( n \geq 4 \) and assume that it is true up to \( n - 1. \) Then we have
\[ \text{deg} X_{n-1}^{\alpha_n + \beta_n} X_{n-2}^{\beta_n-1} \beta_{n-2} - \text{deg} X_{n-1}^{\alpha_n + \beta_n-1} \beta_{n-1} \]
\[ = \text{deg} \beta_{n-2} - \text{deg} \beta_{n-1} + b_n q_{n-2} + q_{n-1} = q_{n-2} + (-1)^{n-1}, \]
by the induction hypothesis, and this is positive. Hence we get
\[ \deg \beta_n = a_n q_{n-1} + b_n q_{n-2} + \deg \beta_{n-2}, \]
which yields the last equality in the lemma.

Now we can prove that the polynomials \( \alpha_n, \alpha_n + \beta_n, \) and \( \pi_n \) converge to the power series
\[ \lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} (\alpha_n + \beta_n) = \lim_{n \to \infty} \pi_n = x^{-a_1 - b_1} \lim_{n \to \infty} P_n^*(x) \]
in \( |x| < 1 \) as \( n \to \infty \). Indeed, it follows from Lemma 2 and Lemma 3 that in (21), \( \deg \alpha_{n-1} < \text{ord} \beta_{n-1} \) and \( \lim_{n \to \infty} \deg \alpha_n = \infty \), so that \( \alpha_n \) and \( \alpha_n + \beta_n \) converge to a power series \( \lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} (\alpha_n + \beta_n) \) in \( |x| < 1 \). Furthermore, \( \deg \alpha_n < \deg \beta_n \) because \( \text{ord} \beta_n - \deg \alpha_n = a_1 + (1/2) (1 + (-1)^n) \), by Lemma 3. So, in the equation \( \pi_n = \alpha_{n-3} + (\alpha_n - \alpha_{n-3}) + \beta_n - X_n \alpha_n \) we have \( \deg \alpha_{n-3} < \text{ord}(\alpha_n - \alpha_{n-3}) = \text{ord} \beta_n < \text{ord} \beta_n \) and \( \deg \alpha_{n-3} < 2q_{n-2} < q_n = \text{ord} X_n \alpha_n \). Hence \( \lim_{n \to \infty} \pi_n = \lim_{n \to \infty} \alpha_n \).

Noticing that \( \lim_{n \to \infty} Q^*_n(x) = 1/(1 - x) \), we have by (13) and (14)
\[ \sum_{k=1}^{\infty} \left( ([k+1] \theta + \phi) - [k\theta + \phi] \right) x^k = x^{a_1 + b_1} \lim_{n \to \infty} (\alpha_n + \beta_n). \]

Therefore we obtain the following

**Theorem 3.** Let \( \theta \) be irrational with \( 0 < \theta < 1 \) and \( -1 < \phi < 0 \). We assume the condition (10). Then we have
\[ \left\{ [k+1] \theta + \phi - [k\theta + \phi] \right\}_{k=1}^{\infty} = \lim_{n \to \infty} u_n v_n, \]
where \( u_n, v_n \in \{0, 1\}^* \) are given by
\[ u_0 = 0, \quad u_1 = 0 \cdots 0, \quad v_0 = 0 \cdots 01, \]
\[ u_n = u_{n-1} v_{n-1} \cdots v_{n-1}, \quad v_n = v_{n-1} \cdots v_{n-1} v_{n-2} \text{ for } n \geq 2. \]

Here \( a_n \) is the \( n \)th partial quotient of the simple continued fraction of \( \theta \) and \( b_n \) is defined by (7).

\( u_n \) and \( v_n \) in Theorem 3 can be described by a chain of substitutions as in the following.
**Corollary 4.** Let \( \theta \) and \( \phi \) be as in Theorem 3. Let \( \sigma_{ij} \) be the substitution on \( \{a, b, c\}^* \) defined by

\[
\sigma_{ij} : \begin{cases} 
    a \rightarrow a b \cdots b, \\
    \quad \text{\( i \) times} \\
    b \rightarrow b \cdots bc, \\
    \quad \text{\( j \) times} \\
    c \rightarrow b.
\end{cases}
\]

Then \( u_n, v_n \ (n \geq 1) \) in Theorem 3 are given by

\[
u_1 = \sigma_{b_1 a_1-1}(a), \quad v_1 = \sigma_{a_1 a_1-1}(b) \quad \text{with} \quad a = \varepsilon, \ b = 0, \ c = 1,
\]

\[
u_n = \sigma_{b_n a_n}(a), \quad v_n = \sigma_{b_n a_n}(b)
\]

with

\[
a = u_{n-1}, \ b = v_{n-1}, \ c = v_{n-2} \ (n \geq 2).
\]

Corollary 4 is illustrated in Figs. 3 and 4.

If \( \theta, 0 < \theta < 1 \), is quadratic irrational and the expansion (9) of \( \phi, -1 < \phi < 0 \), is periodic, we can write

\[
\theta = [0; a_1, \ldots, a_j, a_{j+1}, \ldots, a_{j+l}],
\]

\[
\phi = 0, b_1 \cdots b_j b_{j+1} \cdots b_{j+l},
\]

where \( a_{j+1}^*, \ldots, a_{j+l}^* \) and \( b_{j+1}^*, \ldots, b_{j+l}^* \) indicate periods of these expansions. Let \( \sigma \) be the substitution on \( \{a, b, c\}^* \) defined by

\[
\sigma = \sigma_{b_1 a_1-1} \circ \sigma_{b_2 a_2} \circ \cdots \circ \sigma_{b_j a_j},
\]

where \( \sigma \) denotes the identity map when \( j = 0 \). Putting

\[
u(a, b, c) = \sigma(a), \quad v(a, b, c) = \sigma(b), \quad w(a, b, c) = \sigma(c)
\]

and replacing \( a, b, c \) by 0, 0, 1, respectively, we get

\[
u = u(0, 0, 1), \quad v = v(0, 0, 1), \quad w = w(0, 0, 1) \in \{0, 1\}^*.
\]

Fig. 3. The word \( u_n v_n \ (n \geq 2) \).
We then define the substitution \( \tau \) on \( \{u, v, w\}^* \) by

\[
\tau = \sigma_{b_{j+1}a_{j+1}} \circ \sigma_{b_{j+2}a_{j+2}} \circ \cdots \circ \sigma_{b_{j+1}a_{j+1}}
\]

which is of the form (17).

We remark that, if \( \theta > 0 \) is quadratic irrational, then the expansion (9) of \( \phi \) is periodic if and only if \( \phi \) is in the field \( \mathbb{Q}(\theta) \) (see [7, Theorem 1]). Thus we have the following.

**Corollary 5.** Let \( \theta \) and \( \phi \) be as in Theorem 3. If \( \theta \) is quadratic irrational and \( \phi \) is in the field \( \mathbb{Q}(\theta) \), then under the notations as above

\[
\{(k + 1) \theta + \phi \} - \{k \theta + \phi \}\]

where \( \omega(u, v, w; \tau) \) is the fixed point of \( \tau \) prefixed by \( u \).

Remark. Our results in this section can be applied to a kind of billiard problem. Let \( S \) be the unit square with four sides \( a = \{(0, y)|0 \leq y \leq 1\} \), \( b = \{(x, 0)|0 \leq x \leq 1\} \), \( a' = \{(1, y)|0 \leq y \leq 1\} \), and \( b' = \{(x, 1)|0 \leq x \leq 1\} \). We consider a particle starting at a point \( P \) in \( S \) with constant velocity along a vector \( v = (1, \theta) \), where \( \theta > 0 \) is irrational, and is reflected at each side specularly. Writing down the names \( a, b, a', b' \) of the sides which the particle hits in the order of collision, we have an infinite sequence \( \omega \) of \( a, b, a', b' \). The problem is to describe \( \omega \) in terms of \( \theta \) and the coordinates of the starting point \( P \). We note that \( a \) and \( a' \) appear in \( \omega \) alternately and the same is true for \( b \) and \( b' \). Thus we may identify \( a' \) with \( a \) and \( b' \) with
PROPERTIES OF A CERTAIN POWER SERIES

b, and get a new sequence \( \omega^* \) of \( a \) and \( b \) only. By symmetry with respect to the sides, the sequence \( \omega^* \) remains unchanged, if we replace \( S \) by the torus and imagine that the particle does not reflect at the sides but passes through them. Moreover we may assume that \( 0 < \theta < 1 \), replacing \( a \) by \( b \) if necessary, and that the starting point \( P \) is on the side \( a \), neglecting the first symbol \( a \) in \( \omega^* \) if \( P \) is on \( b \). Now let us consider \( S \) to be the square \([0, 1) \times [-1, 0)\) in the \((x, y)\)-plane, and set \( v = (1, \theta) \) with \( 0 < \theta < 1 \), \( P = (0, \phi) \) with \(-1 < \phi < 0\). We attach the symbols \( a \) and \( b \) to the intersection points of the line \( y = \theta x + \phi \) with the lines \( x = k (1 \leq k \in \mathbb{Z}) \) and \( y = m (0 \leq m \in \mathbb{Z}) \), respectively. Tracing these symbols \( a \) and \( b \) along the line \( y = \theta x + \phi \), we obtain the sequence \( \omega^* \) described above. If we denote the sequence \{\((k+1)\theta + \phi\) - \([k\theta + \phi]\)\}_{k=1}^{\infty} \) of \( 0, 1 \) by \( \omega(0, 1) \), we have \( \omega(a, ab) = \omega^* \). Therefore a solution of our problem can be found in Theorem 2 and Theorem 3 and their corollaries.

3. IRRATIONALITY MEASURES

In this section, we shall estimate the irrationality measure for the number \( f(\theta, \phi; 1/a, 1/b) \), where \( a, b \) are integers with \( |a| \geq 2, b \neq 1 \). For an irrational number \( \alpha \), let \( M(\alpha) \) denote the set of all \( \mu > 0 \) for which there exists \( Q_0 = Q_0(\alpha, \mu) > 0 \) such that \( |a - P/Q| > Q^{-\mu} \) for all integers \( Q (\geq Q_0) \) and \( P \). The irrationality measure \( \mu(\alpha) \) of \( \alpha \) is defined by

\[
\mu(\alpha) = \inf M(\alpha) \quad (\geq 2).
\]

If \( M(\alpha) = \phi \), \( \mu(\alpha) = \infty \) and \( \alpha \) is called a Liouville number. We note that

\[
\mu((\alpha x + b)/(\alpha x + d)) = \mu(\alpha) \quad (22)
\]

for all integers \( a, b, c, d \) with \( ad - bc \neq 0 \).

**Theorem 4.** Let \( \theta > 0 \) be irrational and \( \phi \) be real. If the expansion of \( \phi \) defined by (9) is terminating, we have

\[
\mu(f(\theta, \phi; 1/a, 1/b)) = 1 + \lim_{n \to \infty} [a_n; a_{n-1}, \ldots, a_1] > 2
\]

for all integers \( a, b \) with \(|a| \geq 2, b \neq 0\), where \( \theta = [a_0; a_1, a_2, \ldots] \). In particular, the number \( f(\theta, \phi; 1/a, 1/b) \) is transcendental.

**Remark.** If we choose \( \phi = 0 \), \( (Z \ni) a \geq 2 \), and \( b = 1 \), we have the result obtained independently by Adams and Davison [1], Böhmer [2], and Bundschuh [3].
Proof. We may assume $0 < \theta < 1$, $-1 < \phi \leq 0$, and the condition (10). Then it follows from Corollary 1 with (22) that

$$\mu(f(\theta, \phi; 1/a, 1/b)) = \mu(\alpha),$$

where $\alpha = [0; A_1, A_2, \ldots]$. This implies that $\alpha$ is irrational.

By assumption $b_n = 0$ for all large $n$, and so $\sigma_m = \tau_m = \tau_n$ for all large $m, n$. Hence

$$A_n = a^{q_{n-2}}b^{p_{n-1}}(\frac{a^{q_{n-1}}b^{p_{n-1}}}{a^{q_{n-1}}b^{p_{n-1}}} - 1) = \epsilon \in \mathbb{Z}$$

for all large $n$, so that there is an integer $D$ for which $DP_n, DQ_n \in \mathbb{Z}$ ($n \geq 1$). Let $P, Q$ be given integers with $(P, Q) = 1$. We may assume that $Q$ is sufficiently large. Since $|Q_n|$ is monotone increasing for large $n$ and $\lim_{n \to \infty} |Q_n| = \infty$, there is an $n = n(Q)$ such that

$$|Q_n| \leq 4DQ < |Q_{n+1}|.$$

We assume first that $P/Q \neq P_n/Q_n$. Then

$$|DQ_n(\alpha - P/Q)| \geq |D(PQ_n - QP_n)|/Q - D |Q_n\alpha - P_n| \geq 1/(2Q),$$

noticing that $D(PQ_n - QP_n) \neq 0$ is an integer and

$$1/(2 |Q_{n+1}|) \leq |Q_n\alpha - P_n| \leq 2/|Q_{n+1}|$$

for all large $n$. Hence we get

$$|\alpha - P/Q| > 1/(2DQ |Q_n|) \geq 1/(8D^2Q^2).$$

If $P/Q = P_n/Q_n$, we have by the first inequality of (23)

$$|\alpha - P/Q| > 1/2(4DQ)^{-\log |Q_{n+1}|Q_n|/\log(4DQ)}$$

with

$$\lim_{n \to \infty} \log |Q_nQ_{n+1}| \log(4DQ)$$

$$\leq 1 + \lim_{n \to \infty} \log |Q_{n+1}| \log |Q_n|$$

$$= 1 + \lim_{n \to \infty} \frac{q_{n+1}}{q_n} \log |a| + p_{n+1} \log |b| = 1 + \lim_{n \to \infty} \frac{q_{n+1}}{q_n}. $$
Thus we obtain
\[ \mu(\alpha) \leq 1 + \lim_{n \to \infty} q_n + 1/q_n. \]

On the other hand, the second inequality of (23) yields
\[ |\alpha - DP_n/(DQ)| \leq 2(DQ_n)^{-\log |Q_n+1/Q_n|/\log |DQ_n|} \]
and therefore
\[ \mu(\alpha) \geq 1 + \lim_{n \to \infty} q_n + 1/q_n. \]

Noticing that \( \lim_{n \to \infty} q_n + 1/q_n > 1 \), the last statement follows from Roth's theorem.

**Theorem 5.** Let \( \theta > 0 \) be irrational and \( \phi \) be real. Then there is an absolute constant \( C > 1 \) such that
\[ C^{-1} \lim_{n \to \infty} a_n \leq \mu(f(\theta, \phi; 1/a, 1/b)) \leq C \lim_{n \to \infty} a_n \]
for all integers \( a, b \) with \( |a| \geq 2, b \neq 0 \). In particular, \( f(\theta, \phi; 1/a, 1/b) \) is a Liouville number if and only if the partial quotients \( a_n \) of \( \theta \) are unbounded.

**Proof.** It is enough to estimate \( \mu(\alpha) \). We may assume that the expansion (9) of \( \phi \) is non-terminating, namely \( b_n \geq 1 \) for all \( n \geq 1 \), and so \( A_n \) is a rational number with
\[ a^\tau_n b^\sigma_n = (a^n b^n - 1) A_n \in \mathbb{Z}. \]

Putting
\[ D_n = \left| a^\tau_n b^\sigma_n \prod_{v=1}^{n-1} (a^v b^v - 1) \right| \quad (n \geq 1), \]
we have \( D_n A_1 \cdots A_n \in \mathbb{Z} \), so that \( D_n P_n, D_n Q_n \in \mathbb{Z} \). Here
\[ \log D_n = \left( \sum_{v=1}^{n-1} q_v + \tau_n - 1 \right) \log |a| + \left( \sum_{v=1}^{n-1} p_v + \sigma_n - 1 \right) \log |b| + O(1), \quad (24) \]
\[ \log |Q_n| = (q_n + \tau_n) \log |a| + (p_n + \sigma_n) \log |b| + O(1). \quad (25) \]

Hence \( |Q_n|/D_n \) is monotone increasing for all large \( n \) and \( \lim_{n \to \infty} |Q_n|/D_n = \infty \). Now let \( P, Q \) be given integers with \( (P, Q) = 1 \). We may assume that \( Q \) is sufficiently large. Then there is an integer \( n = n(Q) \) such that
\[ |Q_n|/D_n = Q < |Q_n+1|/D_n. \]

The rest of the proof is similar to that of the preceding theorem.
4. TRANSCENDENCE AND ALGEBRAIC INDEPENDENCE

If \( \phi = 0 \), transcendence and algebraic independence of the values of the function \( f(\theta, 0; x, y) \) at some algebraic points are established in \([6, 9, 12]\). If the partial quotients \( \{a_n\} \) of \( \theta \) are unbounded, a stronger result on algebraic independence can be obtained, even for any real \( \phi \), by the method developed in \([12]\).

**Theorem 6.** Let \( \theta > 0 \) have an unbounded partial quotient in its simple continued fraction and let \( \phi \) be real. Denote the \( n \)-th convergent of \( \theta \) by \( p_n/q_n \). If \( \alpha_1, \beta_1, \ldots, \alpha_n, \beta_n \) are algebraic numbers with

\[
0 < |\alpha_i| \cdot |\beta_i|^{\theta} < 1 \quad \text{and} \quad \alpha_i^{q_i} \beta_i^{p_i} \neq 1 \quad (k \geq 0) \quad (i = 1, \ldots, n)
\]

such that at least one of \( \alpha_i/\alpha_j \) and \( \beta_i/\beta_j \) is not a root of unity for \( i \neq j \), then \( f(\theta, \phi; \alpha_1, \beta_1), \ldots, f(\theta, \phi; \alpha_n, \beta_n) \) are algebraically independent.

We prepare some notations and lemmas. In what follows \( K \) denotes an algebraic number field. An equivalence class of non-trivial valuations on \( K \) is called a prime on \( K \). \( S_K \) and \( S_\infty \) denote the set of all primes and the set of all infinite primes on \( K \), respectively. For every prime \( v \) on \( K \) lying above a prime \( p \) on \( \mathbb{Q} \), we choose a valuation \( | \cdot |_v \) such that

\[
|\alpha|_v = |\alpha|^{|K_v: Q_p|}_p \quad (\alpha \in \mathbb{Q}),
\]

where \( K_v \) and \( Q_p \) denote the completions of \( K \) at \( v \) and \( \mathbb{Q} \) at \( p \), respectively. Then we have the product formula

\[
\prod_{v \in S_K} |\alpha|_v = 1 \quad (\alpha \in K, \alpha \neq 0).
\]

For any projective point \( x = (x_0: x_1: \cdots: x_n) \) in \( P^n(K) \), we define the **height** of \( x \) by

\[
H(x) = H_K(x) = \prod_{v \in S_K} \max(|x_0|_v, |x_1|_v, \ldots, |x_n|_v),
\]

which is well-defined because of the product formula. We put

\[
h(\alpha) = h_K(\alpha) = H(1: \alpha) \quad (\alpha \in K).
\]

Then we have the **fundamental inequality**

\[
-\log h(\alpha) \leq \sum_{v \in S} \log |\alpha|_v \leq \log h(\alpha) \quad (\alpha \in K, \alpha \neq 0),
\]
where $S$ is any subset of $S_K$. If $\alpha \in K$, then $h(\alpha) = 1$ if and only if $\alpha$ is a root of unity or $0$, $h(\alpha) = h(\alpha^{-1})$, and $h(\alpha^m) = h(\alpha)^m$. Furthermore, if $\alpha_1, \ldots, \alpha_m \in K$,

$$h(\alpha_1 + \cdots + \alpha_m) \leq m^d h(\alpha_1) \cdots h(\alpha_m), \quad d = [K : \mathbb{Q}],$$

$$h(\alpha_1 \cdots \alpha_m) \leq h(\alpha_1) \cdots h(\alpha_m).$$

Let $S$ be a finite subset of $S_K$ including $S_\infty$ and let $c, d$ be constants with $c > 0$, $d \geq 0$. A projective point $x \in P^n(K)$ is called $(c, d, S)$-admissible if its homogeneous coordinates $x_0, x_1, \ldots, x_n$ can be chosen such that all $x_i$ are $S$-integers, i.e., $|x_i|_v \leq 1$ for $v \not\in S$, and

$$\prod_{v \in S} \prod_{i=0}^n |x_i|_v \leq cH(x)^d.$$

The following theorem is due to Evertse [5]: Let $c, d$ be constants with $c > 0$, $0 < d < 1$. Then there are only finitely many $(c, d, S)$-admissible points $x = (x_0 : x_1 : \cdots : x_n) \in P^n(K)$ satisfying

$$x_0 + x_1 + \cdots + x_n = 0$$

but

$$x_{i_0} + x_{i_1} + \cdots + x_{i_s} \neq 0$$

for each proper, non-empty subset $\{i_0, i_1, \ldots, i_s\}$ of $\{0, 1, \ldots, n\}$.

**Lemma 4.** Let $\theta$ be irrational. If $\alpha$ and $\beta$ are nonzero elements in an algebraic number field $K$ such that at least one of $\alpha$ and $\beta$ is not a root of unity, then

$$|\alpha|_v > |\beta|^\theta_v$$

for some $v \in S_K$.

**Proof.** Assume that $|\alpha|_v \leq |\beta|^\theta_v$ for all $v \in S_K$. Then by the product formula $|\alpha|_v = |\beta|_v^\theta$ for all $v \in S_K$, and so $|\alpha|_v = |\beta|_v = 1$ for all $v \in S_K \setminus S_\infty$, since $\theta$ is irrational and $|\alpha|_v$, $|\beta|_v$ are some powers of a rational prime provided $v \in S_K \setminus S_\infty$. Hence $\alpha$ and $\beta$ are units in $K$. We can write

$$\alpha = \zeta \eta_1^{j_1} \cdots \eta_r^{j_r} \quad (j_1, \ldots, j_r \in \mathbb{Z}),$$

$$\beta = \zeta' \eta_1^{j_1'} \cdots \eta_r^{j_r'} \quad (j_1', \ldots, j_r' \in \mathbb{Z}),$$

where $\zeta$, $\zeta'$ are roots of unity and $\eta_1, \ldots, \eta_r$ with $r + 1 = |S_\infty|$ are generators.
For different primes \( v_1, \ldots, v_r \in S, \) we have
\[
|\alpha|_{v_i} = |\eta_1|_{v_i}^{\frac{n}{v_i}} \cdots |\eta_r|_{v_i}^{\frac{n}{v_i}} \quad (i = 1, \ldots, r),
\]
so that
\[
(j_1 j_2 \theta) \log |\eta_1|_{v_i} \cdots (j_r j_r \theta) \log |\eta_r|_{v_i} = 0 \quad (i = 1, \ldots, r)
\]
with
\[
\begin{vmatrix}
\log |\eta_1|_{v_i} & \cdots & \log |\eta_r|_{v_i} \\
\vdots & \ddots & \vdots \\
\log |\eta_1|_{v_i} & \cdots & \log |\eta_r|_{v_i}
\end{vmatrix} \neq 0.
\]
Hence we get
\[
(j_1 j_2 \theta) = \cdots = (j_r j_r \theta).
\]
Here, by the assumption of the lemma, at least one of \( j_1, \ldots, j_r \) is different from zero, and therefore \( \theta \) is rational, which is a contradiction.

**Lemma 5.** Let \( x_1, \ldots, x_n, \beta_1, \ldots, \beta_n \) be nonzero elements in an algebraic number field \( K \) such that at least one of \( x_i/\alpha_k \) and \( \beta_i/\beta_1 \) is not a root of unity for each \( i \neq 1 \). Let \( \{e_k\}_{k=1}^{\infty} \) and \( \{f_k\}_{k=1}^{\infty} \) be sequences of positive integers with \( \lim_{n \to \infty} e_k = \lim_{n \to \infty} f_k = \infty \) such that \( e_k/f_k \) converges to an irrational number as \( k \to \infty \). Let \( \{A_{ik}\}_{k=1}^{\infty} \) \((i = 1, \ldots, n)\) be \( n \) sequences of elements in \( K \) satisfying the following conditions:

(i) \( A_{ik} \neq 0 \) \((k \geq 1)\)

(ii) \( \lim_{k \to \infty} (\log h(A_{ik}))/f_k = 0 \) \((1 \leq i \leq n)\).

Let \( 0 < r < 1 \). Then we have
\[
\left| \sum_{i=1}^{n} A_{ik} x_i^f \beta_i^e \right| \geq |\alpha_1^f \beta_1^e| \gamma^f
\]
for all large \( k \).

**Proof.** We may assume \( \sqrt{-1} \in K \) and \( |\cdot|^2 = |\cdot|_{v_0} \) for some \( v_0 \in S_\infty \). Let \( S \) be a finite subset of \( S_K \) containing \( S_\infty \) and all divisors of \( \alpha_i, \beta_i \) \((1 \leq i \leq n)\).
We may assume without loss of generality that all \( A_{ik} \) \((1 \leq i \leq n, k \geq 1)\) are algebraic integers, since for each \( k \) there is an integer \( D_k \) with \( 1 \leq D_k \leq \prod_{i=1}^{n} h(A_{ik}) \) such that \( D_k A_{ik} \), \( \ldots, D_k A_{nk} \) are algebraic integers. Then \( A_{ik} x_i^f \beta_i^e \) \((1 \leq i \leq n, k \geq 1)\) are \( S \)-integers.
PROPERTIES OF A CERTAIN POWER SERIES

81

We prove the lemma by induction on \( n \). If \( n = 1 \), the statement follows from (i), (ii), and the fundamental inequality. Let \( n \geq 2 \). We assume that

\[
\sum_{i=1}^{n} A_{ik} \alpha_i^{\epsilon_k} \beta_i^{\epsilon_k} = 0
\]

holds for all \( k \) belonging to an infinite set \( \Omega_1 \) of positive integers. By the induction hypothesis, no proper subsum of the left-hand side of (27) vanishes, provided that \( k \in \Omega_1 \) is large.

In particular, \( A_{ik} \neq 0 \) \((1 \leq i \leq n)\) for all large \( k \in \Omega_1 \). Then, putting

\[
H_k := H(A_{ik} \alpha_i^{\epsilon_k} \beta_i^{\epsilon_k} : \cdots : A_{nk} \alpha_n^{\epsilon_k} \beta_n^{\epsilon_k}),
\]

we have

\[
H_k \geq H(\alpha_1^{\epsilon_1} \beta_1^{\epsilon_1} : \cdots : \alpha_n^{\epsilon_n} \beta_n^{\epsilon_n}) \left/ \prod_{i=1}^{n} h(A_{ik}) \right.
\geq H(1: (\alpha_2/\alpha_1)^{\epsilon_1} (\beta_2/\beta_1)^{\epsilon_1}) \left/ \prod_{i=1}^{n} h(A_{ik}) \right.
\]

for all large \( k \in \Omega_1 \). Here we can find a constant \( C > 1 \) independent of \( k \) such that

\[
H(1: (\alpha_2/\alpha_1)^{\epsilon_1} (\beta_2/\beta_1)^{\epsilon_1}) > C^{\epsilon_k}
\]

holds for all large \( k \). Indeed, it follows from Lemma 4 that

\[
|\alpha_2/\alpha_1|_v > |\beta_2/\beta_1|_v^{-\omega}
\]

for some \( v \in S_K \), where \( \omega = \lim_{k \to \infty} e_k/f_k \). If \( |\beta_2/\beta_1|_v > 1 \), we choose \( \eta > 0 \) such that \( |\alpha_2/\alpha_1|_v > |\beta_2/\beta_1|_v^{-\omega + 2\eta} \). Then

\[
|\alpha_2/\alpha_1|_v \cdot |\beta_2/\beta_1|_v^{\epsilon_k/f_k} \geq |\beta_2/\beta_1|_v^{\epsilon_k} f_k^{-\omega + 2\eta} \geq |\beta_2/\beta_1|_v^n > 1
\]

for all large \( k \). If \( |\beta_2/\beta_1|_v = 1 \), then

\[
|\alpha_2/\alpha_1|_v \cdot |\beta_2/\beta_1|_v^{\epsilon_k/f_k} = |\alpha_2/\alpha_1|_v > 1.
\]

Finally, if \( |\beta_2/\beta_1|_v < 1 \), we choose \( \eta > 0 \) such that \( |\alpha_2/\alpha_1|_v > |\beta_2/\beta_1|_v^{-\omega - 2\eta} \). Then

\[
|\alpha_2/\alpha_1|_v \cdot |\beta_2/\beta_1|_v^{\epsilon_k/f_k} \geq |\beta_2/\beta_1|_v^{\epsilon_k} f_k^{-\omega - 2\eta} \geq |\beta_2/\beta_1|_v^{-\eta} > 1.
\]

In any case, we can choose a constant \( C > 1 \) satisfying (29). Combining (28), (29), and (ii), we have

\[
\lim_{\Omega_1 \ni k \to \infty} H_k = \infty.
\]
It follows from Evertse's theorem that \( (A_{1k} \beta_1^{\epsilon_k} : \cdots : A_{nk} \beta_n^{\epsilon_k}) \) is not \((1, 1/2, S)\)-admissible; namely

\[
\prod_{i=1}^{n} h(A_{ik}) \geq \prod_{v \in S} \prod_{i=1}^{n} |A_{ik}|_v = \prod_{v \in S} \prod_{i=1}^{n} |A_{ik} \beta_i^{\epsilon_k}|_v > H_k^{1/2},
\]

for all large \( k \in \Omega_1 \). This together with (28) and (29) implies that

\[
\left( \prod_{i=1}^{n} h(A_{ik}) \right)^3 > C^k,
\]

for all large \( k \in \Omega_1 \), which contradicts condition (ii). Therefore we have

\[
\sum_{i=1}^{n} A_{ik} \beta_i^{\epsilon_k} \neq 0
\]

(30)

for all large \( k \).

Now we assume that the inequality

\[
\left| \sum_{i=1}^{n} A_{ik} \beta_i^{\epsilon_k} \right| < |a_1^{\epsilon_1} \beta_1^{\epsilon_1}|^{-\epsilon/k}
\]

(31)

holds for all \( k \) belonging to an infinite set \( \Omega_2 \) of positive integers. Let \( \delta_k \) be defined by

\[
\sum_{i=1}^{n} A_{ik} \beta_i^{\epsilon_k} + \delta_k = 0.
\]

(32)

Then \( \delta_k \) is an \( S \)-integer. By the induction hypothesis, (30) and (31), no proper subsum of the left-hand side of (32) vanishes for all large \( k \in \Omega_2 \). Noticing that \( A_{ik} \neq 0 \) \((1 \leq i \leq n)\) for all large \( k \in \Omega_2 \), we have again (28), which together with (29) and (ii) yields \( \lim_{\Omega_2 \ni k \to \infty} H_k = \infty \), so that

\[
H_k \leq H(A_{1k} \beta_1^{\epsilon_k} : \cdots : A_{nk} \beta_n^{\epsilon_k} : \delta_k) \to \infty \quad (\Omega_2 \ni k \to \infty).
\]

It follows from Evertse's theorem that, if \( 0 < \epsilon < 1 \), \( (A_{1k} \beta_1^{\epsilon_k} : \cdots : A_{nk} \beta_n^{\epsilon_k} : \delta_k) \in P^n(K) \) is not \((1, 1 - \epsilon, S)\)-admissible, namely

\[
\prod_{v \in S} \prod_{i=1}^{n} |A_{ik} \beta_i^{\epsilon_k}|_v \cdot \prod_{v \in S} |\delta_k|_v > H_k^{1-\epsilon}
\]

(33)

for all large \( k \in \Omega_2 \). Here we have

\[
\prod_{v \in S} \prod_{i=1}^{n} |A_{ik} \beta_i^{\epsilon_k}|_v \leq \prod_{i=1}^{n} h(A_{ik})
\]
and by (31), (32)
\[
\prod_{v \in S} |\delta_v|_\varepsilon \leq n^d \prod_{i=1}^{n} h(A_{ik}) \cdot H(\alpha_i^{f_k} \beta_i^{e_k} ; \cdots ; \alpha_n^{f_k} \beta_n^{e_k}) \\
\times |\alpha_i^{f_k} \beta_i^{e_k}|^2 \gamma^{2f_k} \left( \max_{1 \leq i \leq n} |\alpha_i^{f_k} \beta_i^{e_k}| \right)^{-2},
\]
so that the left-hand side of the inequality (33) is not greater than
\[
n^d \left( \prod_{i=1}^{n} h(A_{ik}) \right)^2 H(\alpha_i^{f_k} \beta_i^{e_k} ; \cdots ; \alpha_n^{f_k} \beta_n^{e_k}) \gamma^{2f_k}.
\]
This together with (33) and the left-hand side of (28) implies that
\[
n^d \left( \prod_{i=1}^{n} h(A_{ik}) \right)^3 \gamma^{2f_k} \geq H(\alpha_i^{f_k} \beta_i^{e_k} ; \cdots ; \alpha_n^{f_k} \beta_n^{e_k})^{-\varepsilon}
\]
holds for all large \( k \in \Omega_2 \). Therefore, using the condition (ii), we get
\[
2 \log \gamma \geq -\varepsilon \lim_{\Omega_2 \ni k \to \infty} \left( \log H(\alpha_i^{f_k} \beta_i^{e_k} ; \cdots ; \alpha_n^{f_k} \beta_n^{e_k}) \right)/f_k.
\]
Noticing that \( \log H(\alpha_i^{f_k} \beta_i^{e_k} ; \cdots ; \alpha_n^{f_k} \beta_n^{e_k})/f_k \) is bounded and letting \( \varepsilon \to 0 \), we obtain
\[
\log \gamma \geq 0,
\]
which contradicts the assumption \( 0 < \gamma < 1 \).

**Lemma 6.** Let \( \theta = [a_0 ; a_1, a_2, \ldots] \) be the simple continued fraction of an irrational number \( \theta \) with \( p_n/q_n = [a_0 ; a_1, \ldots, a_n] \), and let \( \phi = b_0 \cdot b_1 b_2 \cdots \) be the expansion of \( \phi \) defined by (9). Put
\[
e_k = \sum_{\kappa=1}^{k} b_\kappa p_{\kappa-1} + \rho_k + p_{k-1}, \quad f_k = \sum_{\kappa=1}^{k} b_\kappa q_{\kappa-1} + q_k + q_{k-1}.
\]
Then we have \( \lim_{k \to \infty} e_k = \lim_{k \to \infty} f_k = \infty \) and
\[
e_k/f_k = \theta + O(1/a_k)
\]
as \( k \to \infty \), where the constant implied in the \( O \)-function is absolute.

**Proof.** It follows from (5) that
\[
\sum_{\kappa=1}^{k} b_\kappa p_{\kappa-1} \leq 4p_k, \quad \sum_{\kappa=1}^{k} b_\kappa q_{\kappa-1} \leq 4q_k.
\]
Hence we get
\[
\left( \sum_{k=1}^{\infty} b_k p_{k-1} \right) q_k = b_k p_{k-1} + O(p_{k-1}/q_k) = \theta b_k/a_k + O(1/a_k)
\]
\[
\left( \sum_{k=1}^{\infty} b_k q_{k-1} \right) q_k = b_k/a_k + O(1/a_k).
\]
Therefore we obtain
\[
\frac{e_k}{f_k} = \frac{\theta + \theta b_k/a_k + O(1/a_k)}{1 + b_k/a_k + O(1/a_k)} = \theta + O(1/a_k).
\]

Proof of Theorem 6. We may assume \(0 < \theta < 1\), \(-1 < \phi \leq 0\), and (10), because of (2), (3), and remark 1. Suppose that the numbers
\[
\zeta_i := f(\theta, \varphi; \alpha_i, \beta_i) \quad (1 \leq i \leq n)
\]
are algebraically dependent. Then there is a nonzero polynomial \(P(x)\) of \(x = (x_1, ..., x_n)\) with algebraic coefficients such that
\[
P(\xi) = 0, \quad \xi = (\xi_1, ..., \xi_n).
\]
We may assume that \(P(x)\) is of the minimum total degree among such polynomials and that
\[
\frac{\partial P}{\partial x_i}(\xi) \neq 0 \quad (1 \leq i \leq n).
\]
Let \(K\) be an algebraic number field containing \(\alpha_1, \beta_1, ..., \alpha_n, \beta_n\) and all the coefficients of \(P(x)\). It follows from Theorem 1 that
\[
\xi_i = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{a^{f_k/\alpha_{i}} \beta_{i}^{c_k}}{(1 - \alpha_i^{q_k} \beta_i^{p_k})(1 - \alpha_i^{q_k-1} \beta_i^{p_k})} \quad (1 \leq i \leq n),
\]
where \(c_k\) and \(f_k\) are given in Lemma 6. We put
\[
\xi_i(k) = \sum_{k=1}^{k-1} (-1)^{k-1} \frac{a^{f_k/\alpha_{i}} \beta_{i}^{c_k}}{(1 - \alpha_i^{q_k} \beta_i^{p_k})(1 - \alpha_i^{q_k-1} \beta_i^{p_k})} \quad (1 \leq i \leq n),
\]
\[
\xi(k) = (\xi_1(k), ..., \xi_n(k)).
\]
Then we have
\[
-P(\xi(k)) = P(\xi) - P(\xi(k)) = \sum_{|J| > 1} \frac{1}{J!} \frac{\partial^{(J)}}{\partial x^J} P(\xi(k))(\xi - \xi(k))^J,
\]
where \( J = (j_1, \ldots, j_n) \) with \( j_k \) being nonnegative integers and \( |J|, J! \), and \((\xi - \xi(k))'\) are defined in the usual way.

Since \( \{a_k\} \) is unbounded, there is an infinite set \( \Omega \) of positive integers such that

\[
\lim_{\Omega \ni k \to \infty} a_k = \infty.
\]

Then we have by Lemma 6

\[
\lim_{\Omega \ni k \to \infty} e_k/f_k = \theta
\]

and \( \lim_{k \to \infty} e_k = \lim_{k \to \infty} f_k = \infty \). In the rest of the proof, we always assume that \( k \) is an element in \( \Omega \). Choosing \( \rho \) such that

\[
\max_{1 \leq i \leq n} |x_i| |\beta_i| \theta < \rho < 1,
\]

we have

\[
|\xi_i - \xi_i(k)| \leq c_1 \rho^{f_k} \quad (1 \leq i \leq n),
\]

where \( c_1, c_2, \ldots \) are positive constants independent on \( k \), so that

\[
|P(\xi(k))| \leq c_2 \rho^{f_k}.
\]  

At the same time, we have

\[
h(P(\xi(k))) \leq c_3 \rho^{q(k) - 1}.
\]  

We note here that

\[
\lim_{k \to \infty} f_k/q_{k-1} = \infty.
\]

By (36), (37), (38), and the fundamental inequality, we get

\[
P(\xi(k)) = 0.
\]

It is easily seen that

\[
\begin{align*}
\xi_i - \xi_i(k) &= \frac{(-1)^{k-1} \alpha_i^{f_k} \beta_i^{e_k}}{(1 - \alpha_i^{q_k} \beta_i^{p_k})(1 - \alpha_i^{q_k-1} \beta_i^{p_k-1})} + O(|\alpha_i^{f_k+1} \beta_i^{e_k+1}|) \\
&= \frac{(-1)^{k-1} \alpha_i^{f_k} \beta_i^{e_k}}{1 - \alpha_i^{q_k-1} \beta_i^{p_k-1}} + O(|\alpha_i^{f_k} \beta_i^{e_k} \rho^{q_k}).
\end{align*}
\]
This together with (35) and (39) leads to
\[
0 = \sum_{i=1}^{n} \frac{\partial P}{\partial x_i}(\zeta(k)) \frac{(-1)^{k-1} \alpha_i^{a_i} \beta_i^{b_i}}{1 - \alpha_i^{a_i-1} \beta_i^{b_i-1}} + O\left(\sum_{i=1}^{n} |\alpha_i^{a_i} \beta_i^{b_i}| \rho^{q_i}\right).
\]

Hence
\[
\left| \sum_{i=1}^{n} A_{ik} \alpha_i^{a_i} \beta_i^{b_i} \right| \leq c_4 \max_{1 \leq i \leq n} |\alpha_i^{a_i} \beta_i^{b_i}| \rho^{q_i}, \tag{40}
\]

where
\[
A_{ik} = \frac{\partial P}{\partial x_i}(\zeta(k)) \frac{(-1)^{k-1}}{1 - \alpha_i^{a_i-1} \beta_i^{b_i-1}} \in K.
\]

We remark that \( A_{ik} \neq 0 \) (\( i = 1, \ldots, n \)) for all large \( k \), by (34), and
\[
\lim_{k \to \infty} (\log h(A_{ik}))/f_k = 0 \quad (1 \leq i \leq n),
\]

by (38). Thus all the conditions in Lemma 5 are satisfied. Therefore, choosing \( \gamma \) with \( \rho < \gamma^3 < 1 \), we get
\[
\left| \sum_{i=1}^{n} A_{ik} \alpha_i^{a_i} \beta_i^{b_i} \right| \geq \max_{1 \leq i \leq n} |\alpha_i^{a_i} \beta_i^{b_i}| \gamma^{f_k}
\]

for all large \( k \). This inequality and (40) yield
\[
\gamma^{f_k} \leq c_4 \rho^{a_k}.
\]

Since \( \lim_{k \to \infty} f_k/q_k \leq 3 \), we obtain
\[
\gamma^3 \leq \rho,
\]

which is a contradiction.

References