Real quadratic flexible division algebras

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Abstract

In this paper we give a new process called vectorial isotopy, in order to classify the eight-dimensional real quadratic flexible division algebras, and we solve the isomorphism problem. © 1999 Published by Elsevier Science Inc. All rights reserved.

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1. Introduction

An algebra $A$ over a field $K$ is called a division algebra if, for all nonzero $x$ in $A$, the operators $L_x : y \mapsto xy, R_x : y \mapsto yx$, are invertible in the associative and unital algebra $\text{End}_K(A)$. It is well known [5] that an associative finite-dimensional real division algebra is either $\mathbb{R}, \mathbb{C}$ or the Hamilton quaternions real division algebra $\mathbb{H}$. Bruck and Kleinfeld [4] proved that an alternative finite-dimensional real

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division algebra is either associative, or the Cayley–Dickson octonions real division algebra \( \mathbb{O} \). One of the fundamental results about finite-dimensional real division algebras, proved by Kervaire [8] and Milnor–Bott [3], states that if the real space \( \mathbb{R}^n \) possess a bilinear product without zero divisors, then \( n = 1, 2, 4 \) or \( 8 \). However, the determination of all finite-dimensional real division algebras remains an open problem even in dimension 4 (see [1] for two-dimensional case). Our work has its origin in [9], where Osborn established the basis of quadratic algebra theory, and determined all quadratic four-dimensional real division algebras and a particular class of quadratic (non-alternative) eight-dimensional real division algebras. Next, with the use of Kervaire–Milnor–Bott’s result, Benkart, Britten and Osborn [2], reduced the determination of flexible finite-dimensional real division algebras to noncommutative Jordan ones. In [6], it was proved that a noncommutative Jordan finite-dimensional real division algebra \( A \) is quadratic, and if it satisfies the identity \( (x, x, [x, y]) = 0 \), it is alternative. In particular, if \( A \) is a Jordan algebra, then \( A \) is either \( \mathbb{R} \) or \( \mathbb{C} \). If a noncommutative Jordan finite-dimensional real division algebra \( A \) satisfies the Osborn property (that is two elements not in the same subalgebra of dimension two generate a four-dimensional subalgebra), then \( A \) is either \( \mathbb{R}, \mathbb{C}, \mathbb{H}^{(\lambda)} \) or \( \mathbb{O}^{(\lambda)} \), \( \lambda \in \mathbb{R} - \{ \frac{1}{2} \} \) [10]. In [7] we gave a process which generalizes the Cayley–Dickson one, and allows us to determine a new family of quadratic flexible eight-dimensional real division algebras, which contains strictly the algebras obtained from \( \mathbb{R} \) by mutations and Cayley–Dickson processes: \( (E_{-1, \mu}(\mathbb{H}^{(\lambda)}))^{(\mu)}; \lambda, \mu \in \mathbb{R} - \{ \frac{1}{2} \} \). But this does not suffice to determine all of them [11]. Here we prove firstly, the existence of a four-dimensional subalgebra for such an algebra, we give secondly a new construction method of real quadratic flexible division algebras of dimension \( n \geq 2 \) from a given one called vectorial isotopy, and with the use of the first result we establish the main result. All quadratic flexible eight-dimensional real division algebras are obtained from \( \mathbb{O} \) by this process. We give also a necessary and sufficient condition for the isomorphism of two such algebras.

2. Notations and requirements

**Definition 2.1.** Let \( A \) be an algebra over a field \( K \) of characteristic zero and \( \alpha \in K \). We denote by \( A^{(\alpha)} \) the \( \alpha \)-mutation of \( A \), i.e. the space \( A \) with product: \( x, y = \alpha xy + (1 - \alpha)yx; \ x, y \in A \). If \( x, y, z \) are elements of \( A \), \( [x, y] \) is the commutator \( xy - yx \) of \( x \) and \( y \), and \( (x, y, z) \) the associator \( (xy)z - x(yz) \) of \( x, y \) and \( z \). \( A \) is said to be flexible if it satisfies the identity \( (x, y, x) = 0 \). The algebra \( A \) is called noncommutative Jordan if it is flexible and satisfies the identity \( (x^2, y, x) = 0 \). The algebra \( A \) is said to be quadratic if it contains an identity element 1 and if 1, \( x, x^2 \) are linearly dependent for all \( x \) in \( A \). A quadratic and flexible algebra is a noncommutative Jordan algebra. A quadratic algebra \( A \) is said to be a Cayley algebra if there exists a multiplicative involution \( - \) over \( A \),
such that $x + \bar{x}$ and $x\bar{x}$ are both in $K$. A symmetric bilinear form $(,)$ over $A$ is called a trace form if $(ab, c) = (a, bc)$ for all $a, b, c$ in $A$. We denote by $(x_1, \ldots, x_n)$ the subspace of $A$ spanned by $x_1, \ldots, x_n \in A$.

It is well known that a quadratic $K$-algebra $A$ is obtained from an anti-commutative one, $(V, \wedge)$ and a bilinear form $(,)$ over $V : A = K \oplus V$ with its natural $K$-space structure and product $(x + x)(y + y) = (x\beta + (x, y)) + (\alpha y + \beta x + x \wedge y)$. We denote also by $(,)$, the bilinear form: $A \times A \to K (x + x, y + y) \to x\beta + (x, y)$, and call it the associated bilinear form of $A$. $(V, \wedge)$ is called the associated anti-commutative algebra of $A$, its elements are called vectors and those of $K$ scalars. We denote $A$ by $(V, (,), A)$.

The following preliminary results are useful.

**Lemma 2.2** [9]. Let $B = (V, (,), \wedge)$ be a quadratic $K$-algebra. Then
1. $B$ is a Cayley algebra if and only if $(,)$ is symmetric,
2. $B$ is flexible if and only if $(,)$ is symmetric and the following equivalent statements hold,
   (i) $(,)$ is a trace form over $B$;
   (ii) $(,)$ is a trace from over $(V, \wedge)$;
   (iii) $(x \wedge y, x) = 0$ for every $x, y \in V$.

**Lemma 2.3** [9]. Let $A = (V, (,), \wedge)$ be a quadratic $K$-algebra with the property that the subalgebra generated by any nonzero element is a field. Then the following statements are equivalent
1. $A$ has no zero divisors;
2. $A$ contains no three-dimensional subalgebras;
3. for any two linearly independent vectors $x, y$ of $A$, the vectors $x, y, x \wedge y$ are linearly independent;
4. there do not exist two linearly independent vectors $x, y$ of $A$ such that $x \wedge y = x$ or $x \wedge y = 0$.

**Remark 2.4.** In Lemma 2.3, the fact that the subalgebra generated by any nonzero element is a field does not imply the absence of zero divisors in $A$. Indeed, the quadratic algebra $H^{(1/2)}$ has zero divisors, and the subalgebra generated by any nonzero element is either $\mathbb{R}$ or $\mathbb{C}$.

### 3. Vectorial isotopy

**Definition 3.1** (Generalization of the Cayley–Dickson process). Let $(B, -)$ be a Cayley algebra over $K$, and $\gamma, x, \delta \in K$. Then the $K$-space $B \times B$ with the product: $(x, y)(x', y') = (x, x' + \gamma y'y, yx' + y'x + \frac{1}{2}\delta[y', y])$ is a Cayley algebra over $K$ [7]. We denote it by $E_{\gamma, x, \delta}(B)$ and call it the generalized Cayley–Dickson extension of $(B, -)$ of index $(\gamma, x, \delta)$. We denote by $E_{\gamma, 1, \delta}(B)$ the algebra $E_{\gamma, 1, \delta}(B)$.
Remark 3.2. In [7] we showed that the real algebra \( E_{-1,\delta}(\mathbb{H}) \) is a quadratic flexible algebra and that is a division algebra if and only if \( |\delta| < 2 \). In [11], we gave an example of a real quadratic flexible division algebra of dimension 8 not obtained from \( \mathbb{R} \) by mutation and generalized Cayley extension processes.

We introduce now the Vectorial isotopy process. Let \((V, \wedge)\) be a (nontrivial) finite-dimensional real anti-commutative algebra, \((/\) a negative definite symmetric bilinear form over \(V\) and \(\varphi\) an automorphism of \(V\). We put \(x\Delta y = \varphi^*(\varphi(x) \wedge \varphi(y));\ x, y \in V\), where \(\varphi^*\) is the adjoint automorphism of \(\varphi\), and denote by \((V, (/), \wedge), (V, (/), \Delta)\) respectively, the real Cayley algebras constructed from the anti-commutative algebras \((V, \wedge), (V, \Delta)\) and the symmetric bilinear form \((/\).

Proposition 3.3. \((V, (/), \Delta)\) is a flexible division algebra if and only if \((V, (/), \wedge)\) is a flexible division algebra.

Proof. If \((V, (/), \wedge)\) is flexible, then \((/\) is a trace form over \((V, \wedge)\) and for every \(x, y \in W\) we have: \(0 = (\wedge \varphi(y) / \varphi(x)) = (\varphi^*(\varphi(x) \wedge \varphi(y)) / x = (x\Delta y / x)\) i.e. \((V, (/), \Delta)\) is flexible. If, moreover, \((V, (/), \wedge)\) is a division algebra, then the property (iv) of Lemma 2.3 is satisfied by the algebra \((V, (/), \wedge)\) as well as by the algebra \((V, (/), \Delta)\) taking into account the trace property of \((/\). Then \((V, (/), \Delta)\) is a division algebra. The converse is similar taking into account that: \(x \wedge y = (\varphi^*)^{-1}(\varphi^{-1}(x)\Delta\varphi^{-1}(y))\) for all \(x, y \in V\). □

Remark 3.4. 1. We shall say that the algebra \((V, (/), \Delta)\) is obtained from \(A = (V, (/), \wedge)\) and the automorphism \(\varphi\) by vectorial isotopy and denote it by \(A\). This notion of vectorial isotopy is an equivalence relation in the class \(\mathcal{C}_n\); \(n \geq 2\) of \(n\)-dimensional real Cayley algebras, whose real space \(V = \mathbb{R}^{n-1}\) of associated vectors is provided with the same negative definite symmetric bilinear form \((/\). Moreover, for an algebra \(A\) in \(\mathcal{C}_n\) and \(\varphi, \psi\) automorphisms of the associated space of vectors we have \((A(\varphi))(\psi) = A(\varphi\psi)\).

2. The real quadratic flexible division algebras of dimension 2 and 4 are obtained respectively from the real algebras \(\mathbb{C}, \mathbb{H}\) by vectorial isotopy. Indeed, the four-dimensional real quadratic flexible division algebras are the mutations \(\mathbb{H}^{(\lambda)}\) of \(\mathbb{H}\) where \(\lambda \in \mathbb{R}\) with \(\lambda \neq \frac{1}{2}\) and we have \(\mathbb{H}^{(\lambda)} = \mathbb{H}(\varphi)\) where \(\varphi\) is the homothety \((2\lambda - 1)V\) of the real space \(V\) of vectors of \(\mathbb{H}\). Moreover \(\mathbb{H}^{(\lambda)} \cong \mathbb{H}^{(\mu)}\) if and only if \(\lambda = \mu\) or \(\lambda = 1 - \mu\) [10].

Note 3.5. If \(A = (V, (/), \wedge)\) is a quadratic algebra and \(f\) an automorphism of the space \(V\), then the mapping \(\tilde{f} : A \rightarrow A\) given by \(\alpha + u \mapsto \alpha + f(u),\ (\alpha \in K, u \in V)\) is an automorphism of the space \(A\), which we call the natural extension of \(f\) to \(A\).

We obtain the following results.
Proposition 3.6. Let $A$ and $B$ be two real Cayley algebras in $\mathcal{L}_n$ and let $f$ be an automorphism of the real space $V$. Then

1. $A$ and $B$ are isomorphic if and only if they are isometrically vectorially isotopic;
2. $f \in \text{Aut}(A)$ if and only if $A(f) = A$ and $f$ is an isometry of the Euclidean space $(V, -(\cdot))$.

Proof. 1. If $g: A = (V, (\cdot), A) \rightarrow B = (V, (\cdot), A)$ is an algebra isomorphism then for all $x, y \in A$, we have $(x/y) + g(xA) = g((x/y) + xA) = g(x/y) = g(x)/g(y) = (g(x)/g(y)) + g(x) \wedge g(y)$. It follows that $(g(x)/g(y)) = (x/y)$ and $g(xA) = g(x) \wedge g(y)$, i.e. $g$ is an isometry of the space $A$, which is the natural extension of an isometry $g_0$ of the Euclidean space $(V, -(\cdot))$. Moreover, $xA = g_0^{-1}(g_0(x) \wedge g_0(y)) = g_0^*(g_0(x) \wedge g_0(y))$ for every $x, y \in V$, i.e. $A = B(g_0)$. Conversely if $A = B(h)$, where $h$ is an isometry of the Euclidean space $(V, -(\cdot))$, then the natural extension $h$ of $h$ to $\mathbb{R} \oplus V$ is an isomorphism of the algebra $A$ into the algebra $B$.

2. Consequence of 1. □

Proposition 3.7. $f: E_{-1,-\delta}(\mathbb{H}) \rightarrow E_{-1,-\delta}(\mathbb{H}) \quad (x,y) \mapsto (x,-y); \quad \delta \in \mathbb{R}$, is an algebra isomorphism, i.e. $E_{-1,-\delta}(\mathbb{R})$ and $E_{-1,\delta}(\mathbb{R})$ are vectorially isotopic.

Proof. $f((x,y)(x',y')) = f((xx' - \delta y'y' + y'x + \frac{1}{2}\delta[y',y])) = (xx' - (y')(-y); (-y)x + \frac{1}{2}\delta[(-y'),(-y)]) = (x,-y)(x',-y') = f(x,y)f(x',y'); x,y,x',y' \in \mathbb{H}$. □

Proposition 3.8. Let $\mathcal{O} = (V, (\cdot), \wedge)$ be the Cayley–Dickson real algebra and $\varphi$ an automorphism of $V$. We denote $\mathcal{O}(\varphi)$ by $(V, (\cdot), \wedge)$. Then $\mathcal{O}(\varphi) = \mathcal{O}$ if and only if $\varphi$ is an automorphism of $\mathcal{O}$.

Proof. The sufficiency of the condition follows from Proposition 3.6.2. We suppose, then, that $\mathcal{O}(\varphi) = \mathcal{O}$. We have $x \wedge y = \varphi^*(\varphi(x) \times \varphi(y)) = x \times y$. If $\{x_1, \ldots, x_3\}$ is an orthonormal basis of $V$ formed by eigenvectors of $\varphi \varphi^*$ whose associated eigenvalues are respectively $\lambda_1, \ldots, \lambda_7$, then

1. $\lambda_i > 0$,
2. $(\varphi^*(x_i)/\varphi^*(x_j)) = -\lambda_i \delta_{ij}$; $\delta_{ij}$ being the Kronecker symbol,
3. $\varphi^*(x_i) \times \varphi^*(x_j) = \lambda_i \lambda_j \varphi^*(x_i \times x_j)$.

We suppose $i \neq j$, then

$$
\varphi^*(\lambda_i x_i \times \varphi^*(x_i \times x_j)) = \varphi^*(x_i) \wedge \varphi^*(x_i \times x_j) = \varphi^*(x_i) \times \varphi^*(x_i \times x_j) = (\lambda_i \lambda_j)^{-1}(\varphi^*(x_j))^2 \varphi^*(x_i) (by \ 3.) = -\lambda_j^{-1} \varphi^*(x_j) (by \ 2.), \ so \ that
\lambda_i x_i \times \varphi^*(x_i \times x_j) = -\lambda_j^{-1} x_j, \ i.e. \ \varphi^*(x_i \times x_j) = (\lambda_i \lambda_j)^{-1} x_i \times x_j.
$$
Consequently \( x_i, x_i \times x_1, \ldots, x_i \times x_{i-1}, x_i \times x_{i+1}, \ldots, x_i \times x_7 \) is an orthonormal basis of \( V \) formed by eigenvectors of \( \varphi \varphi^* \) whose associated eigenvalues are respectively

\[
\lambda_i, (\lambda_i^2)^{-1}, \ldots, (\lambda_i^2)^{-1}, (\lambda_{i+1})^{-1}, \ldots, (\lambda_{7})^{-1}.
\]

If \( k \neq i, j \) we have

\[
\varphi \varphi^* ((x_i \times x_k) \times (x_i \times x_j)) = ((\lambda_i^2)^{-1} (\lambda_{k}^2)^{-1})^{-1} (x_i \times x_k) \times (x_i \times x_j).
\]

If the eigenvectors \( x_i \) and \( x_j \times x_k \) are not orthogonal to each other then they have the same associated eigenvalue and we have \( \lambda_i = (\lambda_j^2)^{-1} \). If \( x_i \) and \( x_j \times x_k \) are orthogonal, we have

\[
((x_i \times x_k) \times (x_i \times x_j))/x_k \times x_j = -(x_i x_k)(x_i x_j)/x_i x_j
\]

since \( x_i \times x_k \) and \( x_i \times x_j \) are orthogonal. By the Moufang identity, we obtain

\[
((x_i \times x_k) \times (x_i \times x_j))/x_k \times x_j = -(x_i x_k)(x_i x_j)/x_i x_j = -(x_i x_k)(x_i x_j)/x_k x_j
\]

and \( x_k \times x_j \) are orthogonal. Then the eigenvectors \( x_k \times x_j \) are orthogonal and \( (x_i \times x_k) \times (x_i \times x_j) \) are not orthogonal (i.e. they have the same associated eigenvalue) and we have \( (\lambda_i \lambda_j \lambda_k)^2 = 1 \) i.e. \( \varphi \varphi^* = \text{Id}_V \) (by 1.), so that \( \bar{\varphi} \) is an automorphism.

In what follows we shall denote \( \mathcal{G}_2 \) for the automorphism group of \( O \).

**Corollary 3.9.** Let \( O = (V, (\cdot), \times) \) be the Cayley–Dickson real algebra and \( \varphi, \psi \) automorphisms of the space \( V \). Then

1. \( O(\varphi) \simeq O \) if and only if \( \varphi \in \mathcal{C}_7(\mathbb{R}) \) (the orthogonal group of isometries of the Euclidean space \( \mathbb{R}^7 \)),
2. \( O(\varphi) = O(\psi) \) if and only if \( \varphi \psi^{-1} \in \mathcal{G}_2 \),
3. \( O(\varphi) \simeq O(\psi) \) if and only if there exists \( g \in \mathcal{C}_7(\mathbb{R}) \) such that \( \varphi g \psi^{-1} \in \mathcal{G}_2 \).

**Proof.** 1. \( \bar{\varphi} : O(\varphi) \to O \) is an algebra isomorphism. Conversely, there exists \( f \in \mathcal{C}_7(\mathbb{R}) \) such that \( O = (O(\varphi))(f) = O(\varphi f) \) i.e. \( \varphi f \in \mathcal{C}_7(\mathbb{R}) \). Then \( \varphi = (\varphi f)f^{-1} \in \mathcal{C}_7(\mathbb{R}) \).

2. \( \varphi \psi^{-1} \in \mathcal{G}_2 \iff O(\varphi \psi^{-1}) = O \) (by Proposition 3.8) \( \iff O(\varphi) = O(\psi) \).

3. \( O(\varphi) \simeq O(\psi) \iff \exists f \in \mathcal{C}_7(\mathbb{R})/O(\varphi) = (O(\psi))(f) \) (by Proposition 3.6.1.)

The following result has great importance in this paper.

**Proposition 3.10.** The real algebras \( E_{1,\delta}(\mathbb{H}) \) where \( |\delta| < 2 \) are obtained from the Cayley–Dickson real algebras \( O = (V, (\cdot), \times) \), by vectorial isotopy.
Proof. By Proposition 3.7, we can suppose \( \delta \geq 0 \). Let \( \alpha \in [\frac{1}{2}, 1] \); then the endomorphism \( \varphi \) of \( V \) whose matrix with respect to the canonical basis \( \mathcal{B} = \{e_1, \ldots, e_7\} \) given by

\[
\begin{pmatrix}
(2\alpha - 1)^{-1} & 0 & (1 - \alpha^2)^{1/2} \\
(2\alpha - 1)^{-1} & 0 & (1 - \alpha^2)^{1/2} \\
0 & 0 & 1 \\
0 & \alpha' & 0 \\
\alpha' & 0 & \alpha \\
\alpha' & 0 & \alpha \\
0 & 0 & 1
\end{pmatrix}
\]

where \( \alpha' = (1 - \alpha^2)^{1/2}(\alpha + 1)^{-1}(2\alpha - 1)^{-1} \), is an automorphism. We have the multiplication table of \( A = O(\varphi) \):

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<td>\beta e1</td>
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<td>-e3 - \lambda e7</td>
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where \( \beta = 2(\alpha + 1)^{-1}(2\alpha - 1)^{-2} > 0 \) and \( \lambda = (4\alpha^2 - 1)(1 - \alpha^2)^{1/2} \geq 0 \). We consider next, the automorphism \( \psi \) of \( V \) such that \( M(\psi, \mathcal{B}) = \text{diag}\{\beta^{-1/3}, \beta^{-1/3}, \beta^{-1/3}, \beta^{1/6}, \beta^{1/6}, \beta^{1/6}, \beta^{1/6}\} \). The multiplication table of the algebra \( A(\psi) = O(\varphi \psi) \) with respect to \( B \) is the same as that of the algebra \( E_{-1, \delta}(H) \) where \( \delta = \lambda \beta^{1/2} \geq 0 \). Moreover \( \delta^2 = 2(1 - \alpha)(2\alpha + 1)^2 \) is a continuous function of \( \alpha \), and strictly decreasing over \([\frac{1}{2}, 1]\); by the intermediate value theorem, it takes all values of the interval \([\delta^2_1, \lim_{\alpha \to 1/2+} \delta^2_2] = [0, 4]\). \( \square \)

4. Existence of a four-dimensional subalgebra

Let now \( A = (W, (\cdot, \cdot), \wedge) \) be a real quadratic, flexible division algebra, of dimension 8, so that the mapping \( (\cdot, \cdot) : W \times W \to \mathbb{R}, (x, y) \mapsto (x, y) = -(x/y) \) is a
positive definite trace form over \( W \), which provides \( W \) with an Euclidian space structure \((W, -\langle \cdot, \cdot \rangle)\). If \( u \in W \) with \( u \neq 0 \) we denote by \( W(u) \) the orthogonal subspace of \( u \) in \( W \) and by \( \| u \| \) the norm \( \langle u, u \rangle^{1/2} \) of \( u \). We have the following preliminary results.

**Lemma 4.1.** The linear transformation \( L_u^* : W \rightarrow W \) \( v \mapsto u \wedge v \) is anti-symmetric with respect to \( \langle \cdot, \cdot \rangle \) and induces a bijection \( f_u : W(u) \rightarrow W(u) \). Moreover, the symmetric linear transformation \( f_u^2 \) is negative definite with respect to \( \langle \cdot, \cdot \rangle \).

**Proof.** The fact that \( L_u^* \) is anti-symmetric is a consequence of the trace property of \( \langle \cdot, \cdot \rangle \). For the same reason and by 4 of Lemma 2.3, \( L_u^* \) induces a bijection \( f_u \) of \( W(u) \) to itself. If \( y_1 \in W(u) \) with norm one is an eigenvector of \( f_u^2 \), whose associated eigenvalue \( \lambda_1 \), we have: \( \lambda_1 = \lambda_1 \| y_1 \|^2 = -\langle \lambda_1 y_1, y_1 \rangle = -(u \wedge (u \wedge y_1))/y_1 = (u \wedge y_1/u \wedge y_1) = -\| u \wedge y_1 \|^2 < 0 \) \((u \wedge y_1) \) is also an eigenvector of \( f_u^2 \) with the same associated eigenvalue). \[\]

**Lemma 4.2.** \( f_u^2 \) has at least three distinct eigenvalues.

**Proof.** If \( y_2 \) is an eigenvector of \( f_u^2 \) with norm one, orthogonal to \( y_1 \) and \( u \wedge y_1 \), whose associated eigenvalue is \( \lambda_2 \), then the eigenvector \( u \wedge y_2 \) is orthogonal to \( y_1 \) and \( u \wedge y_1 \). We construct then an orthogonal basis \( y_1, u \wedge y_1; y_2, u \wedge y_2; y_3, u \wedge y_3 \), of \( W(u) \) formed by eigenvectors of \( f_u^2 \) whose associated eigenvalues are respectively \( \lambda_1, \lambda_1; \lambda_2, \lambda_2; \lambda_3, \lambda_3 \). \[\]

**Note 4.3.** The orthonormal eigenvectors \( y_1, y_2, y_3 \) can be extended to an orthonormal basis of \( W(u) \) completed by the orthonormal eigenvectors

\[
z_1 = (-\lambda_1)^{-1/2} u \wedge y_1, \quad z_2 = (-\lambda_2)^{-1/2} u \wedge y_2, \quad z_3 = (-\lambda_3)^{-1/2} u \wedge y_3.
\]

**Remark 4.4.** If \( \lambda \) is an eigenvalue of \( f_u^2 \) then the corresponding eigenspace \( E_\lambda \) is of dimension 2, 4 or 6 and splits as an orthogonal direct sum of two-dimensional subspaces stable under \( f_u \).

**Lemma 4.5.** There exist two orthonormal vectors \( u_0, v_0 \in W \) such that \( v_0 \) is an eigenvector of \( f_{u_0}^2 \) and \( u_0 \) is an eigenvector of \( f_{v_0}^2 \) with the same associated eigenvalue.

**Proof.** Let \( S \) be the unital sphere of \( W \) and \( H : S \times S \rightarrow \mathbb{R}^+ \) the mapping defined by \( H((x, y)) = \| x \wedge y \| \). \( H \) is continuous over the compact \( S \times S \) then, there exist \( u_0, v_0 \in S \) such that \( \| u_0 \wedge v_0 \| = \text{sup} H(S \times S) \). According to Note 4.3, there is an orthonormal basis \( y_1, z_1, y_2, z_2, y_3, z_3 \) of \( W(u_0) \) for which the matrix of \( f_{u_0} \) is...
\[
\begin{pmatrix}
0 & -(-\lambda_1)^{1/2} \\
(-\lambda_1)^{1/2} & 0 \\
0 & -(-\lambda_2)^{1/2} \\
(-\lambda_2)^{1/2} & 0 \\
0 & -(-\lambda_3)^{1/2} \\
(-\lambda_3)^{1/2} & 0
\end{pmatrix},
\]

where \(-\lambda_i < 0\). If \(v = \sum_{i=1}^{3} (a_i y_i + b_i z_i)\) is an arbitrary element of \(S\), we have

\[
f_{w_0}(v) = \sum_{1 \leq i \leq 3} (a_i(-\lambda_i)^{1/2} z_i - b_i(-\lambda_i)^{1/2} y_i)
\]

and

\[
\|f_{w_0}(v)\|^2 = -\sum_{1 \leq i \leq 3} \lambda_i (a_i^2 + b_i^2) \leq \sup\{-\lambda_i / 1 \leq i \leq 3\} \sum_{1 \leq i \leq 3} (a_i^2 + b_i^2)
\]

\[
= \sup\{-\lambda_i / 1 \leq i \leq 3\}.
\]

This shows that the maximum of \(H\) is reached on the vector of \(E_\lambda\) of norm one, where \(-\lambda = \sup\{-\lambda_i / 1 \leq i \leq 3\}\). Indeed, let \(E\) be a two-dimensional subspace of \(E_\lambda\), stable under \(f_{w_0}\) and \(v_0 = a_{w_0} y_{w_0} + b_{w_0} z_{w_0}\) a vector of \(E\) of norm one, then

\[
f_{w_0}(v_0) = (a_{w_0}(-\lambda_{w_0})^{1/2} z_{w_0} - b_{w_0}(-\lambda_{w_0})^{1/2} y_{w_0})
\]

i.e.

\[
\|f_{w_0}(v_0)\|^2 = -\lambda_{w_0}(a_{w_0}^2 + b_{w_0}^2) = -\lambda.
\]

Consequently, \(v_0\) is an eigenvector of \(f_{w_0}\), and the same considerations are valid for \(f_{w_0}\). This establishes the first statement of the Lemma. Since \(\|u_0 \wedge v_0\|^2 = -\lambda\), the second statement is established as well. \(\square\)

**Remark 4.6.** 1. In fact the maximum of \(H\) is reached on the compact

\[
U = \{(u, v) \in S \times S / \langle u, v \rangle = 0\} = S \times S \cap \{,\}^{-1}\{0\},
\]

and we put \(M = \sup H(S \times S) = \sup H(U) = \|u_0 \wedge v_0\|, m = \inf H(U) = \|u_1 \wedge v_1\| > 0, (u_1, v_1) \in U\). Then the subalgebra \(\mathbb{R}[u_0, v_0]\) of \(A\) generated by \(u_0\) and \(v_0\) has a basis \(\{1, u_0, v_0, u_0 \wedge v_0\}\), and is isomorphic to \(\mathbb{H}^{(1/2)(1+\sqrt{\lambda_{w_0}})} = \mathbb{H}^{(1/2)(1+M)}\). Analogously, one has \(\mathbb{R}[u_1, v_1] \cong \mathbb{H}^{(1/2)(1+m)}\).

2. If \((u, v) \in U\) where \(u, v \in \mathbb{R}[u_0, v_0]\) then \(\mathbb{R}[u, v] = \mathbb{R}[u_0, v_0] \cong \mathbb{H}^{(1/2)(1+M)}\) i.e. \(\|u \wedge v\| = M\).

The first proposition of the following theorem is one of our key results:

**Theorem 4.7.** Let \(A\) be a real quadratic flexible division algebra of dimension 8. Then \(A\) has a four-dimensional subalgebra. Moreover the following statements are equivalent:
1. $A$ satisfies the Osborn property,
2. Any two four-dimensional subalgebras of $A$ are isomorphic.

**Proof.** According to the above notation, the subalgebra $\mathbb{R}[u_0, v_0]$ is four-dimensional. We now prove the equivalence 1. $\iff$ 2.:

2. $\implies$ 1. The subalgebras $\mathbb{H}\{(1/2)(1+i)^{M}\}$ and $\mathbb{H}\{(1/2)(1+m)\}$ are isomorphic if and only if $m = M$ [10]. Thus for every $(u, v) \in U$ one has $m = \|u \wedge v\| = M$ i.e. the subalgebra $\mathbb{R}[u, v]$ is four-dimensional. Then $A$ satisfies the Osborn property.

1. $\implies$ 2. Let $v = \sum_{1 \leq i \leq 3} (a_i y_i + b_i z_i)$ be a vector of $A$ such that $(u_0, v) \in U$; then $v$ and $f_{u_0}^2(v) = \sum_{1 \leq i \leq 3} \lambda_i (a_i y_i + b_i z_i)$ are linearly dependent because $u_0$ and $v$ satisfy the Osborn property. Then $f_{u_0}^2$ has only one eigenvalue, and we have $M = \|u_0 \wedge v_0\| = \|u_0 \wedge v\|$. Let now $(u, v) \in U$; there exists $v_0 \in W(u) \cap \mathbb{R}[u_0, v_0] - \{0\}$ and $u_0 \in W(v_0') \cap \mathbb{R}[u_0, v_0] - \{0\}$, which can be chosen of norm one, and one has $M = \|u_0 \wedge v_0\|$ (by 4.6.2.) $= \|u \wedge v\|$ (because $(u, v_0') \in U$) $= \|u \wedge v\|$. Then $\mathbb{R}[u, v] \cong \mathbb{H}\{(1/2)(1+M)\}$.

**Remark 4.8.** All real division algebras of dimension 8 which we have seen in the literature contain a four-dimensional subalgebra, but the existence of a four-dimensional subalgebra for a real quadratic division algebra of dimension 8 seems still an open problem.

5. Classification of the eight-dimensional real quadratic flexible division algebras

Let $A = (W, (/), \wedge)$ be a real quadratic flexible division algebra of dimension 8 and $B$ a four-dimensional subalgebra of $A$. There is an orthonormal basis $1, u; y_1, z_1$ of $B$, which can be extended, according to Note 4.3, to an orthonormal basis $\mathcal{B} = \{1, u; y_1, z_1; y_2, z_2; y_3, z_3\}$ of $A$ such that $u \wedge y_i = a_i z_i$, $u \wedge z_i = -a_i y_i$; $i = 1, 2, 3$ and $y_1 \wedge z_1 = a_1 u$, where $a_i$ are parameters $> 0$. Similarly there exist parameters $\theta_{ij}$, $\omega_{ij}$, $\pi_{ij}$, $\sigma_{ijk}$, $\beta_{ijk}$, $\gamma_{ijk}$, $\eta_{ijk}$, $\lambda_{ijk}$, $\mu_{ijk}$; $i, j, k \in \{1, 2, 3\}$, such that

$$y_i \wedge y_j = \theta_{ij} u + \sum_{1 \leq k \leq 3} (\alpha_{ijk} y_k + \beta_{ijk} z_k),$$

$$y_i \wedge z_j = \omega_{ij} u + \sum_{1 \leq k \leq 3} (\gamma_{ijk} y_k + \eta_{ijk} z_k),$$

$$z_i \wedge z_j = \pi_{ij} u + \sum_{1 \leq k \leq 3} (\lambda_{ijk} y_k + \mu_{ijk} z_k).$$

The trace form property of $(/)$ and anti-commutativity of $\wedge$ shows that $\alpha_{ijk}$ and $\mu_{ijk}$ are alternative in $i, j$ and $k$, $\beta_{ijk} = -\beta_{ikj} = \gamma_{jik}$, $\eta_{ijk} = -\eta_{ikj} = \lambda_{jik}$, $\theta_{ij} = \pi_{ij} = \beta_{i11} = \eta_{11j} = 0$, $\omega_{ii} = a_i$ and $\omega_{ij} = 0$ if $i \neq j$. This reduces the previous parameters to:
which we denote respectively by

\[ a, b, c, \alpha, \beta, \gamma, \mu, \lambda, \eta, \sigma, \delta, v, \pi, \rho, \theta, \omega, \pi', \theta', \omega'. \]

With respect to the basis \( \mathcal{B} \) we obtain a first multiplication table of \( A \), restricting us at the upper superior triangular part, in view of the anti-commutativity property of \( \wedge \):

\[
\begin{array}{cccc}
1 & u & y_1 & z_1 & y_2 & z_2 \\
\hline
1 & u & y_1 & z_1 & y_2 & z_2 \\
u & -1 & az_1 - ay_1 & bz_2 & -by_2 \\
y_1 & -1 & au & \pi_* z_2 + \alpha y_3 + \beta z_3 & -\pi_* y_2 + \gamma y_3 + \mu z_3 \\
z_1 & -1 & \pi_* z_2 + \beta y_3 + \eta z_3 & -\pi_* y_2 + \sigma y_3 + \delta z_3 \\
y_2 & -1 & bu + \pi_* y_1 + \pi_* z_1 + \theta_* y_3 + \theta_* z_3 & -1 \\
z_2 & -1 & & & \\
y_3 & & & & \\
z_3 & & & & \\
\end{array}
\]

We shall denote \( (y_i; z_i), \ i = 2, 3 \), by \( H_i \). If \( \pi' = (\pi_*^2 + \pi^2)^{1/2} \neq 0 \), we put \( y'_i = \pi'^{-1}(\pi y_i - \pi z_i) \) and \( z'_i = \pi'^{-1}(\pi_* y_i + \pi_* z_i) \). Now \( y'_i \) and \( z'_i \) are orthonormal and we have \( u \wedge y'_i = az'_i \), \( u \wedge z'_i = -ay'_i \), \( y'_1 \wedge z'_1 = y_1 \wedge z_1 \) and \( L_y(H_2) \subset H_3 \). These allow us to suppose \( \pi' = 0 \). If \( \theta' = (\theta^2 + \theta^3)^{1/2} \neq 0 \), we put \( y'_3 = \theta'^{-1}(\theta y_3 - \theta z_3) \) and \( z'_3 = \theta'^{-1}(\theta_* y_3 + \theta_* z_3) \). Now \( y'_3 \) and \( z'_3 \) are orthonormal
and we have \( u \wedge y_3 = cz'_3 \), \( u \wedge z'_3 = -cy'_3 \), \( y'_3 \wedge z'_3 = y_3 \wedge z_3 \), and \( L_{v'}(H_2) \subset \langle y'_1; z_1; z'_1 \rangle \). These allow us to suppose \( \theta' = 0 \) and, for the same reason, \( \omega' = 0 \). Thus we reduce the multiplication table of \( A \) to 16 parameters (Table 1).

Conversely, a direct computation shows that an algebra whose multiplication is given by Table 1, is quadratic and flexible. Later we shall need the following preliminary result:

**Lemma 5.1.** If \( v = 0 \), the conditions \( \beta y - \alpha \mu > 0, \beta \lambda - \alpha \eta > 0 \) and \( \gamma \lambda - \alpha \sigma > 0 \) are necessary for \( A \) to be a division algebra.

**Proof.** We suppose \( A \) is a division algebra and distinguish two cases.

Case 1: If \( x = 0 \), then \( \beta y \neq 0 \) and it has: \( (\beta x u - cy_1)(xy_2 + y_3) = (\beta b x_2 + c y_3)z_2 \) and \( (\gamma x u - by_3)(xy_1 + y_3) = (\gamma a x^2 + b \lambda)z_1 \) for each \( x \in \mathbb{R} \). As the left-hand terms in the last equality are non-zero, the trinomials \( \beta b x_2 + c y_3 \) and \( \gamma a x^2 + b \lambda \) in \( x \) have negative discriminant, i.e. \( \beta y, \gamma \lambda, \beta \lambda = \gamma - (\beta y)(\gamma \lambda) \) are all positive.

Case 2: If \( x \neq 0 \), we consider the automorphism \( \varphi \) of \( W \) whose matrix with respect to the basis \( B_0 = \{ u; y_1, z_1; y_2, z_2, y_3, z_3 \} \) is \( I - \lambda x^{-1}e_{23} - \gamma x^{-1}e_{45} - \beta x^{-1}e_{57} \), and obtain the multiplication table of \( A(\varphi) \) (Table 2) where

\[
\begin{align*}
\mu' &= \mu - \beta y x^{-1}, \\
\sigma' &= \sigma - \gamma \lambda x^{-1}, \\
\eta' &= \eta - \beta \lambda x^{-1}
\end{align*}
\]

As \( A(\varphi) \) is a division algebra, we have \( \sigma' \neq 0 \). Moreover, the equalities

\[
(xu + z_1)(\sigma'y_2 + (bx + \pi)c)x + \sigma'y' + \pi p)z_3,
\]

\[
(xu + y_1)(-az_2 + bx y_3) = (bcx^2 + (b \rho + \pi c)x + \sigma' y' + \pi \rho)z_3
\]

hold for each \( x \in \mathbb{R} \). As the left terms in the last three equalities are non-zero, the trinomials \( bcx^2 + (b \rho + \pi c)x + \sigma' y' + \pi \rho \), \( bcx^2 - \alpha \mu' \), \( acc^2 + \omega x + \sigma' \mu' \) in \( x \) have negative discriminant, i.e. \( \beta y - \alpha \mu, \beta \lambda - \alpha \eta, \gamma \lambda - \alpha \sigma > 0 \).

**Note 5.2.** According to Table 2, there exists \( \varepsilon \in \{ 1, -1 \} \) such that

\[
|x| = \varepsilon x, \quad |\mu'| = -\varepsilon \mu', \quad |\sigma'| = -\varepsilon \sigma' \quad \text{and} \quad |\eta'| = -\varepsilon \eta'.
\]

**Lemma 5.3.** If \( v = \pi = \theta = 0 \), \( A \) is a division algebra if and only if \( \beta y - \alpha \mu, \beta \lambda - \alpha \eta, \gamma \lambda - \alpha \sigma \) are all positive and \( c(x \delta - \beta \sigma - \lambda \mu + \gamma \eta)^2 + b(\beta y - \alpha \mu)\rho^2 + a(\beta \lambda - \alpha \eta)\omega^2 < 4c(\beta \lambda - \alpha \eta)(\mu \sigma - \gamma \delta) \).

**Proof.** We suppose that \( A \) is a division algebra and distinguish two cases.

Case 1: \( a = b = c = 1 \).

(i) If \( x \neq 0 \) we consider the automorphisms \( \varphi \) (of Lemma 5.1) and the automorphism \( \psi \) such that \( M(\psi, B_0) = \text{diag}\{ 1, q, q^{-1}, r, r^{-1}, s, s^{-1} \} \), where
### Table 1

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J.A. Cuenca Mira et al. / Linear Algebra and its Applications 290 (1999) 1-22
We obtain the multiplication table of the algebra $A_1 = (A(\varphi))(\psi)$:

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where

$$\alpha^* = -\alpha^2(\beta \gamma - \alpha \mu)(\beta \lambda - \alpha \eta)(\gamma \lambda - \alpha \sigma) < 0, \quad \rho' = -\alpha \rho((\beta \lambda - \alpha \eta)(\gamma \lambda - \alpha \sigma))^{-1/2},$$

$$\delta^* = -\alpha^3 \delta'(\beta \gamma - \alpha \mu)^{-1}(\beta \lambda - \alpha \eta)^{-1}(\gamma \lambda - \alpha \sigma)^{-1}, \quad \omega' = -\alpha \omega((\beta \gamma - \alpha \mu)(\gamma \lambda - \alpha \sigma))^{-1/2}.$$

We consider finally the automorphism $f$ of $W$ such that $M(f, B_0) = \text{diag}\{-(-\alpha^*)^{1/6}, (\alpha^*)^{-1/3}, (-\alpha^*)^{1/6}, (\alpha^*)^{-1/3}, (-\alpha^*)^{1/6}, (\alpha^*)^{-1/3}, (-\alpha^*)^{1/6}\}$. We obtain the multiplication table of $A_2 = A_1(f)$:

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<tr>
<td>$y_2$</td>
<td>$-1$</td>
<td>$u$</td>
<td>$y_1$</td>
<td>$z_1$</td>
<td>$-z_1 - \omega^*z_3$</td>
<td>$-y_1 + \delta^*z_1 + \omega^*y_3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$z_2$</td>
<td>$-1$</td>
<td>$z_1 - \omega^*z_3$</td>
<td>$-y_1 + \delta^*z_1 + \omega^*y_3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y_3$</td>
<td>$-1$</td>
<td>$u - \rho'z_1 - \omega^*z_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$z_3$</td>
<td>$-1$</td>
<td>$z_3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
where $\delta_0 = (-\alpha^*)^{1/2} \delta^*$. We put then $e_0 = u, e_i = y_i, e_{i+4} = -z_i, i = 1, 2, 3,$ and obtain with respect to the basis $1, e_1, \ldots, e_7$ the following table:

<table>
<thead>
<tr>
<th></th>
<th>$e_1$</th>
<th>$e_2$</th>
<th>$e_3$</th>
<th>$e_4$</th>
<th>$e_5$</th>
<th>$e_6$</th>
<th>$e_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$e_1$</td>
<td>$e_2$</td>
<td>$e_3$</td>
<td>$e_4$</td>
<td>$e_5$</td>
<td>$e_6$</td>
</tr>
<tr>
<td>$e_1$</td>
<td>-1</td>
<td>$e_2$</td>
<td>$e_3$</td>
<td>$e_4$</td>
<td>$e_5$</td>
<td>$-e_7$</td>
<td>$e_6$</td>
</tr>
<tr>
<td>$e_2$</td>
<td>-1</td>
<td>$e_3$</td>
<td>$e_4$</td>
<td>$e_5$</td>
<td>$e_6$</td>
<td>$-e_7$</td>
<td>$e_7$</td>
</tr>
<tr>
<td>$e_3$</td>
<td>-1</td>
<td>$e_4$</td>
<td>$e_5$</td>
<td>$e_6$</td>
<td>$e_7$</td>
<td>$-e_7$</td>
<td>$e_7$</td>
</tr>
<tr>
<td>$e_4$</td>
<td>-1</td>
<td>$e_5$</td>
<td>$e_6$</td>
<td>$e_7$</td>
<td>$-e_7$</td>
<td>$e_7$</td>
<td>$e_7$</td>
</tr>
<tr>
<td>$e_5$</td>
<td>-1</td>
<td>$e_6$</td>
<td>$e_7$</td>
<td>$-e_7$</td>
<td>$e_7$</td>
<td>$e_7$</td>
<td>$e_7$</td>
</tr>
<tr>
<td>$e_6$</td>
<td>-1</td>
<td>$e_7$</td>
<td>$-e_7$</td>
<td>$e_7$</td>
<td>$e_7$</td>
<td>$e_7$</td>
<td>$e_7$</td>
</tr>
</tbody>
</table>

If $\rho = \omega = 0$, then $A_2 \simeq E_{-1,0}(H)$, and is a division algebra if and only if $|\delta_0| < 2$ [7]. So

$$
\delta_0^3 < 4 \iff \alpha^2 \delta^2 (\beta\gamma - \alpha\mu)^{-1} (\beta\lambda - \alpha\eta)^{-1} (\gamma\lambda - \alpha\sigma)^{-1} < 4
$$

$$
\iff (\alpha^2 \delta - (\gamma\eta + \alpha\mu + \beta\sigma)\alpha + 2\beta\lambda\gamma)^2 < 4(\beta\gamma - \alpha\mu)(\beta\lambda - \alpha\eta)(\gamma\lambda - \alpha\sigma)
$$

$$
\iff (\alpha\delta - \beta\sigma - \lambda\mu + \gamma\eta) + 2\gamma(\beta\lambda - \alpha\eta))^2 < 4(\beta\lambda - \alpha\eta)(\sigma\mu - \gamma\delta).
$$

If $(\rho, \omega) \neq (0, 0)$ we put: $\delta_1 = (\rho^2 + \delta_0^2)^{1/2}, \delta_2 = (\delta_0^2 + \omega^2)^{1/2}$ and

$$
eq 1(\delta,e_1 + \rho\delta e_4) \quad \text{if } \rho \neq 0(= e_1 \text{ if } \rho = 0),$$

$$
eq 2(\delta,e_2 + \delta_1 e_2 + \omega_1 \delta_1 e_4) \quad \text{if } \rho \neq 0(= \delta_2^{-1}(\delta,e_2 + \omega_1 e_4) \text{ if } \rho = 0),$$

$$
eq 3(\delta,e_3 + \omega_1 e_3 + \rho \delta e_6),$$

$$
eq 4(\delta_2^{-1}(\delta,e_4 + \omega_1 e_4),$$

$$
eq 5(\delta,e_5 + \omega_1 e_5 + \rho \delta e_6) \quad \text{if } \rho \neq 0(= \delta_2^{-1}(\delta,e_5 + \delta e_5) \text{ if } \rho = 0),$$

$$
eq 6(\delta_1^{-1}(\rho \delta e_5 + \delta e_6) \quad \text{if } \rho \neq 0(= e_6 \text{ if } \rho = 0),$$

$$
eq 7.$$
Then $A_2 \simeq E_{-1,0,1}(4)$, and is a division algebra if and only if $|\delta_2| < 2$. We have

$$\delta_2^2 = \omega^2 + \rho^2 + \delta_2^2 = \alpha^2 \omega^2 (\beta \gamma - \alpha \mu)^{-1} (\gamma \lambda - \alpha \sigma)^{-1} + x^2 \rho^2 (\beta \lambda - \alpha \eta)^{-1}$$

$$\times (\gamma \lambda - \alpha \sigma)^{-1} + x^4 \delta^2 (\beta \gamma - \alpha \mu)^{-1} (\beta \lambda - \alpha \eta)^{-1} (\gamma \lambda - \alpha \sigma)^{-1}.$$ 

So

$$\delta_2^2 < 4 \iff x^2 (\beta \gamma - \alpha \eta) \omega^2 + \alpha^2 (\beta \lambda - \alpha \mu) \rho^2 + x^4 \delta^2$$

$$< 4 (\beta \gamma - \alpha \mu) (\beta \lambda - \alpha \eta) (\gamma \lambda - \alpha \sigma)$$

$$\iff (\alpha \delta - \beta \sigma - \lambda \mu + \gamma \eta)^2 + (\beta \lambda - \alpha \eta) \omega^2 + (\beta \gamma - \alpha \mu) \rho^2$$

$$< 4 (\beta \lambda - \alpha \eta) (\sigma \mu - \gamma \delta).$$

As $A$ is a division algebra if and only if: $\beta \lambda - \alpha \eta, \beta \gamma - \alpha \mu, \gamma \lambda - \alpha \sigma$ are all positive and $A_2$ is a division algebra, the result is established in this first situation.

(ii) If $x = 0$, we put $y_3 = z_3, z'_3 = -y_3$ and obtain the following table:

Thus $A$ is a division algebra if and only if $\beta \lambda > 0, \beta \gamma > 0, \mu \eta - \beta \delta > 0$ and $(-\beta \sigma - \lambda \mu + \gamma \eta)^2 + \beta \lambda \omega^2 + \beta \gamma \rho^2 < 4 \beta \lambda (\sigma \mu - \gamma \delta)$ i.e. $\beta \lambda > 0, \beta \gamma > 0, \gamma \lambda > 0$. 

and \((-\beta \sigma - \lambda \mu + \gamma \eta)^2 + \beta \lambda \omega^2 + \beta \gamma \rho^2 < 4 \beta \lambda (\sigma \mu - \gamma \delta)\) because \((-\beta \sigma - \lambda \mu + \gamma \eta)^2 - 4 \gamma \lambda (\mu \eta - \beta \delta) = (-\beta \sigma - \lambda \mu + \gamma \eta)^2 - 4 \beta \lambda (\sigma \mu - \gamma \delta) < 0\) i.e. \(\mu \eta - \beta \delta > 0\).

**Case 2:** \(a, b, c > 0\). We consider the automorphism \(g\) of \(W\) such that
\[
M(g, \phi_0) = \text{diag}\{1, a^{-1/2}, a^{-1/2}, b^{-1/2}, b^{-1/2}, c^{-1/2}, c^{-1/2}\},
\]
and obtain the following table:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>u</th>
<th>y_1</th>
<th>z_1</th>
<th>y_2</th>
<th>z_2</th>
<th>y_3</th>
<th>z_3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>u</td>
<td>y_1</td>
<td>z_1</td>
<td>y_2</td>
<td>z_2</td>
<td>y_3</td>
<td>z_3</td>
</tr>
<tr>
<td>u</td>
<td>-1</td>
<td>z_1</td>
<td>-y_1</td>
<td>z_2</td>
<td>-y_2</td>
<td>z_3</td>
<td>-y_3</td>
<td></td>
</tr>
<tr>
<td>y_1</td>
<td>-1</td>
<td>u</td>
<td>\alpha_0 y_3 + \beta_0 z_3</td>
<td>\beta_0 y_3 + \beta_0 z_3</td>
<td>-\alpha_0 y_2 - \gamma_0 z_2</td>
<td>-\beta_0 y_2 - \mu_0 z_2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>z_1</td>
<td>-1</td>
<td>\lambda_0 y_3 + \eta_0 z_3</td>
<td>\sigma_0 y_3 + \delta_0 z_3</td>
<td>-\lambda_0 y_2 - \sigma_0 z_2 + \rho_0 z_3</td>
<td>-\eta_0 y_2 - \delta_0 z_2 - \rho_0 y_3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>y_2</td>
<td>-1</td>
<td>u</td>
<td>\gamma_0 y_1 + \lambda_0 z_1</td>
<td>\beta_0 y_1 + \eta_0 z_1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>z_2</td>
<td>-1</td>
<td>\gamma_0 z_1 + \sigma_0 z_3 + \omega_0 z_3</td>
<td>\mu_0 z_1 + \delta_0 z_3 + \omega_0 z_3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>y_3</td>
<td>-1</td>
<td>u + \rho_0 z_1 + \omega_0 z_2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>z_3</td>
<td>-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where \(\alpha_0 = d \alpha, \beta_0 = d \beta, \gamma_0 = d \gamma, \mu_0 = d \mu, \lambda_0 = d \lambda, \eta_0 = d \eta, \sigma_0 = d \sigma, \delta_0 = d \delta, \omega_0 = b^{-1/2} c^{-1} \omega, \rho_0 = a^{-1/2} c^{-1} \rho\) with \(d = (abc)^{-1/2}\). These get us to the first situation. Thus \(A(g)\) is a division algebra if and only if: \(\beta \lambda - \alpha \eta, \beta \gamma - \alpha \mu, \gamma \lambda - \alpha \sigma\) are all positive, and
\[
c(\xi \delta - \beta \sigma - \lambda \mu + \gamma \eta)^2 + a(\beta \lambda - \alpha \eta) \omega^2 + b(\beta \gamma - \alpha \mu) \rho^2 < 4 c(\beta \lambda - \alpha \eta)(\mu \sigma - \gamma \delta).
\]
The proof is finished taking into account that \(A\) is a division algebra if any only if \(A(g)\) is a division algebra. \(\square\)

**Corollary 5.4.** Let \(\alpha, \delta, \lambda, \mu\) be real arbitrary parameters, then the real algebra \((E_{-1,0,0}((\mathbb{H})^{(1)}))^{(\mu)}\) is quadratic and flexible. Moreover it is a division algebra if and only if \(\lambda, \mu \neq \frac{1}{2}, \alpha > \frac{1}{2}\) and \((2 \alpha - 1) \delta^2 < 4\).

**Proof.** The first statement is established in [7]. \(\square\)

**Lemma 5.5.** If \(v = \pi = 0\) and \(\theta \neq 0\), then \(A\) is a division algebra if and only if \(\beta \gamma - \alpha \mu, \beta \lambda - \alpha \eta, \gamma \lambda - \alpha \sigma\) are all positive \(c(\gamma \lambda - \alpha \sigma) \theta^2 + b(\beta \lambda - \alpha \eta) \omega^2 > b \alpha \rho \omega \theta\) and \(bc(\xi \delta - \beta \sigma - \lambda \mu + \gamma \eta)^2 + ab(\beta \lambda - \alpha \eta) \omega^2 + b^2(\beta \gamma - \alpha \mu) \rho^2 + ac(\gamma \lambda - \alpha \sigma) \theta^2 < ab \alpha \rho \omega \theta + 4 bc(\beta \lambda - \alpha \eta)(\mu \sigma - \gamma \delta)\).

**Proof.** We suppose that \(A\) is a division algebra and shall distinguish the following two cases:
**Case 1:** If \( x \neq 0 \), we consider the algebra, \( A_1 = A(\varphi) \) where \( \varphi \) is the automorphism of \( W \) defined in Lemma 5.1 and we put \( \omega_0 = (\omega^2 + \theta^2)^{1/2} \), \( u' = y_1, \ y_2' = \omega_0^{-1}(\omega y_2 + \theta y_3), \ y_3' = -\varepsilon y_2, \ z_1' = -z_1, \ z_2' = \omega_0^{-1}(-\theta y_2 + \omega y_3) \) (according to the notations 1.2). We obtain the following table:

<table>
<thead>
<tr>
<th></th>
<th>( u' )</th>
<th>( y_1' )</th>
<th>( z_1' )</th>
<th>( y_2' )</th>
<th>( z_2' )</th>
<th>( y_3' )</th>
<th>( z_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( u' )</td>
<td>( y_1' )</td>
<td>( z_1' )</td>
<td>( y_2' )</td>
<td>( z_2' )</td>
<td>( y_3' )</td>
<td>( z_3 )</td>
</tr>
<tr>
<td>( u' )</td>
<td>-1 ( ax_1 )</td>
<td>-( ay_1 )</td>
<td>( ayz_2 )</td>
<td>-( axz_2 )</td>
<td>( ay_2z_3 )</td>
<td>-( ax_2z_3 )</td>
<td>-( u' )</td>
</tr>
<tr>
<td>( y_1' )</td>
<td>-1 ( ay_2z_3 )</td>
<td>( a_0y_1 + \beta_0z_3 )</td>
<td>( a_0y_1 + \mu_0z_3 )</td>
<td>-( a_0y_2 - \gamma_0z_2 )</td>
<td>-( \beta_0y_2 - \mu_0z_2 )</td>
<td>-( \beta_0y_2 - \mu_0z_2 )</td>
<td>-( \beta_0y_2 - \mu_0z_2 )</td>
</tr>
<tr>
<td>( z_1' )</td>
<td>-1 ( \lambda_0y_1 + \eta_0z_3 )</td>
<td>( \sigma_0y_1 + \delta_0z_3 )</td>
<td>-( \lambda_0y_2 - \sigma_0z_2 + \varepsilon\delta z_3 )</td>
<td>-( \eta_0y_2 - \delta_0z_2 - \varepsilon\delta y_1 )</td>
<td>-( \beta_0y_2 - \mu_0z_2 )</td>
<td>-( \beta_0y_2 - \mu_0z_2 )</td>
<td>-( \beta_0y_2 - \mu_0z_2 )</td>
</tr>
<tr>
<td>( y_2' )</td>
<td>-1 ( \gamma_0y_1 + \sigma_0z_1 )</td>
<td>( \beta_0y_1 + \eta_0z_1 )</td>
<td>( \beta_0y_1 + \eta_0z_1 )</td>
<td>-( \mu_0y_1 + \delta_0z_2 - \varepsilon\delta y_1 )</td>
<td>-( \beta_0y_2 - \mu_0z_2 )</td>
<td>-( \beta_0y_2 - \mu_0z_2 )</td>
<td>-( \beta_0y_2 - \mu_0z_2 )</td>
</tr>
<tr>
<td>( z_2' )</td>
<td>-1 ( \gamma_0y_1 + \sigma_0z_1 )</td>
<td>( \beta_0y_1 + \eta_0z_1 )</td>
<td>( \beta_0y_1 + \eta_0z_1 )</td>
<td>-( \mu_0y_1 + \delta_0z_2 - \varepsilon\delta y_1 )</td>
<td>-( \beta_0y_2 - \mu_0z_2 )</td>
<td>-( \beta_0y_2 - \mu_0z_2 )</td>
<td>-( \beta_0y_2 - \mu_0z_2 )</td>
</tr>
<tr>
<td>( y_3' )</td>
<td>-1 (</td>
<td>u'</td>
<td>y_1' )</td>
<td>( \alpha_0y_1 + \beta_0z_1 )</td>
<td>( \beta_0y_1 + \eta_0z_1 )</td>
<td>( \beta_0y_1 + \eta_0z_1 )</td>
<td>( \beta_0y_1 + \eta_0z_1 )</td>
</tr>
<tr>
<td>( z_3 )</td>
<td>-1 (</td>
<td>u'</td>
<td>y_1' )</td>
<td>( \alpha_0y_1 + \beta_0z_1 )</td>
<td>( \beta_0y_1 + \eta_0z_1 )</td>
<td>( \beta_0y_1 + \eta_0z_1 )</td>
<td>( \beta_0y_1 + \eta_0z_1 )</td>
</tr>
</tbody>
</table>

where \( \alpha_0 = -\omega_0^{-1}ob, \ \beta_0 = \omega_0^{-1}b\theta c, \ \gamma_0 = \omega_0^{-1}e\theta b, \ \mu_0 = \omega_0^{-1}eoc, \ \lambda_0 = -\omega_0^{-1}e\sigma, \ \eta_0 = -\omega_0^{-1}e(\eta\theta' + \theta\rho), \ \sigma_0 = -\omega_0^{-1}e\sigma\rho \) and \( \delta_0 = \omega_0^{-1}(e\theta' - e\rho) \). Thus \( A_1 \) is a division algebra if and only if:

\[
\beta_0\lambda_0 - \alpha_0\eta_0, \ \beta_0\gamma_0 - \alpha_0\mu_0, \ \gamma_0\lambda_0 - \alpha_0\sigma_0 \text{ are all positive, and } |\mu'|(\alpha_0\delta_0 - \beta_0\sigma_0 - \lambda_0\mu_0 + \gamma_0\eta_0)^2 + a(\beta_0\lambda_0 - \alpha_0\eta_0)(\varepsilon\omega_0)^2 + |\sigma|(\beta_0\gamma_0 - \alpha_0\mu_0)(\varepsilon\delta)^2 < 4|\mu'|(\beta_0\lambda_0 - \alpha_0\eta_0)(\mu_0\sigma_0 - \gamma_0\delta_0) \text{ i.e. } A \text{ is a division algebra if and only if } \beta_0 \lambda_0 - \alpha_0 \eta_0, \ \beta_0 \gamma_0 - \alpha_0 \mu_0, \ \gamma_0 \lambda_0 - \alpha_0 \sigma_0 \text{ are all positive.}
\]

Thus \( A_1 \) is a division algebra if and only if:

\[
\beta_0 \lambda_0 - \alpha_0 \eta_0, \ \beta_0 \gamma_0 - \alpha_0 \mu_0, \ \gamma_0 \lambda_0 - \alpha_0 \sigma_0 \text{ are all positive, and } |\mu'|(\alpha_0\delta_0 - \beta_0\sigma_0 - \lambda_0\mu_0 + \gamma_0\eta_0)^2 + a(\beta_0\lambda_0 - \alpha_0\eta_0)(\varepsilon\omega_0)^2 + |\sigma|(\beta_0\gamma_0 - \alpha_0\mu_0)(\varepsilon\delta)^2 < 4|\mu'|(\beta_0\lambda_0 - \alpha_0\eta_0)(\mu_0\sigma_0 - \gamma_0\delta_0) \text{ i.e. } A \text{ is a division algebra if and only if } \beta_0 \lambda_0 - \alpha_0 \eta_0, \ \beta_0 \gamma_0 - \alpha_0 \mu_0, \ \gamma_0 \lambda_0 - \alpha_0 \sigma_0 \text{ are all positive.
\]

**Case 2:** If \( x = 0 \), the condition \( \gamma \neq 0 \) is necessary for \( A \) being a division algebra. We consider then the automorphism \( \varphi' \) of \( W \) whose matrix with respect of the basis \( B_0 \) is:

\[
\begin{pmatrix}
1 & -\theta b^{-1} \\
-\beta_0 y' & \lambda_0 y' & -b(\beta_0 - \gamma_0)\omega^2 > b\alpha_0 \varphi \omega b & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & -\sigma_0 y' & -\omega_0 y' \\
-\beta_0 y' & \lambda_0 y' & -b(\beta_0 - \gamma_0)\omega^2 > b\alpha_0 \varphi \omega b & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & -\mu_0 y' & -\gamma_0 y' \\
-\beta_0 y' & \lambda_0 y' & -b(\beta_0 - \gamma_0)\omega^2 > b\alpha_0 \varphi \omega b & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & -\mu_0 y' & -\gamma_0 y' \\
-\beta_0 y' & \lambda_0 y' & -b(\beta_0 - \gamma_0)\omega^2 > b\alpha_0 \varphi \omega b & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & -\mu_0 y' & -\gamma_0 y' \\
-\beta_0 y' & \lambda_0 y' & -b(\beta_0 - \gamma_0)\omega^2 > b\alpha_0 \varphi \omega b & 0
\end{pmatrix}
\]
We obtain the multiplication table of the algebra $A(\varphi')$:

<table>
<thead>
<tr>
<th></th>
<th>$z_2$</th>
<th>$u$</th>
<th>$y_2$</th>
<th>$y_1$</th>
<th>$y_3$</th>
<th>$z_3$</th>
<th>$z_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$z_2$</td>
<td>$u$</td>
<td>$y_2$</td>
<td>$y_1$</td>
<td>$y_3$</td>
<td>$z_3$</td>
</tr>
<tr>
<td>$z_2$</td>
<td>-1</td>
<td>$by_2$</td>
<td>$-bu$</td>
<td>$-yy_3$</td>
<td>$\gamma y_1$</td>
<td>$\delta_1 z_1$</td>
<td>$-\delta_1 z_3$</td>
</tr>
<tr>
<td>$u$</td>
<td>-1</td>
<td>$bz_2$</td>
<td>$a z_1$</td>
<td>$c z_3$</td>
<td>$-c y_3 - a \omega y^{-1} z_1$</td>
<td>$-a y_1 + a \omega y^{-1} z_3$</td>
<td></td>
</tr>
<tr>
<td>$y_2$</td>
<td>-1</td>
<td>$-\beta z_3$</td>
<td>$\lambda z_1$</td>
<td>$\beta y_1 + \eta_1 z_1$</td>
<td>$-\lambda y_3 - \eta_1 z_3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y_1$</td>
<td>-1</td>
<td>$-\gamma z_2$</td>
<td>$-\beta y_2 + ab^{-1} \theta z_1$</td>
<td>$a u - ab^{-1} \theta z_3$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y_3$</td>
<td>-1</td>
<td>$c u + \rho z_1$</td>
<td>$\lambda y_2 - \rho z_3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$z_3$</td>
<td>-1</td>
<td>$\delta_1 z_2 - a \omega y^{-1} u + \eta_1 y_2$</td>
<td>$+ab^{-1} \theta y_1 + \rho y_3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$z_1$</td>
<td>-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where $\delta_1 = \delta - \sigma \mu y^{-1}$ and $\eta_1 = \eta - (\lambda \mu + \beta \sigma) y^{-1}$. As $A(\varphi')$ is a division algebra, we have $\gamma \delta_1 = \gamma \delta - \sigma \mu < 0$, because the equality $(xz_2 + u)(-cy_1 + x\gamma z_3) = (\gamma \delta_1 x^2 - \omega ax - ac)z_1$ holds for each $x \in \mathbb{R}$ and $\delta_1 \neq 0$.

Then there exists $e \in \{1, -1\}$ such that: $|\delta_1| = e \delta_1, |\gamma| = -e \gamma$. We put $\rho_1 = (a^2 b^{-2} \theta^2 + \rho^2)^{1/2}, \ y'_1 = \rho_1^{-1}(ab^{-1} \theta y_1 + \rho y_3), \ y'_2 = \rho_1^{-1}(\rho y_1 - ab^{-1} \theta y_3), \ z'_3 = e' z_3$. If $(\omega, \eta_1) \neq (0, 0)$, we put again $\eta' = (a^2 \omega^2 y^{-2} + \eta_1^2)^{1/2}, \ u' = \eta'^{-1}(\eta_1 u + a \omega y^{-1} y_2)$, and $y'_2 = \eta'^{-1}(-a \omega y^{-1} u + \eta_1 y_2)$, we obtain the following table:

<table>
<thead>
<tr>
<th></th>
<th>$u'$</th>
<th>$y'_2$</th>
<th>$y'_1$</th>
<th>$y'_3$</th>
<th>$z'_1$</th>
<th>$z'_2$</th>
<th>$z'_3$</th>
<th>$z'_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$z_2$</td>
<td>$u'$</td>
<td>$y'_2$</td>
<td>$y'_1$</td>
<td>$y'_3$</td>
<td>$z'_1$</td>
<td>$z_1$</td>
</tr>
<tr>
<td>$z_2$</td>
<td>-1</td>
<td>$by'_2$</td>
<td>$-bu'$</td>
<td>$-yy'_3$</td>
<td>$\gamma y'_1$</td>
<td>$\delta_1</td>
<td>z_1$</td>
<td>$-</td>
</tr>
<tr>
<td>$u'$</td>
<td>-1</td>
<td>$bz_2$</td>
<td>$a z'_1 + \beta_1 z_1$</td>
<td>$\gamma z'_1 + \mu_1 z_1$</td>
<td>$-\beta y'_1 - \gamma y'_3$</td>
<td>$\beta y'_1 - \mu y'_3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y'_2$</td>
<td>-1</td>
<td>$-\beta z'_3$</td>
<td>$\lambda_1 z'_1 + \gamma_1 z_1$</td>
<td>$\sigma_1 z'_1 + \delta z_1$</td>
<td>$-\beta y'_2 - \gamma' y'_3$</td>
<td>$-\lambda y'_3 - \sigma z'_1 + \gamma y'_2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y'_1$</td>
<td>-1</td>
<td>$\gamma z_2$</td>
<td>$z_1 u' + \lambda_1 y'_1$</td>
<td>$\beta_1 u' + \gamma_1 y'_2$</td>
<td>$\beta_1 u' + \sigma_1 y'_1 + \epsilon_1 \rho z_1$</td>
<td>$\mu_1 u' + \delta z_2 - \epsilon \rho z_3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y'_3$</td>
<td>-1</td>
<td>$\gamma z_2$</td>
<td>$z_1 u' + \lambda_1 y'_1$</td>
<td>$\beta_1 u' + \gamma_1 y'_2$</td>
<td>$\beta_1 u' + \sigma_1 y'_1 + \epsilon_1 \rho z_1$</td>
<td>$\mu_1 u' + \delta z_2 - \epsilon \rho z_3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$z_3$</td>
<td>-1</td>
<td>$z_1</td>
<td>z_2 + e' \eta' y_2 + e' \rho_1 y'_3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$z_1$</td>
<td>-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where

$\alpha_1 = -\eta'^{-1} \rho_1^{-1}(ab^{-1} \epsilon \theta \eta_1 + a \beta \rho \omega y^{-1}), \ \beta_1 = \eta'^{-1} \rho_1^{-1} \epsilon'(a p \eta_1 - a^2 b^{-1} \omega y^{-1} \theta \lambda), \ \gamma_1 = \eta'^{-1} \rho_1^{-1} \epsilon'(c p \eta_1 - a^2 b^{-1} \omega y^{-1} \theta \beta), \ \mu_1 = \eta'^{-1} \rho_1^{-1}(a^2 b^{-1} \theta \eta_1 + a \omega y^{-1} \rho \lambda), \ \lambda_1 = \eta'^{-1} \rho_1^{-1}(a^2 b^{-2} \omega y^{-1} \rho - \eta_1 \beta), \ \eta_1 = -\eta'^{-1} \rho_1^{-1} \epsilon'(a^2 \omega y^{-1} \rho + ab^{-1} \eta_1 \theta \lambda), \ \sigma_1 = -\eta'^{-1} \rho_1^{-1} \epsilon'(a \omega y^{-1} \rho + ab^{-1} \eta_1 \theta \beta), \ \delta_2 = \eta'^{-1} \rho_1^{-1}(-a^2 b^{-1} \omega y^{-1} \theta + \eta_1 \rho \lambda).
Thus $A(\varphi')$ is a division algebra if and only if: 
\[ |\delta_1|\left(|\alpha_1\delta_2 - \beta_1\sigma_1 - \lambda_1\mu_1 + \gamma_1\eta_2|^2 + b(b_1, \lambda_1 - \alpha_1\eta_2)(e_1\rho_1)^2 + |\beta_1\gamma_1 - \alpha_1\mu_1)(e_1\eta_1)^2\right) < 4|\delta_1|\left(\beta_1\lambda_1 - \alpha_1\eta_2\right)\left(\sigma_1\mu_1 - \gamma_1\delta_2\right) \]
and $\beta_1\lambda_1 - \alpha_1\eta_2, \beta_1\gamma_1 - \alpha_1\mu_1, \gamma_1\lambda_1 - \alpha_1\sigma_1$ are all positive, i.e. $A$ is a division algebra if and only if
\[ \begin{align*}
bc(-\beta\sigma - \lambda\eta)^2 + ab\beta\lambda\omega^2 + b^2\beta\gamma\rho^2 + ac\gamma\lambda\theta^2 \\
< 4bc\beta\lambda(\alpha\sigma - \gamma\delta) \quad \text{and} \quad \beta\lambda, \beta\gamma, \gamma\lambda \text{ are all positive. In particular this hold if} \eta^* = 0. \quad \square
\end{align*} \]

**Theorem 5.6.** Let $A$ be a real quadratic, flexible division algebra of dimension 8. Then there is a basis $1, u, y_1, z_1, y_2, z_2, y_3, z_3$ of $A$, parameters $a, b, c > 0$ and thirteen other $\alpha, \beta, \gamma, \mu, \lambda, \eta, \sigma, \delta, v, \pi, \rho, \theta, \omega$ for which the multiplication table of $A$ is given by Table 1. Moreover an algebra whose multiplication is given by Table 1 is quadratic and flexible. It is a division algebra if and only if $\beta\lambda - \alpha\eta, \gamma\lambda - \alpha\sigma$ are all positive, $bc(\alpha\delta - \beta\sigma - \lambda\mu + \gamma\eta)^2 + ab(\beta\lambda - \alpha\eta)\omega^2 + ac(\gamma\lambda - \alpha\sigma)\theta^2 + (\beta\gamma - \alpha\mu)(b\rho - c\pi)^2 + b^2(\sigma\eta - \lambda\delta)v^2 + bv(\alpha\delta - \beta\sigma + \lambda\mu - \gamma\eta)(b\rho - c\pi) < a\omega\theta(\alpha(\beta\rho - c\pi) - b\lambda\nu) + 4bc(\beta\lambda - \alpha\eta)(\mu\sigma - \gamma\delta)$ and one of the following cases hold:

1. $v = 0 = \theta$ and $\beta\gamma - \alpha\mu > 0$,
2. $v \neq 0, \theta \neq 0, \beta\gamma - \alpha\mu > 0$ and $c(\gamma\lambda - \alpha\sigma)\theta^2 + b(\beta\lambda - \alpha\eta)\omega^2 > \alpha\omega\theta(b\rho - c\pi)$,
3. $v \neq 0, \theta = 0$ and $(\beta\gamma - \alpha\mu)(b\rho - c\pi)^2 + b^2(\sigma\eta - \lambda\delta)v^2 + bv(\alpha\delta - \beta\sigma + \lambda\mu - \gamma\eta)(b\rho - c\pi) > 0$,
4. $v\theta \neq 0, c(\gamma\lambda - \alpha\sigma)\theta^2 + b(\beta\lambda - \alpha\eta)\omega^2 > \alpha\omega\theta(\alpha(\beta\rho - c\pi) - b\lambda\nu)$ and $(\beta\gamma - \alpha\mu)(b\rho - c\pi)^2 + b^2(\sigma\eta - \lambda\delta)v^2 + bv(\alpha\delta - \beta\sigma + \lambda\mu - \gamma\eta)(b\rho - c\pi) > 0$.

**Proof.** We have to establish only the last assertion. We distinguish the following two cases.

**Case 1: $\pi = 0$**. We can suppose $v \neq 0$ and by putting $\rho' = (v^2 + \rho^2)^{1/2}$, $y'_1 = \rho'^{-1}(\rho y_1 - v z_1)$ and $z'_1 = \rho'^{-1}(\rho z_1 + v y_1)$ we obtain the following table:

<table>
<thead>
<tr>
<th></th>
<th>$1$</th>
<th>$u$</th>
<th>$y'_1$</th>
<th>$z'_1$</th>
<th>$y_2$</th>
<th>$z_2$</th>
<th>$y_3$</th>
<th>$z_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$1$</td>
<td>$u$</td>
<td>$y'_1$</td>
<td>$z'_1$</td>
<td>$y_2$</td>
<td>$z_2$</td>
<td>$y_3$</td>
<td>$z_3$</td>
</tr>
<tr>
<td>$u$</td>
<td>$-1$</td>
<td>$a_2'$</td>
<td>$-a_2'y_1'$</td>
<td>$b_2'$</td>
<td>$-b_2'y_2$</td>
<td>$c_2'$</td>
<td>$-c_2'y_3$</td>
<td></td>
</tr>
<tr>
<td>$y'_1$</td>
<td>$-1$</td>
<td>$au$</td>
<td>$a_0 y_3 + b_2 z_3$</td>
<td>$u_0 y_3 + \mu_2 z_3$</td>
<td>$-2 a_0 y_2 - \gamma_0 z_2$</td>
<td>$-b_2 y_2 - \mu_2 z_2$</td>
<td>$-a_2'y_1'$</td>
<td>$-b_2'y_2$</td>
</tr>
<tr>
<td>$z'_1$</td>
<td>$-1$</td>
<td>$a_0 y_3 + \eta_0 z_3$</td>
<td>$\sigma_0 y_3 + \delta_0 z_3$</td>
<td>$-a_0 y_2 - \sigma_0 z_2 + \rho' z_3$</td>
<td>$-\eta_0 y_2 - \delta_0 z_2 - \rho' y_3$</td>
<td>$-a_2'y_1'$</td>
<td>$-\eta_0 y_2 + \rho' y_3$</td>
<td></td>
</tr>
<tr>
<td>$y_2$</td>
<td>$-1$</td>
<td>$bu + \theta_2$</td>
<td>$b_0 y_1' + \lambda_0 z_1'$</td>
<td>$\beta_0 y_1' + \eta_0 z_1'$ + $\theta_2$</td>
<td>$b_2 y_1' + \lambda_2 z_1'$</td>
<td>$\beta_2 y_1' + \eta_2 z_1'$ + $\theta_2$</td>
<td>$a_2'y_1'$</td>
<td>$a_2'y_2 + \omega_2$</td>
</tr>
<tr>
<td>$z_2$</td>
<td>$-1$</td>
<td>$\gamma_0 y_1' + \sigma_0 z_1'$ + $\omega_2$</td>
<td>$\mu_0 y_1' + \delta_0 z_1'$ + $\theta_2 - \omega_2$</td>
<td>$a_2'y_1'$</td>
<td>$a_2'y_2 + \omega_2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y_3$</td>
<td>$-1$</td>
<td>$cu + \rho' z_1 + \omega_2$</td>
<td>$-1$</td>
<td>$a_2'y_1'$</td>
<td>$a_2'y_2 + \omega_2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$z_3$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$a_2'y_1'$</td>
<td>$a_2'y_2 + \omega_2$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where
\[ a_0 = \rho'^{-1}(\alpha\rho - \lambda\mu), b_0 = \rho'^{-1}(\beta\rho - \eta\nu), \gamma_0 = \rho'^{-1}(\gamma\rho - \sigma\nu), \mu_0 = \rho'^{-1}(\mu\rho - \delta\nu), \]
\[ \lambda_0 = \rho'^{-1}(\alpha\nu + \lambda\rho), \eta_0 = \rho'^{-1}(\beta\nu + \eta\rho), \sigma_0 = \rho'^{-1}(\gamma\nu + \sigma\rho), \delta_0 = \rho'^{-1}(\mu\nu + \delta\rho). \]
Thus $A$ is a division algebra if and only if: $b_0\lambda_0 - a_0\eta_0, b_0\gamma_0 - a_0\mu_0, \gamma_0\lambda_0 - a_0\sigma_0$ are all positive, $bc(\zeta_0\sigma_0 - b_0\eta_0 - \lambda_0\mu_0 + \gamma_0\eta_0)^2 + ac(\gamma_0\lambda_0 - a_0\sigma_0)\theta^2 + ab(b_0\lambda_0 - a_0\eta_0)\sigma^2 + b^2(b_0\gamma_0 - a_0\mu_0)\rho^2 < ab\zeta_0\theta^2 + 4bc(b_0\lambda_0 - a_0\eta_0) (\mu_0\sigma_0 - \gamma_0\delta_0)$, and one of the following situations hold:

1' $\theta = 0$,
2' $\theta \neq 0, c(\gamma_0\zeta_0 - a_0\sigma_0)\theta^2 + b(b_0\lambda_0 - a_0\eta_0)\sigma^2 > b\zeta_0\rho^2\theta$.

This gives $\beta\lambda - \alpha\eta, \gamma\lambda - \alpha\sigma$ are all positive, $bc(\zeta_0 - \beta\sigma - \lambda\mu + \eta\eta)^2 + b^2((\beta\gamma - \zeta\mu)\rho^2 + (\zeta\delta - \beta\sigma - \gamma\eta + \lambda\mu)\rho\sigma + (\sigma\eta - \lambda\delta)v^2) + ab((\beta\lambda - \alpha\eta)\sigma^2 + ac(\gamma\lambda - \alpha\sigma)\beta^2 < ab\zeta_0\theta^2 + 4bc(\beta\lambda - \alpha\eta)(\mu\sigma - \gamma\delta)$ and one of the two last cases of the theorem.

Case 2: $\pi$ arbitrary. We consider the automorphism $h$ of $W$ whose matrix with respect to the basis $\mathcal{B}_0$ is $I - \pi b^{-1}e_{13}$. The multiplication table of the algebra $A(h)$ is obtained from Table 1 by putting $\pi = 0$ and changing $\rho$ by $\rho - b^{-1}c\pi$. As $A$ is a division algebra if and only if $A(h)$ is a division algebra, the proof is finished after taking account the first case.

We state now the main result:

**Theorem 5.7.** The real quadratic flexible division algebras of dimension 8 are obtained from the Cayley–Dickson real algebra $O = (W, (\cdot), \wedge)$ by vectorial isotopy, and are isomorphic to $O(s)$, where $s$ is any positive definite symmetric automorphism of the Euclidian space $(W, -\langle \cdot, \cdot \rangle)$. Moreover, $O(s') \simeq O(s)$ ($s, s'$ being two positive definite symmetric automorphism of the Euclidian space $(W, -\langle \cdot, \cdot \rangle)$ if and only if there exists $f \in \mathcal{G}_2$ such that $s' = f^{-1}sf$ (i.e. $s'$ and $s$ are in the same orbit for the operation $\mathcal{G}_2 \times E \to E, (f, g) \mapsto f^{-1}g^f$ of the group $\mathcal{G}_2$ on the set $E$ of positive definite symmetric automorphisms of the Euclidean space $(W, -\langle \cdot, \cdot \rangle)$).

**Proof.** Taking into account Proposition 3.10, Lemma 5.1, Lemma 5.3, Lemma 5.5 and Theorem 5.6 we have shown that all of the real quadratic flexible division algebras of dimension 8 are obtained from the Cayley–Dickson real algebra $O = (W, (\cdot), \wedge)$ by vectorial isotopy. If $A = (W, (\cdot), \Delta)$ is a real quadratic flexible division algebra of dimension 8, then there is an automorphism $\varphi$ of $W$ such that $A = O(\varphi)$. By the polar decomposition theorem, $\varphi$ can be decomposed as a product $sr$ of a positive definite symmetric automorphism $s$ of the Euclidian space $(W, -\langle \cdot, \cdot \rangle)$ and an isometry $r$ of $(W, -\langle \cdot, \cdot \rangle)$. Thus $A = O(\varphi) = O(sr) = O(s)(r) \simeq O(s)$ (by 3.6.(1)). Let now $s$ and $s'$ be two positive definite symmetric automorphisms of the Euclidian space $(W, -\langle \cdot, \cdot \rangle)$ and an isometry $r$ of $(W, -\langle \cdot, \cdot \rangle)$. Then $O(s') \simeq O(s) \iff \exists \varphi \in \mathcal{G}_2(\mathbb{R})$ such that $s'\varphi s^{-1} \in \mathcal{G}_2$ (by Corollary 3.9 (3)) $\iff \exists f \in \mathcal{G}_2$ such that $s'^{-1}(f_{\wedge}W)s \in C_{7}(\mathbb{R}) \iff \exists f \in \mathcal{G}_2$ such that $s'^2 = (f_{\wedge}W)s^2(f_{\wedge}W) = ((f_{\wedge}W)^{-1}s(f_{\wedge}W))^2 \iff \exists f \in \mathcal{G}_2$ such that $s' = (f_{\wedge}W)^{-1}s(f_{\wedge}W)$. □
Corollary 5.8. The finite-dimensional real quadratic flexible division algebras of dimension $\geq 2$, are obtained from $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ by vectorial isotopy.

References