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Temporal stability of small disturbances in MHD Jeffery–Hamel flows

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Abstract

In this paper, the temporal development of small disturbances in magnetohydrodynamic (MHD) Jeffery–Hamel flows is investigated, in order to understand the stability of hydromagnetic steady flows in convergent/divergent channels at very small magnetic Reynolds number R_m. A modified form of normal modes that satisfy the linearized governing equations for small disturbance development asymptotically far downstream is employed [A. McAlpine, P.G. Drazin, On the spatio-development of small perturbations of Jeffery–Hamel flows, Fluid Dyn. Res. 22 (1998) 123–138]. The resulting fourth-order eigenvalue problem which reduces to the well known Orr–Sommerfeld equation in some limiting cases is solved numerically by a spectral collocation technique with expansions in Chebyshev polynomials. The results indicate that a small divergence of the walls is destabilizing for plane Poiseuille flow while a small convergence has a stabilizing effect. However, an increase in the magnetic field intensity has a strong stabilizing effect on both diverging and converging channel geometry.
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1. Introduction

Theoretical study of steady flow of an electrically conducting fluid in channels of varying width finds applications in engineering and biological systems, e.g. control of liquid metal flows, crystal growth, design of medical diagnostic devices which make use of the interaction of magnetic fields with tissue fluids, etc. In a pioneering work, Hartmann and Lazarus [1] investigated steady laminar flow between two parallel stationary and insulating plates under the influence of a transverse magnetic force. They observed that the flow rate decreases with increase in the magnetic field intensity. Since then, this work has received much attention and has been extended in numerous ways; see Makinde and Alagoa [2]. The monograph by Moreau [3] discusses some of these extensions and technological applications, and gives an ample survey of the literature.

The small disturbance stability of hydromagnetic steady flow between two parallel plates at a very small magnetic Reynolds number ($R_m \ll 1$) has been analysed by Lock [4], Makinde and Motsa [5] and Makinde [6] (for plane Poiseuille flow), Kakutani [7] (for plane Couette flow) and Makinde and Motsa [8] (for generalized plane Couette

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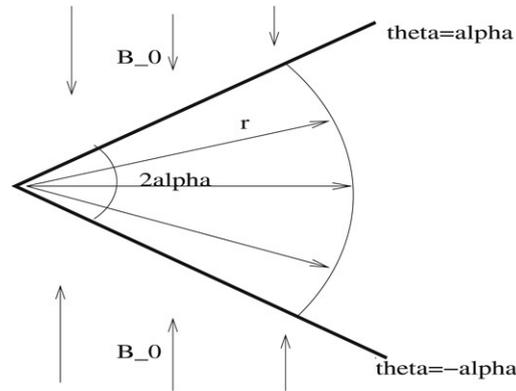
Fig. 1. Geometry of the problem.

flow). Their results show that magnetic field has stabilizing effects on the flow. Takashima [9,10] re-examined the problems treated by Lock [4] and Kakutani [7] for all values of the magnetic Reynolds number. His results show that for large flow Reynolds number R, the magnetic Reynolds number (R_m) may not be negligible and in that case the magnetic field has a destabilizing effect on the flow.

Meanwhile, the steady flow in convergent/divergent channels forms the classic theory of Jeffery and Hamel [11, 12]. Discussions on Jeffery and Hamel flows may be found in Batchelor [13] and Fraenkel [14]. The bifurcations and stability of these flows have been extensively studied; see for instance [15,16]. Furthermore, McAlpine and Drazin [17] used the idea of Tam [18] to present a new linear theory that achieves a separation of variables which is valid downstream. Their results indicate that 'a small divergence of the walls is an astonishingly strong destabilizing influence on plane Poiseuille flow and a small convergence has a strong stabilizing influence'.

The main objective of this study is to investigate the temporal development of small disturbances in hydromagnetic Jeffery–Hamel flows at small magnetic Reynolds number. This extends the theoretical study of McAlpine and Drazin [17] to MHD flows. The paper is structured as follows. In Section 2, the problem is formulated and the solution for the steady basic flow is obtained. The eigenvalue problem for temporal development of small disturbances is derived in Section 3. In Section 4, the Chebyshev spectral collocation numerical technique is employed to solve the resulting eigenvalue problem and the pertinent results are discussed quantitatively in Section 5.

2. The basic flow analysis

For an analytical study of Jeffery–Hamel MHD flows, we consider the two-dimensional flow of a viscous, incompressible and electrically conducting fluid in the presence of an imposed transverse homogeneous magnetic field. A very small magnetic Reynolds number R_m is assumed and consequently the induced electric and magnetic fields are neglected. Relativistic effects are also neglected and $\mathbf{J}$, the current density, is given by Ohm's law, $\mathbf{J} = \sigma(\mathbf{u} \times \mathbf{B})$, where σ is the electrical conductivity, $\mathbf{u}$ is the velocity field and $\mathbf{B}$ the magnetic flux density. Take plane polar coordinates (r, θ) such that the flow is driven between two impermeable planes at rest with angle $\theta = \pm\alpha$ by a steady line source or sink of strength Q at the intersection $r = 0$ of the two planes as shown in Fig. 1.

Now, in polar coordinates the velocity components are

$$u_r = \frac{\partial \psi}{r \partial \theta}, \qquad u_\theta = -\frac{\partial \psi}{\partial r}, \tag{1}$$

where ψ is a stream function. Then the equation governing two-dimensional motion is

$$\frac{\partial \omega}{\partial t} + \frac{1}{r} \frac{\partial(\psi, \omega)}{\partial(\theta, r)} = \nu \nabla^2 \omega + \frac{\sigma B_0^2}{\rho r^4} \frac{\partial^2 \psi}{\partial \theta^2}, \tag{2}$$

with

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad \text{(Ampere's law)}, \tag{3}$$

$$\nabla . \mathbf{B} = 0, \tag{4}$$

$$\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \quad \text{(Faraday's law)}, \tag{5}$$

$$\nabla . \mathbf{J} = 0, \tag{6}$$

where $\omega = -\nabla^2 \psi$ is the vorticity, $\nabla^2 = \partial^2/\partial r^2 + \partial/r\partial r + \partial^2/r^2\partial\theta^2$ the Laplacian, ρ the density, ν the kinematic viscosity, $\mathbf{E}$ the induced electric field, and μ_0 the magnetic permeability. The boundary conditions for no slip or penetration at the stationary walls with flux Q between the walls may be expressed as

$$\psi = \pm\frac{1}{2}Q, \quad \frac{\partial\psi}{\partial\theta} = 0 \quad \text{at } \theta = \pm\alpha. \tag{7}$$

Thus we seek a steady, purely radial, basic flow with velocity $u_r = \pm\frac{1}{2}Q\mathrm{d}\Psi/r\mathrm{d}\theta$ and stream function $\psi = \pm\frac{1}{2}Q\,\Psi(\theta)$. Then Eq. (2) becomes

$$\frac{\mathrm{d}^4\Psi}{\mathrm{d}y^4} + 4\alpha^2\left(1 - \frac{H^2}{4}\right)\frac{\mathrm{d}^2\Psi}{\mathrm{d}y^2} + 2R\alpha\frac{\mathrm{d}\Psi}{\mathrm{d}y}\frac{\mathrm{d}^2\Psi}{\mathrm{d}y^2} = 0, \tag{8}$$

where $y = \theta/\alpha$, $R = Q/2\nu$ is the flow Reynolds number, $H = \sqrt{\sigma B_0^2/\rho\nu}$ is the Hartmann number. The boundary conditions in Eq. (7) now become

$$\Psi = 1, \quad \frac{\mathrm{d}\Psi}{\mathrm{d}y} = 0 \quad \text{at } y = \pm1. \tag{9}$$

Note that when $H = 0$, Eqs. (8) and (9) reduce to the Jeffery–Hamel problem. For $0 < H < 2$, any solution Ψ can be mapped to a Jeffery–Hamel solution. However, such a mapping does not exist for $H > 2$.

In order to obtain the basic flow radial velocity $U(y) = \partial\Psi/\partial y$, it is convenient to seek a solution of Eqs. (8) and (9) as a power series of α, (i.e. for small semi-angle). Let

$$\Psi = \sum_{j=0}^{\infty} \alpha^j\,\Psi_j. \tag{10}$$

Substituting the expression (10) into Eqs. (8) and (9) and collecting terms of like powers of α, we obtain:

Zeroth order:

$$\frac{\mathrm{d}^4\Psi_0}{\mathrm{d}y^4} = 0,$$

$$\Psi_0 = 1, \quad \frac{\mathrm{d}\Psi_0}{\mathrm{d}y} = 0, \quad \text{at } y = \pm1. \tag{11}$$

First order:

$$\frac{\mathrm{d}^4\Psi_1}{\mathrm{d}y^4} + 2R\frac{\mathrm{d}\Psi_0}{\mathrm{d}y}\frac{\mathrm{d}^2\Psi_0}{\mathrm{d}y^2} = 0,$$

$$\Psi_1 = 0, \quad \frac{\mathrm{d}\Psi_1}{\mathrm{d}y} = 0, \quad \text{at } y = \pm1. \tag{12}$$

Second order:

$$\frac{\mathrm{d}^4\Psi_2}{\mathrm{d}y^4} + 4\left(1 - \frac{H^2}{4}\right)\frac{\mathrm{d}^2\Psi_0}{\mathrm{d}y^2} + 2R\left(\frac{\mathrm{d}\Psi_0}{\mathrm{d}y}\frac{\mathrm{d}^2\Psi_1}{\mathrm{d}y^2} + \frac{\mathrm{d}\Psi_1}{\mathrm{d}y}\frac{\mathrm{d}^2\Psi_0}{\mathrm{d}y^2}\right) = 0,$$

$$\Psi_2 = 0, \quad \frac{\mathrm{d}\Psi_2}{\mathrm{d}y} = 0, \quad \text{at } y = \pm1, \tag{13}$$

and so on. Eqs. (11)–(13) are solved to obtain

$$\Psi(y; \alpha, R, H) = \frac{1}{2}y(y^2 - 3) - \frac{3R\alpha}{280}y(y^2 - 5)(y - 1)^2(y + 1)^2$$
$$+ \frac{\alpha^2}{10}\left(1 - \frac{H^2}{4}\right)y(y - 1)^2(y + 1)^2 + \cdots \tag{14}$$

from which we obtain the fluid radial velocity as

$$U(y; \alpha, R, H) = \frac{3}{2}(y^2 - 1) - \frac{3\alpha R}{280}(y^2 - 1)(7y^4 - 28y^2 + 5)$$
$$+ \frac{\alpha^2}{10}\left(1 - \frac{H^2}{4}\right)(y^2 - 1)(5y^2 - 1) + \cdots. \tag{15}$$

In the limit of $\alpha = 0$, the flow becomes that of plane Poiseuille flow between two parallel plates.

3. The temporal stability analysis far downstream

We investigate the temporal stability of the basic steady flow given by Eq. (15) against small disturbances ϕ. Let the solution of Eq. (2) be given as

$$\psi = \frac{1}{2}Q(\Psi + \phi), \tag{16}$$

where $\Psi(\theta)$ is a solution of the MHD Jeffery–Hamel flow and $\phi(r, \theta, t)$ is the perturbation stream function. Substituting Eq. (16) into Eq. (2), we obtain

$$\frac{\partial}{\partial t}(\nabla^2\phi) - \frac{Q}{2}\left(\frac{2}{r^4}\frac{\partial\phi}{\partial\theta}\frac{d^2\Psi}{d\theta^2} - \frac{1}{r}\frac{d\Psi}{d\theta}\frac{\partial}{\partial r}(\nabla^2\phi) + \frac{1}{r^3}\frac{\partial\phi}{\partial r}\frac{d^3\Psi}{d\theta^3} - \frac{1}{r}\frac{\partial\phi}{\partial\theta}\frac{\partial}{\partial r}(\nabla^2\phi) + \frac{1}{r}\frac{\partial\phi}{\partial r}\frac{\partial}{\partial\theta}(\nabla^2\phi)\right)$$
$$- \nu\nabla^2(\nabla^2\phi) + \frac{\sigma B_0^2}{\rho r^4}\frac{\partial^2\phi}{\partial\theta^2} = 0. \tag{17}$$

Neglecting the non-linear terms in ϕ and scaling Eq. (17), we obtain

$$\nabla^2\eta = R\frac{\partial\eta}{\partial t} + \frac{2R}{r^4\alpha^3}\frac{\partial\phi}{\partial y}\frac{d^2\Psi}{dy^2} + \frac{R}{r\alpha}\frac{d\Psi}{dy}\frac{\partial\eta}{\partial r} + \frac{R}{r^3\alpha^3}\frac{\partial\phi}{\partial r}\frac{d^3\Psi}{dy^3} - \frac{H^2}{r^4\alpha^2}\frac{\partial^2\Psi}{\partial y^2}, \tag{18}$$

where r is scaled with an arbitrary length L, t with the inertial time $2L^2/Q$ and the vorticity perturbation $\eta = -\partial^2\phi/\partial r^2 - \partial\phi/r\partial r - \partial^2\phi/r^2\alpha^2\partial y^2$. The boundary conditions to Eq. (18) now become

$$\phi = \frac{\partial\phi}{\partial y} = 0 \quad \text{at } y = \pm 1. \tag{19}$$

Following McAlpine and Drazin [17], the stability problem in Eq. (18) is solved asymptotically for large r by taking modes of the form

$$\phi(r, y, t) = \text{Re}\left\{\exp(ik(\alpha^{-1}\log r - ct/\alpha^2 r^2))f(y)\right\}, \tag{20}$$

where k is a real wavenumber and c is the complex wave velocity. Substituting Eq. (20) into Eq. (18), equating coefficients of $r^{\lambda-4}$ and neglecting lower powers of r for fixed t, we obtain

$$f^{(iv)} - \left[k^2 + (k + 2i\alpha)^2 - H^2\alpha^2\right]f'' + k^2(k + 2i\alpha)^2 f$$
$$= iR\left\{[(k + 2i\alpha)U - kc](f'' - k^2 f) - kU''f + 2i\alpha U'f' + 4i\alpha kc(k + i\alpha)f\right\}, \tag{21}$$

with

$$f(y) = f'(y) = 0 \quad \text{at } y = \pm 1. \tag{22}$$

Note that when $H = 0$, Eq. (21) is equivalent to the one obtained by McAlpine and Drazin [17] and when $H = 0, \alpha \to 0$, Eq. (21) reduces to the well known Orr–Sommerfeld equation.

Eq. (21) together with the boundary conditions (22) furnish one eigenfunction $f(y)$ and one complex eigenvalue $c = c_r(\alpha, k, H, R) + ic_i(\alpha, k, H, R)$ for each set of values α, k, H and R. c_r represents the phase velocity of the prescribed disturbance and the sign of c_i determines whether the disturbance mode is amplified ($c_i > 0$) or damped ($c_i < 0$). For $c_i < 0$ the corresponding flow (U, R) is stable for the given values of α, k, H and $c_i > 0$ denotes instability. The limiting case $c_i = 0$ is a curve called the neutral stability curve and it separates the two regions.

4. Chebyshev spectral collocation method

The solution of the eigenvalue problem (21) and the boundary conditions (22) is expanded as a finite series of Chebyshev polynomials of the form

$$f(y) \approx f_N(y_j) = \sum_{k=0}^{N} \tilde{f}_k T_k(y_j), \quad j = 0, 1, \ldots, N, \tag{23}$$

where T_k is the kth Chebyshev polynomial defined by

$$T_0(y) = 1, \qquad T_1(y) = y, \qquad T_{k+1}(y) - 2yT_k(y) + T_{k-1}(y) = 0, \quad (-1 \le y \le 1), \tag{24}$$

and $y_0, y_1, \ldots, y_N$ are the Gauss–Lobatto collocation points (cf. Canuto et al. [19]) on $[-1, 1]$ defined by

$$y_j = \cos \frac{\pi j}{N}, \quad j = 0, 1, \ldots, N. \tag{25}$$

Note that the Chebyshev polynomials T_j in Eq. (23) do not satisfy the boundary conditions.

Substituting Eq. (23) into Eq. (21) and requiring that Eq. (21) be satisfied at the $N + 1$ collocation points, we obtain $N + 1$ algebraic equations for $N + 1$ unknowns $\tilde{f}_0, \tilde{f}_1, \ldots, \tilde{f}_N$:

$$\mathbf{EF} = c\mathbf{BF} \tag{26}$$

where

$$\mathbf{F}^{\mathrm{T}} = (\tilde{f}_0, \tilde{f}_1, \ldots, \tilde{f}_N) \tag{27}$$

is the transpose of the column vector $\mathbf{F}$. The clamped boundary conditions are incorporated explicitly in the first two and last rows of the matrices $\mathbf{E}$ and $\mathbf{B}$ by setting

$$\mathbf{E}(m, n) = \begin{cases} 1 & m = n = 0; \\ 0 & m = 0, \ n = 1, \ldots, N; \\ \displaystyle\sum_{n=0}^{N} D_{0n} & m = 1, \ n = 0, \ldots, N; \\ \tilde{\mathbf{E}}(m, n) & m = 1, \ldots, N-2, \ n = 0, \ldots, N; \\ \displaystyle\sum_{n=0}^{N} D_{Nn} & m = N-1, \ n = 0, \ldots, N; \\ 0 & m = N, \ n = 1, \ldots, N-1; \\ 1 & m = N, \ n = N; \end{cases} \tag{28}$$

$$\mathbf{B}(m, n) = \begin{cases} 0 & m = 0, 1, N-1, N \ n = 0, \ldots, N; \\ \tilde{\mathbf{B}}(m, n) & m = 2, \ldots, N-2, \ n = 0, \ldots, N; \end{cases} \tag{29}$$

where

$$\begin{aligned} \tilde{\mathbf{E}} &= \frac{1}{iR} \left(D^4 - \left[k^2 + (k + 2i\alpha)^2 - H^2\alpha^2 \right] D^2 + k^2(k + 2i\alpha)^2 \right. \\ &\quad \left. + \left\{ [(k + 2i\alpha)U](D^2 - k^2) - kU^{(2)} + 2i\alpha U^{(1)} D^1 \right\} \right) \end{aligned} \tag{30}$$

$$\mathbf{B} = -k(D^2 - k^2) + 4i\alpha k(k + i\alpha). \tag{31}$$
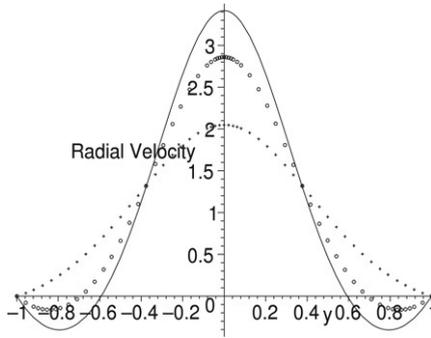
Fig. 2. Fluid radial velocity ($\alpha = \pi/16$; $R = 60$; $-H = 0$; $ooH = 10$; $++H = 20$).

Table 1
Computation showing the eigenvalue of the most unstable mode ($R = 5000$, $k = 1$, $H = 10$)

α	c (wave speed)
0.0000	$0.37483730488027 + 0.00335246344286i$
−0.0001	$0.37118860457946 - 0.00256622178157i$
−0.0005	$0.36265906205604 - 0.02325696160770i$
−0.0010	$1.29399389475173 - 0.03785722101752i$
0.0001	$0.37972132398091 + 0.009751266628183i$
0.0005	$0.42964257952766 + 0.06025461147833i$
0.0010	$0.43644982805659 + 0.35775534265427i$
0.0050	$-147359.4892131027 + 187954.7360134058i$
0.0100	$-256171820.8021082 + 330434285.5203600i$

where I is the $(N + 1) \times (N + 1)$ identity matrix, D is the usual differential matrix (cf. Canuto et al. [19]) and $\mathbf{U}, \mathbf{U}', \mathbf{U}''$ are $(N + 1) \times (N + 1)$ matrices whose only non-zero elements are the diagonal entries $U(y_0), U(y_1), \ldots, U(y_N)$; $U'(y_0), U'(y_1), \ldots, U'(y_N)$ and $U''(y_0), U''(y_1), \ldots, U''(y_N)$ respectively. Using this approach results in the matrix $\mathbf{B}$ being singular. The problem is avoided by employing the idea of Weidmann and Reddy [20] of using Hermite interpolating polynomials that satisfy the boundary conditions

$$\tilde{f}_0 = 0, \qquad \sum_{n=0}^{N} D_{0n} \tilde{f}_n = 0 \quad \text{0n } y = 1 \tag{32}$$

$$\tilde{f}_N = 0, \qquad \sum_{n=0}^{N} D_{Nn} \tilde{f}_n = 0 \quad \text{0n } y = -1. \tag{33}$$

5. Results and discussion

Various properties of the eigensolutions of the generalized eigenvalue problem (21) and (22) found numerically are presented in this section. The numerical solutions have been verified for correctness by comparing with the results obtained by McAlpine and Drazin [17] for $H = 0$. The primary flow $U(y)$ is computed from Eq. (15) and the Chebyshev spectral collocation method is implemented in MATLAB 5.1 to compute the fastest growing mode although there is no reason to believe that more than one mode of the present problem grows for given fixed values of α, k, R and H.

Fig. 2 shows the fluid radial velocity profiles at a semi-angle of $\alpha = \pi/16$ for increasing values of H. The profile for $H = 0$ (Jeffery–Hamel flow) shows flow reversal near both walls, i.e. internal boundary layer separation. An increase in the magnetic field intensity causes a general decrease in the fluid velocity around the centerline and suppresses flow reversal entirely.

The effect of a slight increase in the wall divergence semi-angle α at a fixed magnetic field intensity ($H = 10$) is illustrated in Table 1. The magnitude of the most unstable mode (c_i) increases with increasing positive values of α
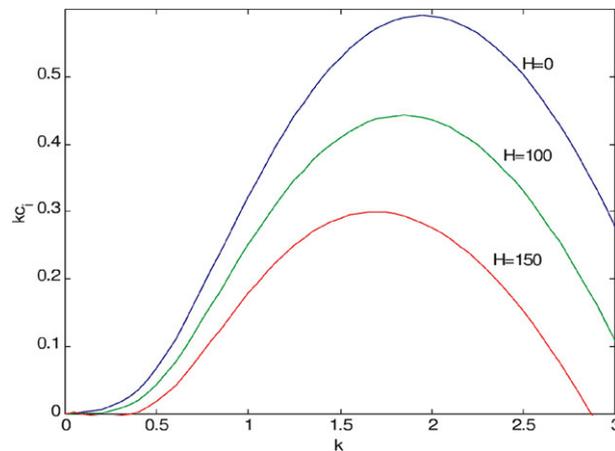
Table 2
Computation showing the eigenvalue of the most unstable mode ($R = 500$, $k = 1$, $\alpha = 0.01$)

H	c (wave speed)
0	$0.44918974044648 + 0.32268364334347i$
10	$0.44959684879326 + 0.32190366044857i$
20	$0.45081131037684 + 0.31957236459263i$
30	$0.45281268411908 + 0.31571557413544i$
40	$0.45556746572778 + 0.31037581750831i$
50	$0.45902990367750 + 0.30361159326864i$

Table 3
Computations showing the critical values at a fixed semi-angle ($\alpha = 0.01$)

H	R_c	k_c
0	200.6790	1.5020
10	200.9717	1.5020
20	201.8508	1.5000
30	203.3221	1.4980
50	208.0783	1.49090

Fig. 3. Growth rate kc_i for $R = 500$, $\alpha = 0.01$.

and decreases with increasing negative values of α. Table 2 shows the effect of increasing the magnetic field intensity on the flow. The most unstable mode (c_i) decreases with increasing value of H which implies that magnetic field has a stabilizing effect on the flow. This confirms that a slight divergence of the channel wall ($\alpha > 0$) has a destabilizing effect on the flow while a slight wall convergence ($\alpha < 0$) stabilizes the flow.

In order to obtain the neutral stability curve, the value of R for which c_i vanishes is sought. The lowest value of the flow Reynolds number on the neutral stability curve gives the critical Reynolds number, R_c, for the onset of instability in the flow field while its corresponding wavenumber gives the critical wavenumber k_c. The effect of increasing the magnetic field intensity at a fixed semi-angle of $\alpha = 0.01$ is to increase the critical Reynolds number and decrease the critical wavenumber as shown in Table 3. This means that the stable region in the (R, k) plane increases as the magnetic field intensity increases (i.e. $H > 0$) as shown in Fig. 4. This is consistent with the results in Table 2. On the other hand, a slight increase in the wall divergence semi-angle decreases the stable region as shown in Table 4.

The growth rate of small disturbances is shown in Fig. 3. It is interesting to note that increasing values of Hartmann number H have the effect of damping the disturbances. This means that the magnetic field acts like a control parameter that eliminates the growth of small disturbances in the flow field.

Table 4
Computations of the critical values at a fixed Hartmann number ($H = 10$)

α	R_c	k_c
0.0001	2819.8205	1.0510
0.0005	1495.2878	1.1260
0.0010	999.0548	1.1860
0.0050	331.6340	1.3940
0.0100	200.9717	1.5020
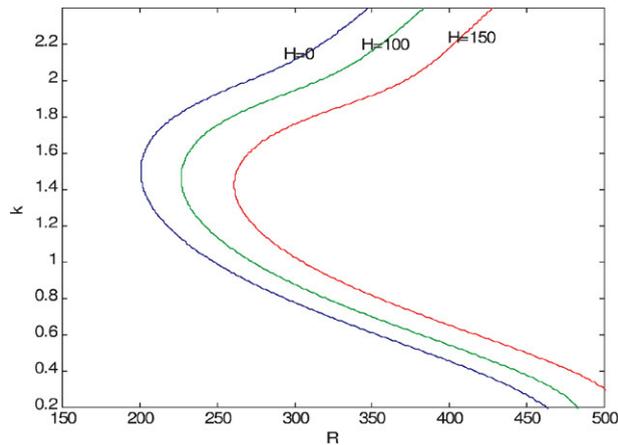
Fig. 4. Marginal stability curve for $\alpha = 0.01$.

Acknowledgements

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