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Parameter Estimation for a Drying System in a Porous Medium

C. COLES
Elon University
Campus Box 2230
Elon, NC 27244, U.S.A.
ccoles@elon.edu

D. MURIO
University of Cincinnati
Department of Mathematical Sciences
Cincinnati, OH 45221, U.S.A.
mmurio@cinci.rr.com

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Abstract—We introduce a stable numerical method for the recovery of the temperature and moisture distributions in a Liukov system with space dependent diffusion coefficients. In this problem, only Cauchy noisy data at the active boundary is given and no information about the amount and/or character of the noise in the data is assumed. The error analysis for the algorithm is discussed and a numerical example is presented. © 2006 Elsevier Ltd. All rights reserved.

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Thermal drying involves the vaporization of moisture within a product by heat and the evaporation of moisture from the medium. Thermal drying in porous medium has important applications in many different fields, including food and environmental engineering. A theoretical model for thermal drying, or more generally simultaneous heat and mass transfer, was developed by Luikov. This model integrates several different physical mechanisms of moisture diffusion. The general Luikov model for thermal drying in a porous medium is

$$\frac{\partial M}{\partial t} = \nabla^2 a_{11} M + \nabla^2 a_{12} T + \nabla^2 a_{13} P,$$
$$\frac{\partial T}{\partial t} = \nabla^2 a_{21} M + \nabla^2 a_{22} T + \nabla^2 a_{23} P,$$
$$\frac{\partial P}{\partial t} = \nabla^2 a_{31} M + \nabla^2 a_{32} T + \nabla^2 a_{33} P,$$

where M, T, and P represent vapor diffusion, thermal diffusion, and hydrodynamic flow respectively, a_{11}, a_{22}, and a_{33} are diffusion coefficients and a_{12}, a_{13}, a_{21}, a_{23}, a_{31}, and a_{32} are

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coupling coefficients. For many agricultural products, knowledge of the diffusion coefficients are still limited. The coupling coefficients account for the combined effect of moisture, temperature, and total pressure gradients on moisture, total mass, and energy transfer. Note that total pressure differences are only significant in relatively high temperatures. Therefore, in most drying applications $a_{13} = a_{23} = a_{33} = 0$. For an overview of drying principles and theory, see [1].

The authors of [2,3] discuss a Luikov system of the form,

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \eta \end{pmatrix} \begin{pmatrix} u_{xx} \\ v_{xx} \end{pmatrix}, \qquad 0 < x < 1, \quad t > 0,$$

with boundary conditions,

$$\begin{aligned} u_x(0,t) &= -Q, \\ v_x(0,t) &= -\mathrm{Pn}\, Q, \\ u_x(1,t) &= -\mathrm{Bi}_q\, u(1,t) + (1-\mathrm{Eo})\mathrm{Ko}\,\mathrm{Lu}\,\mathrm{Bi}_m(v(1,t)-1) + \mathrm{Bi}_q V(t), \\ v_x(1,t) &= \mathrm{Bi}_m(1 - v(1,t)) - \mathrm{Pn}\, u_x(1,t), \end{aligned}$$

where $V(t)$ is a transient function associated with the dry air flow, and initial conditions,

$$\begin{aligned} u(x,0) &= u_0, \\ v(x,0) &= v_0. \end{aligned}$$

The constant coefficients α, β, γ, and η are defined as

$$\begin{aligned} \alpha &= 1 + \mathrm{Eo}\,\mathrm{Ko}\,\mathrm{Lu}\,\mathrm{Pn}\,, \\ \beta &= -\mathrm{Eo}\,\mathrm{Ko}\,\mathrm{Lu}\,, \\ \gamma &= -\mathrm{Lu}\,\mathrm{Pn}\,, \\ \eta &= \mathrm{Lu}\,. \end{aligned}$$

The terms $\mathrm{Lu} = a_m/a$, Pn , Ko , Bi_q, Bi_m, and Q refer to the Luikov number, Possnov number, Kossovitch number, heat Biot, mass Biot, and heat flux, respectively. The coefficients a and a_m represent the thermal diffusivity and the moisture diffusivity of the porous medium. Deterministic, stochastic, and hybrid solutions are introduced in [2] for estimation of parameters in the above problem.

In this paper, we consider nonhomogeneous thermal and moisture diffusivities of the porous medium so that the Luikov number and all the coefficients α, β, γ, and η of the model, are space dependent functions. We will introduce a stable numerical marching scheme based on discrete mollification for the recovery of $u(x,t)$, $v(x,t)$, $u_x(x,t)$, and $v_x(x,t)$ throughout the domain $[0,1] \times [0,1]$ in the (x,t) plane satisfying

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} \alpha(x) & \beta(x) \\ \gamma(x) & \eta(x) \end{pmatrix} \begin{pmatrix} u_{xx} \\ v_{xx} \end{pmatrix}, \qquad 0 < x < 1, \quad t > 0,$$

with boundary conditions,

$$\begin{aligned} u(0,t) &= g_1(t), \\ v(0,t) &= g_2(t), \\ u_x(0,t) &= g_3(t), \\ v_x(0,t) &= g_4(t). \end{aligned}$$

Note that in this problem, g_1, g_2, g_3, and g_4 are only known approximately.

This problem is an inverse problem (Cauchy problem) involving a parabolic system. For other examples of numerical marching schemes based on mollification in parabolic systems see [4,5]. These algorithms do not require any information about the amount and/or characteristics of the noise in the data and the mollification parameters are chosen automatically at each step using the generalized cross validation (GCV) method. For general references to the GCV method, see [6,7].

This paper is organized as follows. Discrete mollification and numerical differentiation will be presented in Section 1. In Section 2, the numerical space marching algorithm is specified. The stability and error estimates for the approach are also presented in this section. Section 3 contains a numerical example of interest.

1. MOLLIFICATION

In this chapter, discrete mollification is introduce and several results related to numerical differentiation are presented. A detailed description of this regularization procedure and its application can be found in [8].

1.1. Discrete Mollification

Let $I = [0, 1]$ and $K = \{x_i : i = 1, 2, \dots, N\} \subset I$ satisfying $0 \le x_1 < x_2 < \dots < x_N \le 1$. Set $s_0 = 0$, $s_N = 1$, and $s_i = (1/2)(x_{i+1} + x_i)$ for $i = 1, 2, \dots, N-1$. Suppose that $G = \{g_i\}_{i=1}^N$ is a discrete function defined on K, then the δ-mollification of G is defined as a convolution with the Gaussian kernel,

$$\rho_\delta(t) = \begin{cases} A_p \delta^{-1} \exp\left(-\dfrac{t^2}{\delta^2}\right), & t \in I_\delta, \\ 0, & t \notin I_\delta, \end{cases}$$

where $I_\delta = [-p\delta, p\delta]$, $\delta > 0$, $p > 0$, and

$$A_p = \left(\int_{-p}^{p} \exp(-s^2)\, ds \right)^{-1}.$$

That is, for every $x \in I_\delta$,

$$J_\delta G(x) = \sum_{i=1}^{N} \left(\int_{s_{i-1}}^{s_i} \rho_\delta(x - s)\, ds \right) g_i.$$

1.2. Numerical Differentiation

The first centered difference operator,

$$\mathbf{D}_0 f(x) = \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x},$$

is defined on

$$\tilde{I}_\delta = [p\delta + \Delta x,\ 1 - p\delta - \Delta x].$$

Let $G^\epsilon = \{g_i + \epsilon_i : |\epsilon_i| < \epsilon,\ i = 1, 2, \dots, N\}$ be a perturbed discrete version of a function g, where ϵ is the maximum noise level. The following lemma, establishes the numerical convergence of centered difference discrete mollified differentiation for a fixed δ.

LEMMA 1.1. *If g is uniformly Lipschitz on I and the discrete functions G and G^ϵ satisfy $\|G - G^\epsilon\|_{\infty,K} \le \epsilon$, then there exist constants C, independent of δ, and C_δ such that $\|J_\delta G^\epsilon - J_\delta g\|_{\infty,I_\delta} \le C(\epsilon + \Delta x)$ and*

$$\left\| \mathbf{D}_0(J_\delta G^\epsilon) - \frac{\partial}{\partial x} J_\delta g \right\|_{\infty,\tilde{I}_\delta} \le C \left(\frac{\epsilon + \Delta x}{\delta} \right) + C_\delta (\Delta x)^2.$$

The proof of Lemma 1.1 can be found in [9].

We define the discrete mollified centered difference $\mathbf{D}_0^\delta(G) = \mathbf{D}_0(J_\delta G)\big|_{\tilde{I}_\delta \cap K}$, by restricting $\mathbf{D}_0(J_\delta G)$ to the grid points of $\tilde{I}_\delta \cap K$. The next theorem establishes a useful upper bound for the operator $\mathbf{D}_0^\delta$.

THEOREM 1.2. *There exists a constant C, independent of δ, such that*

$$\|\mathbf{D}_0^\delta G\|_{\infty, K\cap \tilde{I}_\delta} \le \frac{C}{\delta}\|G\|_{\infty,K}.$$

The proof of this theorem can also be found in [9].

2. THE IDENTIFICATION PROBLEM

The problem is to identify the vapor diffusion, $u(x,t)$, vapor flux $u_x(x,t)$, moisture diffusion $v(x,t)$ and moisture flux $v_x(x,t)$, for all (x,t) throughout the domain $[0,1]\times[0,1]$ satisfying

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} \alpha(x) & \beta(x) \\ \gamma(x) & \eta(x) \end{pmatrix} \begin{pmatrix} u_{xx} \\ v_{xx} \end{pmatrix}, \qquad 0 < x < 1, \quad t > 0,$$

with boundary conditions,

$$u(0,t) = g_1(t),$$
$$v(0,t) = g_2(t),$$
$$u_x(0,t) = g_3(t),$$
$$v_x(0,t) = g_4(t).$$

Note that g_1, g_2, g_3, and g_4 are not known exactly. The available data ${g_1}^\epsilon$, ${g_2}^\epsilon$, ${g_3}^\epsilon$, and ${g_4}^\epsilon$ are discrete noisy functions with maximum noise level ϵ. We define $A(x) = \left(\begin{smallmatrix} \alpha(x) & \beta(x) \\ \gamma(x) & \eta(x)\end{smallmatrix}\right)$ and assume that $|\det(A(x))| \succeq d > 0$ for all $x \in [0,1]$.

We begin by stabilizing the problem using mollification. In this regularization process, a δ-mollification is performed on each of the available data functions, ${g_1}^\epsilon$, ${g_2}^\epsilon$, ${g_3}^\epsilon$, and ${g_4}^\epsilon$. Note that δ-mollifications of ${g_1}^\epsilon$, ${g_2}^\epsilon$, ${g_3}^\epsilon$, and ${g_4}^\epsilon$ are taken with respect to t using δ_u^0, δ_v^0, δ_{ux}^0, and δ_{vx}^0, respectively.

The following numerical marching scheme, along with the mollification method, are applied in order to estimate $u(x,t)$ and $v(x,t)$, as well as derivatives of these functions, throughout $[0,1]\times[0,1]$, where $\tilde{u}(x,t)$ and $\tilde{v}(x,t)$ are the regularized functions.

2.1. The Numerical Space Marching Scheme

Let N_x and N_t be positive integers, $\Delta x = h = 1/N_x$, $\Delta t = k = 1/N_t$, $x_i = ih$, $i = 0,1,\dots,N_x$, and $t_n = nk$, $n = 0,1,\dots,N_t$.

We introduce the following discrete functions.

$R_u^{i,n}$: the discrete approximation to $\tilde{u}(ih,nk)$,
$R_v^{i,n}$: the discrete approximation to $\tilde{v}(ih,nk)$,
$Q_u^{i,n}$: the discrete approximation to $\tilde{u}_x(ih,nk)$,
$Q_v^{i,n}$: the discrete approximation to $\tilde{v}_x(ih,nk)$,
$W_u^{i,n}$: the discrete approximation to $\tilde{u}_t(ih,nk)$,
$W_v^{i,n}$: the discrete approximation to $\tilde{v}_t(ih,nk)$,
$S_u^{i,n}$: the discrete approximation to $\tilde{u}_{xt}(ih,nk)$,
$S_v^{i,n}$: the discrete approximation to $\tilde{v}_{xt}(ih,nk)$.

The space marching algorithm is defined as follows.

1. Select δ_u^0, δ_v^0, δ_{ux}^0, and δ_{vx}^0.
2. Perform mollification of ${g_1}^\epsilon$, ${g_2}^\epsilon$, ${g_3}^\epsilon$, and ${g_4}^\epsilon$. Set the following.
   - $R_u^{0,n} = J_{\delta_u^0}{g_1}^\epsilon(nk)$ and $R_v^{0,n} = J_{\delta_v^0}{g_2}^\epsilon(nk)$.
   - $Q_u^{0,n} = J_{\delta_{ux}^0}{g_3}^\epsilon(nk)$ and $Q_v^{0,n} = J_{\delta_{vx}^0}{g_4}^\epsilon(nk)$.

3. Perform mollified differentiation in time of $J_{\delta_u^0} g_1{}^{\epsilon}(nk)$, $J_{\delta_v^0} g_2{}^{\epsilon}(nk)$, $J_{\delta_{ux}^0} g_3{}^{\epsilon}(nk)$ and $J_{\delta_{vx}^0} g_4{}^{\epsilon}(nk)$. Set the following.
   - $W_u^{0,n} = \mathbf{D}_t(J_{\delta_u^0} g_1{}^{\epsilon}(nk))$ and $W_v^{0,n} = \mathbf{D}_t(J_{\delta_v^0} g_2{}^{\epsilon}(nk))$.
   - $S_u^{0,n} = \mathbf{D}_t(J_{\delta_{ux}^0} g_3{}^{\epsilon}(nk))$ and $S_v^{0,n} = \mathbf{D}_t(J_{\delta_{vx}^0} g_4{}^{\epsilon}(nk))$.

The numerical marching scheme in space is defined in Step 4.

4. Initialize $i = 0$. Do while $i \leq N_x - 1$.
   a. $R_u^{i+1,n} = R_u^{i,n} + h\, Q_u^{i,n}$ and $R_v^{i+1,n} = R_v^{i,n} + h\, Q_v^{i,n}$.
   b. $Q_u^{i+1,n} = Q_u^{i,n} + (h/det(A(ih)))(\eta(ih)W_u^{i,n} - \beta(ih)W_v^{i,n})$.
   c. $Q_v^{i+1,n} = Q_v^{i,n} + (h/det(A(ih)))\,(-\gamma(ih)W_u^{i,n} + \alpha(ih)W_v^{i,n})$.
   d. Select δ_u^{i+1}, δ_v^{i+1}, δ_{ux}^{i+1}, δ_{vx}^{i+1}.
   e. Perform mollified differentiation in time of $R_u^{i+1,n}$, $R_v^{i+1,n}$, $Q_u^{i+1,n}$, $Q_v^{i+1,n}$. Set the following.
      - $W_u^{i+1,n} = \mathbf{D}_t(J_{\delta_u^{i+1}} R_u^{i+1,n})$ and $W_v^{i+1,n} = \mathbf{D}_t(J_{\delta_v^{i+1}} R_v^{i+1,n})$.
      - $S_u^{i+1,n} = \mathbf{D}_t(J_{\delta_{ux}^{i+1}} Q_u^{i+1,n})$ and $S_v^{i+1,n} = \mathbf{D}_t(J_{\delta_{vx}^{i+1}} S_v^{i+1,n})$.
   f. Set $i = i + 1$.

2.2. Stability Analysis

Denote $|Y^i| = \max_n |Y^{i,n}|$ and $\|Y\|_\infty = \max_i |Y^i|$. Theorems 2.1 and 2.2 establish stability and formal convergence, respectively, of the marching scheme presented above.

THEOREM 2.1. *There exists a constant C_0 such that*

$$\max\left\{\left|R_u^L\right|, \left|R_v^L\right|, \left|Q_u^L\right|, \left|Q_v^L\right|\right\} \leq \exp(C_0) \max\left\{\left|R_u^0\right|, \left|R_v^0\right|, \left|Q_u^0\right|, \left|Q_v^0\right|\right\}.$$

PROOF. Using the numerical marching scheme defined in Section 2.1, bounds can be found for the discrete approximations in terms of the initial data.

Note that

$$\left|R_u^{i+1}\right| \leq \left|R_u^i\right| + h\,\left|Q_u^i\right|,$$
$$\left|R_v^{i+1}\right| \leq \left|R_v^i\right| + h\,\left|Q_v^i\right|.$$

Applying Theorem 1.2, there exist constants C_1, C_2, C_3, and C_4 such that $|W_u^i| \leq C_1/(|\delta|_{-\infty})|R_u^i|$ and $|W_v^i| \leq C_2/(|\delta|_{-\infty})|R_v^i|$, where $|\delta|_{-\infty} = \min_i\{\delta_u^i, \delta_v^i, \delta_{ux}^i, \delta_{vx}^i\}$.

Therefore,

$$\left|Q_u^{i+1}\right| \leq \left|Q_u^i\right| + h\frac{C}{|\delta|_{-\infty}}\left(\left|R_u^i\right| + \left|R_v^i\right|\right),$$
$$\left|Q_v^{i+1}\right| \leq \left|Q_v^i\right| + h\frac{C}{|\delta|_{-\infty}}\left(\left|R_u^i\right| + \left|R_v^i\right|\right),$$

where $C = 1/d \max\{C_1\gamma, C_1\eta, C_2\alpha, C_2\beta\}$. Let $C_0 = \max\{1, (2C/|\delta|_{-\infty})\}$. Then,

$$\max\left\{\left|R_u^{i+1}\right|, \left|R_v^{i+1}\right|, \left|Q_u^{i+1}\right|, \left|Q_v^{i+1}\right|\right\} \leq (1 + C_0 h) \max\left\{\left|R_u^i\right|, \left|R_v^i\right|, \left|Q_u^i\right|, \left|Q_v^i\right|\right\}.$$

Calculating L iterations,

$$\begin{aligned}\max\left\{\left|R_u^L\right|, \left|R_u^L\right|, \left|Q_u^L\right|, \left|Q_v^L\right|\right\} &\leq (1 + C_0 h)^L \max\left\{\left|R_u^0\right|, \left|R_v^0\right|, \left|Q_u^0\right|, \left|Q_v^0\right|\right\} \\ &\leq \exp\left(C_0\right) \max\left\{\left|R_u^0\right|, \left|R_v^0\right|, \left|Q_u^0\right|, \left|Q_v^0\right|\right\}.\end{aligned}$$

Thus, the numerical marching scheme is stable.

2.3. Error Estimates

Denoting the error between the calculated discrete functions $R_u^{i,n}$, $Q_u^{i,n}$ and the restriction to the grid of the mollified exact functions $\tilde{u}(ih, nk), \tilde{u_x}(ih, nk)$ by $\Delta R_u^{i,n} = R_u^{i,n} - \tilde{u}(ih, nk)$ and $\Delta Q_u^{i,n} = Q_u^{i,n} - \tilde{u_x}(ih, nk)$, proceeding similarly with the discrete functions related to $v(x,t)$, we define $\Delta_i = \max\{|\Delta R_u^i|, |\Delta R_v^i|, |\Delta Q_u^i|, |\Delta Q_v^i|\}$.

Theorem 2.2. *There exists a constant C_0 such that $\Delta_L \le \exp(C_0)(\Delta_0 + \epsilon + k)$.*

Proof. Define $C_\delta = \max_i\{C_{\delta_u^i}, C_{\delta_v^i}, C_{\delta_{ux}^i}, C_{\delta_{vx}^i}\}$ where $C_{\delta_u^i}$, $C_{\delta_v^i}$, $C_{\delta_{ux}^i}$, and $C_{\delta_{vx}^i}$ represent the upper bound, in magnitude, of higher-order derivatives of the convolution kernels corresponding to the radii of mollification δ_u^i, δ_v^i, δ_{ux}^i, and δ_{vx}^i, respectively, $i = 0, 1, \dots, N_x$. Neglecting the effect of the δ mollification on the already mollified solutions $\tilde{u}$ and $\tilde{v}$ and their partial derivatives, the error estimates for $R_u^{i+1,n}$, $R_u^{i+1,n}$, $Q_u^{i+1,n}$, and $Q_v^{i+1,n}$ are obtain as follows,

$$|\Delta R_u^{i+1}| \le |\Delta R_u^i| + h(|\Delta Q_u^i|)$$

and

$$|\Delta R_v^{i+1}| \le |\Delta R_v^i| + h(|\Delta Q_v^i|).$$

According to Lemma 1.1, there exists constants C_1 and C_2 such that

$$|\mathbf{D}_t(J_{\delta_u^i} R_u^{i,n}) - \tilde{u}_t(ih, nk)| \le \frac{C_1}{|\delta|_{-\infty}} \left(|\Delta R_u^i| + k\right) + C_\delta k^2$$

and

$$|\mathbf{D}_t(J_{\delta_v^i} R_v^{i,n}) - \tilde{v}_t(ih, nk)| \le \frac{C_2}{|\delta|_{-\infty}} \left(|\Delta R_v^i| + k\right) + C_\delta k^2.$$

Thus,

$$\begin{aligned} |\Delta Q_u^{i+1,n}| &= |\Delta Q_u^{i,n} + \frac{h}{det(A(ih))}(\eta(\mathbf{D}_t(J_{\delta_u^i} R_u^{i,n}) - \tilde{u}_t(ih, nk)) \\ &\quad - \beta(\mathbf{D}_t(J_{\delta_v^i} R_v^{i,n}) - \tilde{v}_t(ih, nk))| \\ &\le |\Delta Q_u^{i,n}| + \frac{h}{d\,|\delta|_{-\infty}} \max\{C_1\eta, C_2\beta\} \left(|\Delta R_u^i| + k\right) + C_\delta h k^2 \end{aligned}$$

and

$$\begin{aligned} |\Delta Q_v^{i+1,n}| &= |\Delta Q_v^{i,n} + \frac{h}{det(A(ih))}(\gamma(\mathbf{D}_t(J_{\delta_u^i} R_i^n) - \tilde{u}_t(ih, nk)) \\ &\quad + \alpha(\mathbf{D}_t(J_{\delta_v^i} R_v^{i,n}) - \tilde{v}_t(ih, nk))| \\ &\le |\Delta Q_u^{i,n}| + \frac{h}{d\,|\delta|_{-\infty}} \max\{C_1\gamma,\ C_2\alpha\} \left(|\Delta R_v^i| + k\right) + C_\delta h k^2. \end{aligned}$$

If

$$C_0 = \max\left\{1, \frac{1}{d\,|\delta|_{-\infty}} \max\left\{C_1\eta, C_2\beta\right\}, \frac{1}{d\,|\delta|_{-\infty}} \max\left\{C_1\gamma, C_2\alpha\right\}, C_\delta\right\},$$

then

$$\begin{aligned} \Delta_{i+1} &= \max\left\{\left|\Delta R_u^{i+1}\right|, \left|\Delta R_v^{i+1}\right|, \left|\Delta Q_u^{i+1}\right|, \left|\Delta Q_v^{i+1}\right|\right\} \\ &\le (1 + C_0 h)\left(\max\left\{\left|\Delta R_u^i\right|, \left|\Delta R_v^i\right|, \left|\Delta Q_u^i\right|, \left|\Delta Q_v^i\right|\right\}\right) + C_0 h k^2 \\ &= (1 + C_0 h)\,\Delta_i + C_0 h k^2. \end{aligned}$$

Calculating L iterations,

$$\begin{aligned} \Delta_L &\le (1 + C_0 h)^L (\Delta_0 + \epsilon + k) \\ &\le \exp(C_0)(\Delta_0 + \epsilon + k). \end{aligned}$$

Since $\Delta_0 \le (C/|\delta|_{-\infty})(\epsilon + h + k)$, for fixed δ, as ϵ, h, and k tend to 0 so does Δ_L. This establishes the formal convergence of the numerical method.

3. NUMERICAL EXAMPLE

In this section, the numerical results of an example of interest is presented. To obtain the required data functions $u(0,t)$ and $v(0,t)$ for the inverse problem, it is necessary to solve the direct problem. We set the following dimensionless values for the parameters in Luikov's model,

$$\begin{aligned} \mathrm{Lu} &= 0.8(1+x), \\ \mathrm{Pn} &= 0.32, \\ \mathrm{Ko} &= 65, \\ \mathrm{Eo} &= 0.02, \\ \mathrm{Bi}_q &= 1.7, \\ \mathrm{Bi}_m &= 3.0, \\ Q &= 2.5. \end{aligned}$$

Thus, the system of partial differential equations becomes

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \left((1+x) \begin{pmatrix} 0.7488 & -1.04 \\ -0.256 & 0.8 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} u_{xx} \\ v_{xx} \end{pmatrix}, \qquad 0 < x < 1, \quad 0 < t \leq 1,$$

with boundary conditions,

$$\begin{aligned} u_x(0,t) &= -2.5, \\ v_x(0,t) &= -0.8, \\ u_x(1,t) &= -1.7\, u(1,t) + 152.88(1+x)(v(1,t)-1) + 1.7V(t), \\ v_x(1,t) &= 3(1-v(1,t)) - 0.32\, u_x(1,t), \end{aligned}$$

and initial conditions

$$\begin{aligned} u(x,0) &= 2.5\, x\, (x\, g(0) - 1), \\ v(x,0) &= 1.5 + 0.8\, x\, (x-1). \end{aligned}$$

The functions

$$V(t) = \left(u(1,0) + \frac{v_x(1,0)(1-E_0)\,\mathrm{Ko}\,\mathrm{Lu}}{\mathrm{Bi}_q} \right) \left(-9 + 10e^{t^2} \right)$$

and $g(t) = 3.1 - t$, are chosen to satisfy the required compatibility conditions at $t = 0$ to avoid potential space located patches in the solution for positive times that will render the solution of the inverse problem impossible. See [10].

The numerical solution of the direct problem is computed by the method of lines and the discrete perturbed data functions for the inverse problem are generated by adding random errors to the "exact" computed solutions of the direct problem $g_1 = u(0,t)$ and $g_2 = v(0,t)$ as well as the exact flux functions $g_3 = u_x(0,t) = -2.5$ and $g_4 = v_x(0,t) = -0.8$.

For the inverse problem, the mollification parameter p is set to 3 and all the radii of mollification are chosen automatically using GCV without any prior knowledge about characteristics of the data.

The relative weighted l^2 error for u is calculated as

$$\frac{\left[1/(M+1) \sum_{i=0}^{M} \left| R_u^i - u(ih) \right|^2 \right]^{1/2}}{\left[1/(M+1) \sum_{i=0}^{M} \left| u(ih) \right|^2 \right]^{1/2}}.$$

The relative l^2 errors for u_x, v, and v_x are computed in a similar fashion.

EXAMPLE 3.1. Identify $u(x,t)$, $v(x,t)$, $u_x(x,t)$, and $v(x,t)$ satisfying

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \left((1+x) \begin{pmatrix} 0.7488 & -1.04 \\ -0.256 & 0.8 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} u_{xx} \\ v_{xx} \end{pmatrix}, \qquad 0 < x < 1, \quad 0 < t \leq 1,$$

with boundary conditions,

$$\begin{aligned} u(0,t) &= g_1^{\epsilon}(t), \\ v(0,t) &= g_2^{\epsilon}(t), \\ u_x(0,t) &= g_3^{\epsilon}(t), \\ v_x(0,t) &= g_4^{\epsilon}(t). \end{aligned}$$

Relative l^2 errors for u and v are reported in Table 1 as a function of ϵ and as a function of Δt in Table 2. Both these results and those shown in Figures 1 through 4 emphasize the stability and consistency of the marching scheme. For Table 1 and Figures 1 through 4, $N_x = 100$ and $N_t = 128$. In Table 2 and Figures 1 through 4, $\epsilon = 0.01$.

Table 1.

ϵ	$u(x,t)$	$v(x,t)$
0.001	0.00745	0.04766
0.005	0.00792	0.04435
0.01	0.00953	0.12241

Table 2.

Δt	$u(x,t)$	$v(x,t)$
$\frac{1}{64}$	0.00959	0.12241
$\frac{1}{128}$	0.00953	0.12241
$\frac{1}{256}$	0.00452	0.01227

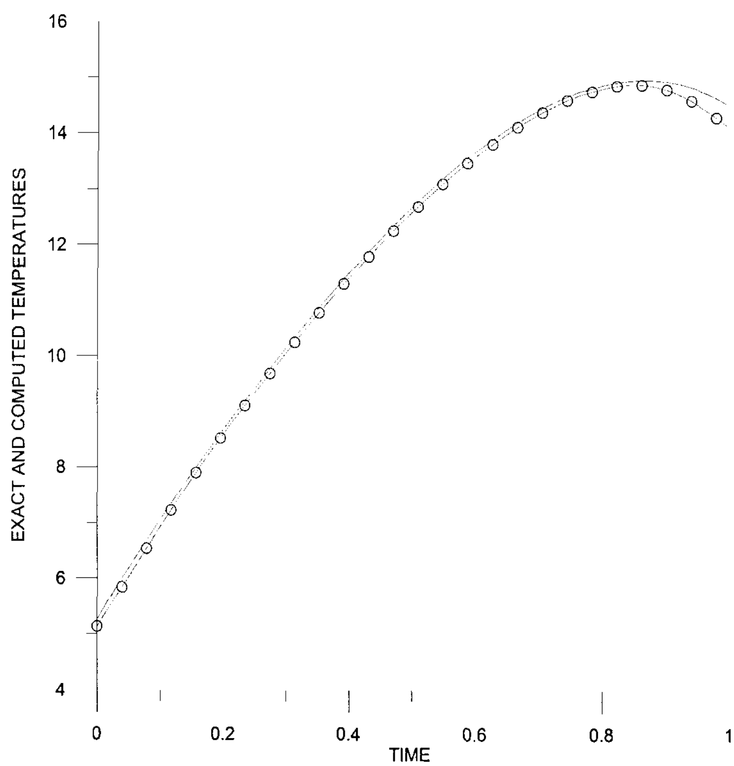

Figure 1. Exact and computed temperatures at $x = 1$.

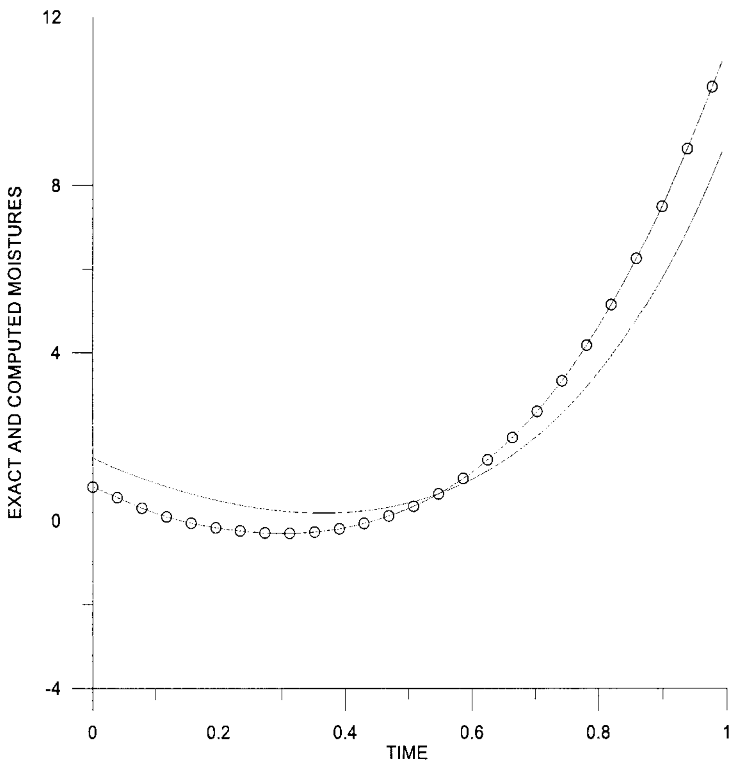

Figure 2. Exact and computed moistures at $x = 1$.

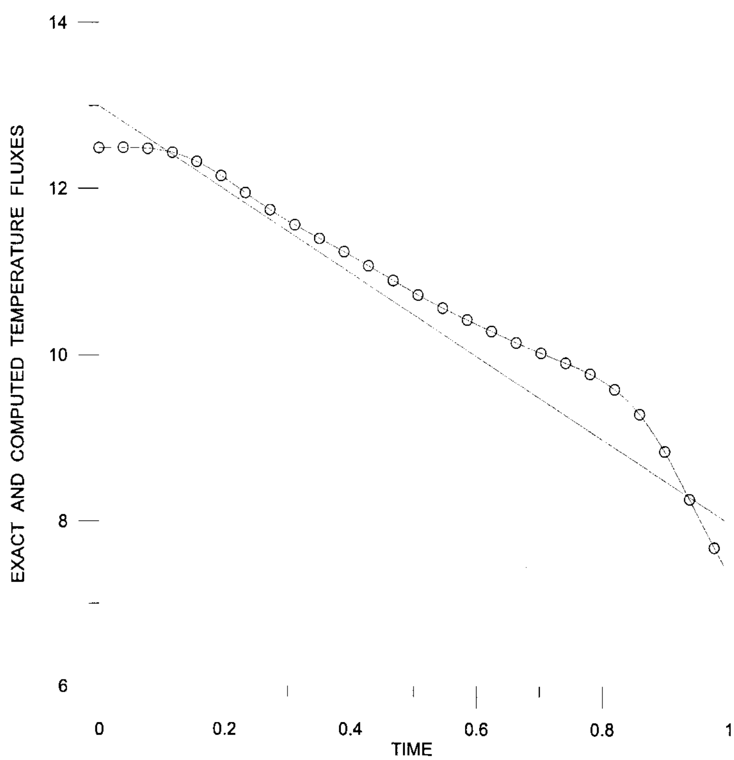

Figure 3. Exact and computed heat fluxes at $x = 1$.

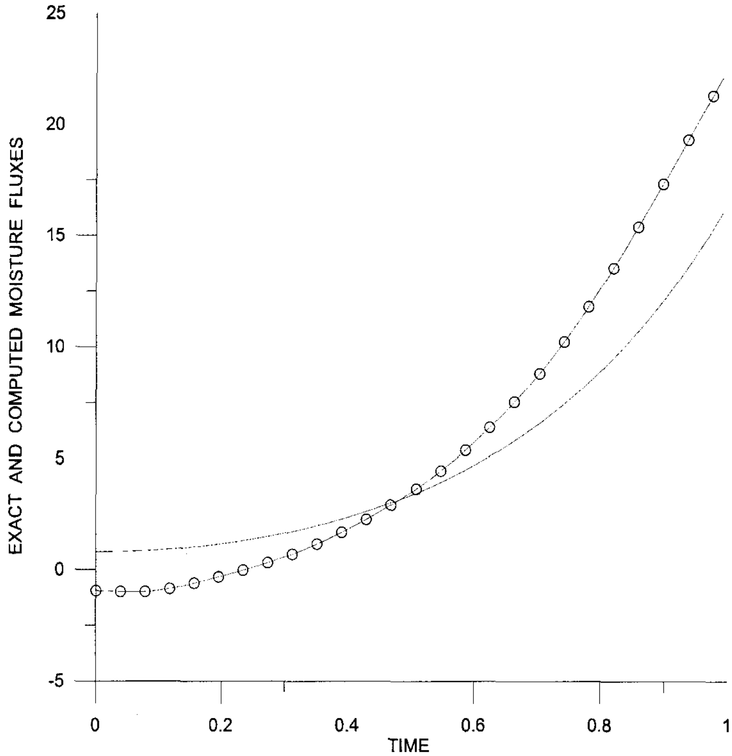

Figure 4. Exact and computed moisture fluxes at $x = 1$.

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