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Convergence and stability of iterative algorithm for a new system of (A, η)-accretive mapping inclusions in Banach spaces ☆

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ABSTRACT

In this paper, we introduce and study a new system of (A, η)-accretive mapping inclusions in Banach spaces. Using the resolvent operator associated with (A, η)-accretive mappings, we suggest a new general algorithm and establish the existence and uniqueness of solutions for this system of (A, η)-accretive mapping inclusions. Under certain conditions, we discuss the convergence and stability of iterative sequence generated by the algorithm. Our results extend, improve and unify many known results on variational inequalities and variational inclusions.

1. Introduction

Variational inequalities theory, as a very effective and powerful tool of the current mathematical technology, has been widely applied to mechanics, physics, optimization and control, economics and transportation equilibrium, engineering sciences, etc. please see [1–31] and the references therein. Because of its wide applications, the classical variational inequality has been generalized in various directions in the past years. Variational inclusion is an important generalization of variational inequality and has been studied by many authors. We also know that one of the most important and interesting problems in the theory of variational inequality is the development of an efficient and implementable algorithm for solving various variational inequalities and variational inclusions. In recent years, many numerical methods have been developed for solving various classes of variational inequalities and variational inclusions in Euclidean spaces or Hilbert spaces, such as the projection methods and its variant forms, linear approximation, descent, and Newton's methods. However, few iterative algorithms have been developed for solving variational inequality and variational inclusion problems in Banach spaces.

Recently, Huang and Fang [32] were the first to introduce the generalized m-accretive mapping and give the definition of the resolvent operator for the generalized m-accretive mappings in Banach spaces. They also showed some properties of the resolvent operator for the generalized m-accretive mappings in Banach spaces. For further works, see Huang [15], Jin and Liu [19] and the references therein. Very recently, inspired and motivated by the works of [8,10,11,15,18,23,30–32], Lan et al. [22] and [24] introduced a new concept of (A, η)-accretive mappings, which generalizes the existing monotone or accretive operators, and studied some properties of (A, η)-accretive mappings and defined resolvent operators associated with (A, η)-accretive mappings. They also studied a class of variational inclusions using the resolvent operator associated with (A, η)-accretive mappings.

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Inspired and motivated by the recent research works in this field, in this paper, we shall introduce and study a new system of (A, η)-accretive mapping inclusions in Banach spaces. Using the resolvent operator associated with (A, η)-accretive mappings, we suggest a new general algorithm and establish the existence and uniqueness of solutions for this system of (A, η)-accretive mapping inclusions. Under certain conditions, we discuss the convergence and stability of iterative sequence generated by the algorithm. Our results extend, improve and unify many known results on variational inequalities and variational inclusions.

2. Preliminaries

Let X be a real Banach space with dual space X^*, $\langle\cdot,\cdot\rangle$ be the dual pair between X and X^*, and 2^X denote the family of all nonempty subsets of X. The generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \{f^* \in X^* : \langle x, f^*\rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\}, \quad \forall x \in X,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that, in general, $J_q(x) = \|x\|^{q-2}J_2(x)$ for all $x \neq 0$ and J_q is single-valued if X^* is strictly convex, and if $X = H$ is a Hilbert space, then J_2 becomes the identity mapping on H.

The modulus of smoothness of X is the function $\rho_X : [0,\infty) \to [0,\infty)$ defined by

$$\rho_X(t) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \le 1, \|y\| \le t\right\}.$$

A Banach space X is called uniformly smooth if

$$\lim_{t\to 0}\frac{\rho_X(t)}{t} = 0.$$

X is called q-uniformly smooth if there exists a constant $c > 0$, such that

$$\rho_X(t) \le ct^q, \quad q > 1.$$

Note that J_q is single-valued if X is uniformly smooth. In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [33] proved the following result:

Lemma 2.1 (*[33]*). *Let X be a real uniformly smooth Banach space. Then X is q-uniformly smooth if and only if there exists a constant $C_q > 0$, such that for all $x, y \in X$,*

$$\|x+y\|^q \le \|x\|^q + q\langle y, J_q(x)\rangle + C_q\|y\|^q.$$

For $i = 1, 2$, let X_i be real q_i-uniformly smooth Banach spaces with norm $\|\cdot\|_i$. Let $\eta_i : X_i \times X_i \to X_i$, $A_i : X_i \to X_i$, $F : X_1 \times X_2 \to X_1$, $G : X_1 \times X_2 \to X_2$ be nonlinear mappings, and let $M : X_1 \times X_1 \to 2^{X_1}$ and $N : X_2 \times X_2 \to 2^{X_2}$ be (A_1, η_1)-accretive and (A_2, η_2)-accretive mappings with respect to the first argument, respectively. Now we consider the following problem:

Find $(x, y) \in X_1 \times X_2$ such that

$$\begin{cases} 0 \in F(x,y) + M(x,x), \\ 0 \in G(x,y) + N(y,y). \end{cases} \tag{2.1}$$

Problem (2.1) is called a system of (A, η)-accretive mapping inclusions.

We remark that for suitable choices of the mappings $F, G, A_1, A_2, \eta_1, \eta_2, M, N$ and the spaces X_1, X_2, problem (2.1) includes many systems of variational inequality (inclusion) problems as special cases, see for example [1,6,8–11,14,15,20,29] and the references therein.

Definition 2.1. Let X_1, X_2 be real Banach spaces. Let Q be a mapping from $X_1 \times X_2 \to X_1 \times X_2$, $(x_0, y_0) \in X_1 \times X_2$ and $(x_{n+1}, y_{n+1}) = f(Q, x_n, y_n)$ define an iterative procedure which yields a sequence of points $\{(x_n, y_n)\}$ in $X_1 \times X_2$, where f is an iterative procedure involving the mapping Q. Let $F(Q) = \{(x,y) \in X_1 \times X_2 : (x,y) = Q(x,y)\} \neq \emptyset$. Suppose that $\{(x_n, y_n)\}$ converges to $(x^*, y^*) \in F(Q)$. Let $\{(u_n, v_n)\}$ be an arbitrary sequence in $X_1 \times X_2$ and $\varepsilon_n = \|\{(u_{n+1}, v_{n+1})\} - f(Q, u_n, v_n)\|$ for each $n \ge 0$. If $\lim_{n\to\infty}\varepsilon_n = 0$ implies that $\lim_{n\to\infty}(u_n, v_n) = (x^*, y^*)$, then the iteration procedure defined by $(x_{n+1}, y_{n+1}) = f(Q, x_n, y_n)$ is said to be Q-stable or stable with respect to Q.

Lemma 2.2 (*[34]*). *Let $\{a_n\}$ be a nonnegative real sequence and $\{b_n\}$ be a real sequence in $[0, 1]$ such that $\sum_{n=0}^{\infty} b_n = \infty$. If there exists a positive integer n_1 such that*

$$a_{n+1} \le (1 - b_n)a_n + b_n c_n, \quad \forall n \ge n_1,$$

where $c_n \ge 0$ for all $n \ge 0$ and $c_n \to 0$ $(n \to \infty)$, then $\lim_{n\to\infty} a_n = 0$.

Definition 2.2. Let $A : X_1 \to X_1$ and $F : X_1 \times X_2 \to X_1$ be single-valued mappings. F is said to be

(i) (α, β)-Lipschitz continuous, if there exist constants $\alpha > 0$ and $\beta > 0$ such that

$$\|F(x_1, y_1) - F(x_2, y_2)\|_1 \leq \alpha\|x_1 - x_2\|_1 + \beta\|y_1 - y_2\|_2, \quad \forall x_1, x_2 \in X_1,\ y_1, y_2 \in X_2.$$

(ii) (a, b)-relaxed cocoercive with respect to A in the first argument if there exist constants $a > 0$ and $b > 0$ such that

$$\langle F(x_1, y) - F(x_2, y), J_{q_1}(A(x_1) - A(x_2))\rangle \geq (-a)\|F(x_1, y) - F(x_2, y)\|_1^{q_1} + b\|x_1 - x_2\|_1^{q_1},$$

for all $x_1, x_2 \in X_1, y \in X_2$.

Definition 2.3. A single-valued mapping $\eta : X \times X \to X$ is said to be τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that $\|\eta(x, y)\| \leq \tau\|x - y\|, \forall x, y \in X$.

Definition 2.4. Let $\eta : X \times X \to X$ and $A : X \to X$ be single-valued mappings. Then set-valued mapping $M : X \to 2^X$ is said to be

(i) accretive if

$$\langle u - v, J_q(x - y)\rangle \geq 0, \quad \forall x, y \in X, u \in M(x), v \in M(y);$$

(ii) η-accretive if

$$\langle u - v, J_q(\eta(x, y))\rangle \geq 0, \quad \forall x, y \in X, u \in M(x), v \in M(y);$$

(iii) strictly η-accretive if M is η-accretive and equality holds if and only if $x = y$;

(iv) r-strongly η-accretive if there exists a constant $r > 0$ such that

$$\langle u - v, J_q(\eta(x, y))\rangle \geq r\|x - y\|^q, \quad \forall x, y \in X, u \in M(x), v \in M(y);$$

(v) α-relaxed η-accretive if there exists a constant $m > 0$ such that

$$\langle u - v, J_q(\eta(x, y))\rangle \geq (-\alpha)\|x - y\|^q, \quad \forall x, y \in X, u \in M(x), v \in M(y).$$

In a similar way, we can define strict η-accretivity and strong η-accretivity of the single-valued mapping A.

Definition 2.5. Let $A : X \to X$, $\eta : X \times X \to X$ be two single-valued mappings. Then a set-valued mapping $M : X \to 2^X$ is called (A, η)-accretive if M is m-relaxed η-accretive and $(A + \rho M)(X) = X$ for every $\rho > 0$.

Remark 2.1. For appropriate and suitable choices of m, A, η and X, it is easy to see that Definition 2.5 includes a number of definitions of monotone operators and accretive operators (see [22]).

In [22], Lan et al. showed that $(A + \rho M)^{-1}$ is a single-valued operator if $M : X \to 2^X$ is an (A, η)-accretive mapping and $A : X \to X$ is a r-strongly η-accretive mapping. Based on this fact, we can define the resolvent operator $R_{\rho,A}^{\eta,M}$ associated with an (A, η)-accretive mapping M as follows:

Definition 2.6. Let $A : X \to X$ be a strictly η-accretive mapping and $M : X \to 2^X$ be an (A, η)-accretive mapping. The resolvent operator $R_{\rho,A}^{\eta,M} : X \to X$ is defined by

$$R_{\rho,A}^{\eta,M}(x) = (A + \rho M)^{-1}(x), \quad \forall x \in X.$$

Lemma 2.3 ([22]). *Let $\eta : X \times X \to X$ be τ-Lipschitz continuous, $A : X \to X$ be a r-strongly η-accretive mapping and let $M : X \to 2^X$ be an (A, η)-accretive mapping. Then the resolvent operator $R_{\rho,A}^{\eta,M} : X \to X$ is $\frac{\tau^{q-1}}{r-\rho m}$-Lipschitz continuous, i.e.,*

$$\|R_{\rho,A}^{\eta,M} - R_{\rho,A}^{\eta,M}(y)\| \leq \frac{\tau^{q-1}}{r - \rho m}\|x - y\|, \quad \forall x, y \in X,$$

where $\rho \in (0, \frac{r}{m})$ is a constant.

3. Main results

Lemma 3.1. *For any given $(x, y) \in X_1 \times X_2$, (x, y) is a solution of problem (2.1) if and only if (x, y) satisfies*

$$\begin{cases} x = R_{\rho_1,A_1}^{\eta_1,M(\cdot,x)}[A_1(x) - \rho_1 F(x, y)], \\ y = R_{\rho_2,A_2}^{\eta_2,N(\cdot,y)}[A_2(y) - \rho_2 G(x, y)], \end{cases} \tag{3.1}$$

where $\rho_1, \rho_2 > 0$ are constants.

Proof. This directly follows from Definition 2.6. □

Based on Lemma 3.1 we suggest the following iterative algorithm for solving problem (2.1) as follows:

Algorithm 3.1. For $i = 1, 2$, assume that η_i, A_i, M, N, F, G and X_i are the same as in problem (2.1). Let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence such that $\alpha_n \in [0, 1]$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. For any given $(x_0, y_0) \in X_1 \times X_2$, define the iterative sequence $\{(x_n, y_n)\}$ by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n R_{\rho_1,A_1}^{\eta_1,M(\cdot,x_n)}[A_1(x_n) - \rho_1 F(x_n, y_n)], \\ y_{n+1} = (1-\alpha_n)y_n + \alpha_n R_{\rho_2,A_2}^{\eta_2,N(\cdot,y_n)}[A_2(y_n) - \rho_2 G(x_n, y_n)], \end{cases} \tag{3.2}$$

for $n = 0, 1, 2, \ldots$.

Let $\{(u_n, v_n)\}$ be any sequence in $X_1 \times X_2$ and define $\{\varepsilon_n\}$ by

$$\varepsilon_n = \|(u_{n+1}, v_{n+1}) - (A_n, B_n)\|_*, \tag{3.3}$$

where

$$A_n = (1-\alpha_n)u_n + \alpha_n R_{\rho_1,A_1}^{\eta_1,M(\cdot,u_n)}(A_1(u_n) - \rho_1 F(u_n, v_n)), \tag{3.4}$$

$$B_n = (1-\alpha_n)v_n + \alpha_n R_{\rho_2,A_2}^{\eta_2,N(\cdot,v_n)}(A_2(v_n) - \rho_2 G(u_n, v_n)), \tag{3.5}$$

for $n = 0, 1, 2, \ldots$.

Theorem 3.1. *For $i = 1, 2$, let X_i be q_i-uniformly smooth Banach space, $\eta_i : X_i \times X_i \to X_i$ be τ_i-Lipschitz continuous, and $A_i : X_i \to X_i$ be r_i-strongly η_i-accretive and γ_i-Lipschitz continuous. Let $F : X_1 \times X_2 \to X_1$ be (a, b)-relaxed cocoercive with respect to A_1 in the first argument and (μ_1, ν_1)-Lipschitz continuous, $G : X_1 \times X_2 \to X_2$ be (c, d)-relaxed cocoercive with respect to A_2 in the second argument and (μ_2, ν_2)-Lipschitz continuous. Let $M : X_1 \times X_1 \to 2^{X_1}$ and $N : X_2 \times X_2 \to 2^{X_2}$ be such that for each fixed $x \in X_1$, $y \in X_2$, $M(\cdot, x)$ and $N(\cdot, y)$ are (A_1, η_1)-accretive and (A_2, η_2)-accretive mappings, respectively. Suppose that there are constants $\xi_1, \xi_2 > 0$ such that*

$$\|R_{\rho_1,A_1}^{\eta_1,M(\cdot,x_1)}(x) - R_{\rho_1,A_1}^{\eta_1,M(\cdot,x_2)}(x)\|_1 \le \xi_1 \|x_1 - x_2\|_1, \quad \forall x, x_1, x_2 \in X_1, \tag{3.6}$$

$$\|R_{\rho_2,A_2}^{\eta_2,N(\cdot,y_1)}(y) - R_{\rho_2,A_2}^{\eta_2,N(\cdot,y_2)}(y)\|_2 \le \xi_2 \|y_1 - y_2\|_2, \quad \forall y, y_1, y_2 \in X_2. \tag{3.7}$$

and $\rho_1 \in \left(0, \frac{r_1}{m_1}\right)$ and $\rho_2 \in (0, \frac{r_2}{m_2})$ such that

$$\begin{cases} l_1\theta_1 + \xi_1 + \rho_2 \mu_2 l_2 < 1, \\ l_2\theta_2 + \xi_2 + \rho_1 \nu_1 l_1 < 1. \end{cases} \tag{3.8}$$

where

$$\theta_1 = (\gamma_1^{q_1} - q_1\rho_1 b + q_1\rho_1 a\mu_1^{q_1} + C_{q_1}\rho_1^{q_1}\mu_1^{q_1})^{\frac{1}{q_1}}$$

$$\theta_2 = (\gamma_2^{q_2} - q_2\rho_2 d + q_2\rho_2 c\nu_2^{q_2} + C_{q_2}\rho_2^{q_2}\nu_2^{q_2})^{\frac{1}{q_2}},$$

$$l_1 = \frac{\tau_1^{q_1-1}}{r_1 - \rho_1 m_1}, \qquad l_2 = \frac{\tau_2^{q_2-1}}{r_2 - \rho_2 m_2}.$$

Then problem (2.1) admits a unique solution.

Proof. For any given $\rho_i > 0$ $(i = 1, 2)$, define $T : X_1 \times X_2 \to X_1$ and $S : X_1 \times X_2 \to X_2$ by

$$\begin{aligned} T(x, y) &= R_{\rho_1,A_1}^{\eta_1,M(\cdot,x)}[A_1(x) - \rho_1 F(x, y)], \\ S(x, y) &= R_{\rho_2,A_2}^{\eta_2,N(\cdot,y)}[A_2(y) - \rho_2 G(x, y)], \end{aligned} \tag{3.9}$$

for all $(x, y) \in X_1 \times X_2$.

For any $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$, it follows from (3.9) and Lemma 2.3 that

$$\begin{aligned} \|T(x_1, y_1) - T(x_2, y_2)\|_1 &\le \|R_{\rho_1,A_1}^{\eta_1,M(\cdot,x_1)}[A_1(x_1) - \rho_1 F(x_1, y_1)] - R_{\rho_1,A_1}^{\eta_1,M(\cdot,x_2)}[A_1(x_2) - \rho_1 F(x_2, y_2)]\|_1 \\ &\le \|R_{\rho_1,A_1}^{\eta_1,M(\cdot,x_1)}[A_1(x_1) - \rho_1 F(x_1, y_1)] - R_{\rho_1,A_1}^{\eta_1,M(\cdot,x_2)}[A_1(x_1) - \rho_1 F(x_1, y_1)]\|_1 \\ &\quad + \|R_{\rho_1,A_1}^{\eta_1,M(\cdot,x_2)}[A_1(x_1) - \rho_1 F(x_1, y_1)] - R_{\rho_1,A_1}^{\eta_1,M(\cdot,x_2)}[A_1(x_2) - \rho_1 F(x_2, y_2)]\|_1 \\ &\le \xi_1\|x_1 - x_2\|_1 + l_1(\|A_1(x_1) - A_1(x_2) - \rho_1(F(x_1, y_1) - F(x_2, y_1))\|_1 \\ &\quad + \rho_1\|F(x_2, y_1) - F(x_2, y_2)\|_1), \end{aligned} \tag{3.10}$$

where $l_1 = \frac{\tau_1^{q_1-1}}{r_1 - \rho_1 m_1}$.

By assumptions, we have

$$\begin{aligned}&\|A_1(x_1)-A_1(x_2)-\rho_1(F(x_1,y_1)-F(x_2,y_1))\|_1^{q_1}\\&\quad\le\|A_1(x_1)-A_1(x_2)\|_1^{q_1}-q_1\rho_1\langle F(x_1,y_1)-F(x_2,y_1),J_{q_1}(A_1(x_1)-A_1(x_2))\rangle\\&\qquad+C_{q_1}\rho_1^{q_1}\|F(x_1,y_1)-F(x_2,y_1)\|_1^{q_1}\\&\quad\le(\gamma_1^{q_1}-q_1\rho_1b+q_1\rho_1a\mu_1^{q_1}+C_{q_1}\rho_1^{q_1}\mu_1^{q_1})\|x_1-x_2\|_1^{q_1}\end{aligned}\tag{3.11}$$

$$\|F(x_2,y_1)-F(x_2,y_2)\|_1\le\nu_1\|y_1-y_2\|_2.\tag{3.12}$$

Combining (3.10)–(3.12), we have

$$\|T(x_1,y_1)-T(x_2,y_2)\|_1\le(l_1\theta_1+\xi_1)\|x_1-x_2\|_1+l_1\rho_1\nu_1\|y_1-y_2\|_2,\tag{3.13}$$

where $\theta_1=(\gamma_1^{q_1}-q_1\rho_1b+q_1\rho_1a\mu_1^{q_1}+C_{q_1}\rho_1^{q_1}\mu_1^{q_1})^{\frac{1}{q_1}}$.

Similarly, we can prove that

$$\|S(x_1,y_1)-S(x_2,y_2)\|_2\le(l_2\theta_2+\xi_2)\|y_1-y_2\|_2+l_2\rho_2\mu_2\|x_1-x_2\|_1.\tag{3.14}$$

where $\theta_2=(\gamma_2^{q_2}-q_2\rho_2d+q_2\rho_2c\nu_2^{q_2}+C_{q_2}\rho_2^{q_2}\nu_2^{q_2})^{\frac{1}{q_2}}$, $\quad l_2=\frac{\tau_2^{q_2-1}}{r_2-\rho_2m_2}$.

By (3.13) and (3.14), we have

$$\begin{aligned}\|T(x_1,y_1)-T(x_2,y_2)\|_1+\|S(x_1,y_1)-S(x_2,y_2)\|_2&\le k_1\|x_1-x_2\|_1+k_2\|y_1-y_2\|_2\\&\le k(\|x_1-x_2\|_1+\|y_1-y_2\|_2),\end{aligned}\tag{3.15}$$

where $k=\max\{k_1,k_2\}$, $k_1=l_1\theta_1+\xi_1+\rho_2\mu_2l_2$, $k_2=l_2\theta_2+\xi_2+\rho_1\nu_1l_1$.

Define the norm $\|\cdot\|_*$ on $X_1\times X_2$ by

$$\|(x,y)\|_*=\|x\|_1+\|y\|_2,\quad (x,y)\in X_1\times X_2.\tag{3.16}$$

It is easy to see that $(X_1\times X_2,\|\cdot\|_*)$ is a Banach space. Define $Q(x,y):X_1\times X_2\to X_1\times X_2$ by

$$Q(x,y)=(T(x,y),S(x,y)),\quad \forall(x,y)\in X_1\times X_2.$$

By (3.8), we know that $0<k<1$. This follows from (3.15) that

$$\|Q(x_1,y_1)-Q(x_2,y_2)\|_*\le k\|(x_1,y_1)-(x_2,y_2)\|_*.$$

This proves that $Q(x,y):X_1\times X_2\to X_1\times X_2$ is a contraction mapping. Hence, by the Banach contraction principle, there exists a unique $(x^*,y^*)\in X_1\times X_2$ such that $Q(x^*,y^*)=(x^*,y^*)$, which implies that

$$\begin{cases}x^*=R_{\rho_1,A_1}^{\eta_1,M(\cdot,x^*)}[A_1(x^*)-\rho_1F(x^*,y^*)],\\ y^*=R_{\rho_2,A_2}^{\eta_2,N(\cdot,y^*)}[A_2(y^*)-\rho_2G(x^*,y^*)].\end{cases}$$

This follows from Lemma 3.1 that (x^*,y^*) is the unique solution of problem (2.1). This completes the proof. □

Theorem 3.2. *For $i=1,2$, let η_i,A_i,M,N,F,G and X_i be the same as in Theorem 3.1 and let conditions (3.6)–(3.8) of Theorem 3.1 hold. Then*

(i) *the sequence (x_n,y_n) generated by Algorithm 3.1 converges strongly to the unique solution (x^*,y^*) of problem (2.1).*
(ii) *if $0<\alpha<\alpha_n$, then $\lim_{n\to\infty}(u_n,v_n)=(x^*,y^*)$ if and only if $\lim_{n\to\infty}\varepsilon_n=0$.*

Proof. This follows from Theorem 3.1 that problem (2.1) has the unique solution (x^*,y^*). By Lemma 3.1, we have

$$\begin{cases}x^*=R_{\rho_1,A_1}^{\eta_1,M(\cdot,x^*)}[A_1(x^*)-\rho_1F(x^*,y^*)],\\ y^*=R_{\rho_2,A_2}^{\eta_2,N(\cdot,y^*)}[A_2(y^*)-\rho_2G(x^*,y^*)].\end{cases}\tag{3.17}$$

From (3.2) and (3.17) and using the same arguments as obtaining (3.10) and (3.14), we have that

$$\|x_{n+1}-x^*\|_1\le(1-\alpha_n)\|x_n-x^*\|_1+\alpha_n((l_1\theta_1+\xi_1)\|x_n-x^*\|_1+l_1\rho_1\nu_1\|y_n-y^*\|_2),\tag{3.18}$$

$$\|y_{n+1}-y^*\|_2\le(1-\alpha_n)\|y_n-y^*\|_2+\alpha_n((l_2\theta_2+\xi_2)\|y_n-y^*\|_2+l_2\rho_2\mu_2\|x_n-x^*\|_1),\tag{3.19}$$

where

$$\theta_1=(\gamma_1^{q_1}-q_1\rho_1b+q_1\rho_1a\mu_1^{q_1}+C_{q_1}\rho_1^{q_1}\mu_1^{q_1})^{\frac{1}{q_1}},\qquad l_1=\frac{\tau_1^{q_1-1}}{r_1-\rho_1m_1},$$

$$\theta_2=(\gamma_2^{q_2}-q_2\rho_2d+q_2\rho_2c\nu_2^{q_2}+C_{q_2}\rho_2^{q_2}\nu_2^{q_2})^{\frac{1}{q_2}},\qquad l_2=\frac{\tau_2^{q_2-1}}{r_2-\rho_2m_2}.$$

By (3.16), (3.18) and (3.19), we obtain

$$\begin{aligned}\|(x_{n+1},y_{n+1})-(x^*,y^*)\|_* &= \|x_{n+1}-x^*\|_1+\|y_{n+1}-y^*\|_2\\ &\le (1-\alpha_n)\|(x_n,y_n)-(x^*,y^*)\|_*+\alpha_n\max\{k_1,k_2\}\|(x_n,y_n)-(x^*,y^*)\|_*\\ &= (1-(1-k)\alpha_n)\|(x_n,y_n)-(x^*,y^*)\|_*, \end{aligned}\tag{3.20}$$

where $k=\max\{k_1,k_2\}$, $k_1=l_1\theta_1+\xi_1+\rho_2\mu_2 l_2$ and $k_2=l_2\theta_2+\xi_2+\rho_1\nu_1 l_1$.

Set

$$a_n=\|(x_n,y_n)-(x^*,y^*)\|_*,\qquad b_n=(1-k)\alpha_n,\qquad c_n=0.$$

This follows from (3.8), $\alpha_n\in[0,1]$ and $\sum_{n=0}^{\infty}\alpha_n=\infty$ that

$$b_n\in[0,1],\quad \sum_{n=0}^{\infty}b_n=\infty.$$

Therefore, Lemma 2.2 and (3.20) imply that $\lim_{n\to\infty}a_n=0$, i.e., $\|(x_n,y_n)-(x^*,y^*)\|_*\to 0$ $(n\to\infty)$. Thus (x_n,y_n) converges strongly to the unique solution (x^*,y^*) of problem (2.1).

Now we prove conclusion (ii). By (3.3)–(3.5), we obtain

$$\begin{aligned}\|(u_{n+1},v_{n+1})-(x^*,y^*)\|_* &\le \|(u_{n+1},v_{n+1})-(A_n,B_n)\|_*+\|(A_n,B_n)-(x^*,y^*)\|_*\\ &\le \varepsilon_n+\|A_n-x^*\|_1+\|B_n-y^*\|_2. \end{aligned}\tag{3.21}$$

As in the proof of inequality (3.18), it follows that

$$\|A_n-x^*\|_1\le(1-\alpha_n)\|u_n-x^*\|_1+\alpha_n[(l_1\theta_1+\xi_1)\|u_n-x^*\|_1+l_1\rho_1\nu_1\|v_n-y^*\|_2]_1,\tag{3.22}$$

$$\|B_n-y^*\|_2\le(1-\alpha_n)\|v_n-y^*\|_2+\alpha_n[(l_2\theta_2+\xi_2)\|v_n-y^*\|_2+l_2\rho_2\mu_2\|u_n-x^*\|_1].\tag{3.23}$$

Since $0<\alpha<\alpha_n$, by (3.22) and (3.23),

$$\begin{aligned}\|(u_{n+1},v_{n+1})-(x^*,y^*)\|_* &\le (1-(1-\max\{k_1,k_2\})\alpha_n)\|(u_n,v_n)-(x^*,y^*)\|_*+\varepsilon_n\\ &\le (1-(1-k)\alpha_n)\|(u_n,v_n)-(x^*,y^*)\|_*+(1-k)\alpha_n\frac{\varepsilon_n}{(1-k)\alpha}, \end{aligned}$$

where $k=\max\{k_1,k_2\}$,$k_1=l_1\theta_1+\xi_1+\rho_2\mu_2 l_2$, $k_2=l_2\theta_2+\xi_2+\rho_1\nu_1 l_1$.

Suppose that $\lim_{n\to\infty}\varepsilon_n=0$. Then from $\sum_{n=0}^{\infty}\alpha_n=\infty$ and Lemma 2.2, we have $\lim_{n\to\infty}(u_n,v_n)=(x^*,y^*)$.

Conversely, if $\lim_{n\to\infty}(u_n,v_n)=(x^*,y^*)$, then

$$\begin{aligned}\varepsilon_n &= \|(u_{n+1},v_{n+1})-(A_n,B_n)\|_*\\ &\le \|(u_{n+1},v_{n+1})-(x^*,y^*)\|_*+\|A_n-x^*\|_1+\|B_n-y^*\|_2\\ &\le \|(u_{n+1},v_{n+1})-(x^*,y^*)\|_*+(1-(1-k)\alpha_n)\|(u_n,v_n)-(x^*,y^*)\|_*\to 0\ (n\to\infty), \end{aligned}$$

i.e., $\lim_{n\to\infty}\varepsilon_n=0$. This completes the proof. □

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