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Note

Distributive laws for commuting equivalence relations

Catherine Huafei Yan

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

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Abstract

In his paper (1942), Ore found necessary and sufficient conditions under which the modular and distributive laws hold in the lattice of equivalence relations on a set S. In the present paper, we consider commuting equivalence relations. It has been proved by Jónsson (1953) that the modular law holds in the lattice of commuting equivalence relations. We give some necessary and sufficient conditions for the distributive law and its dual to hold for commuting equivalence relations. © 1998 Elsevier Science B.V.

1. Introduction

In this note, we find simple necessary and sufficient conditions for the distributive law to hold for commuting equivalence relations.

2. Commuting equivalence relations

An equivalence relation R is a subset of $S \times S$ which is reflexive, symmetric, and transitive.

The set of equivalence relations on a set S is identified with the lattice of partitions $\Pi(S)$ of the set S. Let us denote R_π the equivalence relation associated to the partition π.

We sometimes write $(a,b) \in R_\pi$ as well as $aR_\pi b$.

Two equivalence relations R_π and R_σ are *independent* if for any two blocks $\pi_a \in \pi$ and $\sigma_b \in \sigma$, we have $\pi_a \cap \sigma_b \neq \emptyset$.

Two equivalence relations R_π and R_σ are said to *commute* if $R_\pi \circ R_\sigma = R_\sigma \circ R_\pi$, where $\circ$ represents the composition of relations, i.e., $R \circ T = \{(a,b) \in S \times S$. there exists $c \in S$, such that $(a,c) \in R$, and $(c,b) \in T\}$.

The following facts characterize the commutativity of equivalence relations [2]:

(1) Two equivalence relations R and T commute if and only if $R \circ T$ is an equivalence relation.

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(2) Two equivalence relations R_π and R_σ commute if and only if for every block C of the partition $\pi \vee \sigma$, the restrictions $\pi|_C$, $\sigma|_C$ are independent partitions.

3. The distributive law for commuting equivalence relations

A *linear lattice* is a sublattice of $\Pi(S)$, with the property that the equivalence relations associated with any two partitions in the lattice commute. We denote a linear lattice by $\mathscr{L}$.

It is known that every linear lattice is a modular lattice [3,5,7].

In general, a linear lattice is not distributive. We shall study when the distributive law holds for commuting equivalence relations R_π, R_σ, and R_τ:

$$R_\pi \wedge (R_\sigma \vee R_\tau) = (R_\pi \wedge R_\sigma) \vee (R_\pi \wedge R_\tau). \tag{1}$$

Definition 1. Two partitions π and σ are said to be *associable* if every block of $\pi \vee \sigma$ is a block of either π or σ.

We have trivially:

Proposition 2. *If two partitions π and σ are associable, then R_π and R_σ commute.*

Our first result is:

Theorem 3. *For two commuting equivalence relations R_σ and R_τ, the necessary and sufficient condition that the distributive law* (1) *holds for all R_π which commute with both R_σ and R_τ is that R_σ and R_τ be associable.*

Proof. Assume that R_σ and R_τ are associable. We show that

$$R_\pi \wedge (R_\sigma \vee R_\tau) \leqslant (R_\pi \wedge R_\sigma) \vee (R_\pi \wedge R_\tau).$$

Now $aR_\pi \wedge (R_\sigma \vee R_\tau)b$ implies $aR_\pi b$ and $aR_\sigma \vee R_\tau b$, and hence either $aR_\sigma b$ or $aR_\tau b$. That is, either $aR_\pi \wedge R_\sigma b$ or $aR_\pi \wedge R_\tau b$ holds. This proves that the distributive law holds.

Conversely, if R_σ and R_τ are not associable, then there exists a block σ_1 of σ which overlaps at least two blocks τ_1 and τ_2 in τ, and τ_1 is not entirely contained in σ_1.

Define a partition μ by letting its blocks be $\mu_1, \mu_2, \ldots$, where

$$\mu_1 = (\sigma_1 \cap \tau_2) \cup (\tau_1 - \tau_1 \cap \sigma_1),$$
$$\mu_2 = (\sigma_1 \cap \tau_1) \cup (\tau_2 - \tau_2 \cap \sigma_1),$$
$$\mu_i = \tau_i \quad \text{for } i \in I \text{ where } I \text{ is some index set.}$$

Then R_μ commutes with R_σ and R_τ. But $\sigma_1 \cap \tau_1$ and $\tau_2 - \sigma_1 \cap \tau_2$ are distinct blocks in $(R_\mu \wedge R_\sigma) \vee (R_\mu \wedge R_\tau)$ while they belong to one block in $R_\mu \wedge (R_\sigma \vee R_\tau)$. This contradicts the distributive law. $\square$

Next we fix R_π and R_τ instead of R_σ and R_τ; then we have:

Theorem 4. *For two commuting equivalence relations R_π and R_τ, the necessary and sufficient condition that the distributive law (1) holds for all R_σ which commute with both R_π and R_τ is that R_π and R_τ be associable.*

Proof. Assume that R_π and R_τ are associable. We show that

$$R_\pi \wedge (R_\sigma \vee R_\tau) \leqslant (R_\pi \wedge R_\sigma) \vee (R_\pi \wedge R_\tau).$$

Now $aR_\pi \wedge (R_\sigma \vee R_\tau)b$ means that $aR_\pi b$ and there exists a c such that $aR_\sigma c$, $cR_\tau b$. But

$$cR_\tau b, bR_\pi a \Rightarrow cR_\tau \vee R_\pi a \Rightarrow \text{either } cR_\pi a, \text{ or } cR_\tau a.$$

There are two cases:

(1) $cR_\pi a$. Then

$cR_\pi a$ and $aR_\sigma c$ imply $aR_\pi \wedge R_\sigma c$,

$cR_\pi a$ and $aR_\pi b$ imply $cR_\pi b$,

$cR_\pi b$ and $cR_\tau b$ imply $cR_\pi \wedge R_\tau b$.

Combining the right hand side of the first and the third row, we have $a(R_\pi \wedge R_\sigma) \vee (R_\pi \wedge R_\tau)b$.

(2) $cR_\tau a$. Then

$cR_\tau a$ and $cR_\tau b$ imply $aR_\tau b$,

$aR_\tau b$ and $aR_\pi b$ imply $aR_\pi \wedge R_\tau b$.

Since $R_\pi \wedge R_\tau \leqslant (R_\pi \wedge R_\sigma) \vee (R_\pi \wedge R_\tau)$, so once again we have $a(R_\pi \wedge R_\sigma) \vee (R_\pi \wedge R_\tau)b$.

Conversely, if R_π and R_τ are not associable, there exists a block π_1 of π which overlaps at least two blocks τ_1 and τ_2 in τ, and τ_1 is not entirely contained in π_1. As in the proof of the previous theorem, define a partition μ by letting its blocks be $\mu_1, \mu_2, \ldots$, where

$$\mu_1 = (\pi_1 \cap \tau_2) \cup (\tau_1 - \tau_1 \cap \pi_1),$$

$$\mu_2 = (\pi_1 \cap \tau_1) \cup (\tau_2 - \tau_2 \cap \pi_1),$$

$$\mu_i = \tau_i \quad \text{for } i \in I \text{ where } I \text{ is some index set.}$$

Such a μ commutes with both π and τ. Note that $\pi_1 \cap \tau_1$ and $\pi_1 \cap \tau_2$ are two distinct blocks in both $R_\pi \wedge R_\tau$ and $R_\pi \wedge R_\mu$, but they are one block in $R_\pi \wedge (R_\mu \vee R_\tau)$. This means that $R_\pi \wedge (R_\tau \vee R_\mu) \neq (R_\pi \wedge R_\tau) \vee (R_\pi R_\mu)$. $\square$

Remark. Instead of (1) one could consider the *dual distributive law*

$$R_\pi \vee (R_\sigma \wedge R_\tau) = (R_\pi \vee R_\sigma) \wedge (R_\pi \vee R_\tau). \tag{2}$$

As in the proof of the preceding theorems, one may easily prove:

(1) For two commuting equivalence relations R_σ and R_τ, the necessary and sufficient condition that the dual distributive law (2) holds for all R_π which commute with both R_σ and R_τ is that R_σ and R_τ be associable.

(2) For two commuting equivalence relations R_π and R_τ, the necessary and sufficient condition that the dual distributive law (2) holds for all R_σ which commute with both R_π and R_τ is that R_π and R_τ be associable.

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