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Solutions and Green's functions for some linear second-order three-point boundary value problems☆

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Abstract

In this paper we consider the Green's functions for a second-order linear ordinary differential equation with some three-point boundary conditions. We give exact expressions of the unique solutions for the linear three-point boundary problems by the Green's functions. As applications, we study the iterative solutions for some nonlinear singular second-order three-point boundary value problems.

Keywords: Green's function; Second-order differential equation; Three-point boundary value problem; Explicit solution; Iterative solution

1. Introduction

The Green's function plays an important role in solving boundary value problems of ordinary differential equations. The solutions of some boundary value problems for linear ordinary differential equations can be denoted by its Green's function, see [1–4]. Some boundary value problems for nonlinear differential equations can be transformed into the nonlinear integral equations the kernel of which are the Green's functions of corresponding linear differential equations. The integral equations can be solved to investigate the property of the Green's functions (see [5–9]). The concept, the significance and the development of Green's functions can be seen in [10]. The other study of second-order three-point boundary value problems can be seen in [11–18]. The solutions of the Green's functions diffuse in the literature, there is a lack of uniform method. The undetermined parametric method we use in this paper is a universal method, the Green's functions of many boundary value problems for ordinary differential equations can be obtained by the similar method.

We study respectively the Green's function for the second-order linear differential equation

$$u'' + f(t) = 0, \quad t \in [a, b], \tag{1.1}$$

satisfying the boundary conditions

$$u(a) = ku(\eta), \qquad u(b) = 0; \tag{1.2}$$

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$$u(a) = 0, \qquad u(b) = ku(\eta); \tag{1.3}$$
$$u(a) = ku'(\eta), \qquad u(b) = 0;$$
$$u(a) = 0, \qquad u(b) = ku'(\eta);$$

where k is a given number and $\eta \in (a, b)$ is a given point. In the above-mentioned literature, the boundary conditions all are of one kind (1.3), the boundary conditions of the other kind appear very little.

This paper is organized as follows. In Section 2, we study the Green's function for Eq. (1.1) satisfying the three-point boundary condition (1.2) and give the expression of the unique solution by the Green's function, that incarnate the general method of deriving the Green's function for a class of boundary problems. In Section 3, for some interrelated boundary conditions, we give the Green's function of the problems directly, omitting the particulars of derivation. The correctness of the Green's function needs only direct verification. As applications, in Section 4, we study the uniqueness of the solutions, the iterative methods and the rate of convergence by the iteration for a nonlinear singular second-order three-point boundary value problem.

2. The Green's function of Eq. (1.1) with the boundary condition (1.2)

We have the following conclusions.

Theorem 2.1. *Assume $k(b - \eta) \neq b - a$. Then the Green's function for the second-order three-point linear boundary value problem (1.1), (1.2) is given by*

$$G_1(t, s) = K(t, s) + \frac{k(b - t)}{b - a - k(b - \eta)} K(\eta, s), \tag{2.1}$$

where

$$K(t, s) = \begin{cases} \dfrac{(s - a)(b - t)}{b - a}, & a \le s \le t \le b \\ \dfrac{(t - a)(b - s)}{b - a}, & a \le t < s \le b. \end{cases} \tag{2.2}$$

Proof. It is well known that the Green's function is $K(t, s)$ as in (2.2) for the second-order two-point linear boundary value problem

$$\begin{cases} u'' + f(t) = 0, & t \in [a, b], \\ u(a) = 0, & u(b) = 0, \end{cases} \tag{2.3}$$

and the solution of (2.3) is given by

$$w(t) = \int_a^b K(t, s) f(s) \mathrm{d}s, \tag{2.4}$$

and

$$w(a) = 0, \qquad w(b) = 0, \qquad w(\eta) = \int_a^b K(\eta, s) f(s) \mathrm{d}s. \tag{2.5}$$

The three-point boundary value problem (1.1), (1.2) can be obtained from replacing $u(a) = 0$ by $u(a) = ku(\eta)$ in (2.3). Thus, we suppose the solution of the three-point boundary value problem (1.1), (1.2) can be expressed by

$$u(t) = w(t) + (c + dt)\, w(\eta), \tag{2.6}$$

where c and d are constants that will be determined.

From (2.5), (2.6) we know that

$$u(a) = (c + da) w(\eta),$$
$$u(b) = (c + db) w(\eta),$$
$$u(\eta) = (c + d\eta + 1) w(\eta).$$

Putting these into (1.2) yields

$$\begin{cases} c + da = k(c + d\eta + 1), \\ c + db = 0. \end{cases}$$

Since $k(b-\eta) \neq b-a$, solving the system of linear equations on the unknown numbers c, d, we obtain

$$\begin{cases} c = \dfrac{kb}{b-a-k(b-\eta)}, \\ d = \dfrac{-k}{b-a-k(b-\eta)}, \end{cases}$$

hence, $c + dt = \frac{k(b-t)}{b-a-k(b-\eta)}$. Putting this into (2.6), we obtain the solutions of (1.1) with the boundary condition (1.2) as

$$u(t) = w(t) + \frac{k(b-t)}{b-a-k(b-\eta)} w(\eta).$$

This together with (2.4) implies that

$$u(t) = \int_a^b K(t,s) f(s)\mathrm{d}s + \frac{k(b-t)}{b-a-k(b-\eta)} \int_a^b K(\eta,s) f(s)\mathrm{d}s.$$

Consequently, the Green's function $G_1(t,s)$ for the boundary value problem (1.1), (1.2) is as described in (2.1). $\square$

From Theorem 2.1 we obtain the following corollary.

Corollary 2.1. *If $k(b-\eta) \neq b-a$, then the second-order three-point linear boundary value problem*

$$\begin{cases} u''(t) + f(t) = 0, & t \in [a,b], \\ u(a) = ku(\eta), & u(b) = 0 \end{cases}$$

has an unique solution

$$u_1(t) = \int_a^b G_1(t,s) f(s)\mathrm{d}s,$$

where $G_1(t,s)$ as in (2.1).

Proof. we need only prove the uniqueness. Obviously, $u_1(t)$ satisfies

$$\begin{cases} u_1''(t) + f(t) = 0, & t \in [a,b], \\ u_1(a) = ku_1(\eta), & u_1(b) = 0. \end{cases} \tag{2.7}$$

Assume that $v(t)$ is also a solution of the three-point boundary value problem (1.1), (1.2), that is

$$\begin{cases} v''(t) + f(t) = 0, & t \in [a,b], \\ v(a) = kv(\eta), & v(b) = 0. \end{cases} \tag{2.8}$$

Let

$$z(t) = v(t) - u_1(t), \quad t \in [a,b]. \tag{2.9}$$

That combine with (2.7) and (2.8) implies

$$z''(t) = v''(t) - u_1''(t) = 0, \quad t \in [a,b],$$

therefore

$$z(t) = C_1 t + C_2, \tag{2.10}$$

where C_1, C_2 are undetermined constants. From (2.7), (2.8) and (2.9) we have

$$z(a) = v(a) - u_1(a) = kz(\eta), \tag{2.11}$$

$$z(b) = v(b) - u_1(b) = 0. \tag{2.12}$$

Using (2.10) we obtain

$$z(a) = C_1 a + C_2, \tag{2.13}$$

$$z(b) = C_1 b + C_2, \tag{2.14}$$

$$z(\eta) = C_1 \eta + C_2. \tag{2.15}$$

From (2.11), (2.13) and (2.15) we know that

$$C_1 a + C_2 = k(C_1 \eta + C_2), \tag{2.16}$$

and from (2.12) and (2.14) we know that

$$C_1 b + C_2 = 0. \tag{2.17}$$

Since $k(b-\eta) \neq b-a$, we know that the system (2.16), (2.17) of linear equations on the unknown numbers C_1, C_2, has exactly one solution $\{C_1 = 0, C_2 = 0\}$, therefore $z(t) \equiv 0, t \in [a,b]$, so $v(t) = u_1(t), t \in [a,b]$, that is uniqueness of the solution. □

Corollary 2.2. *Suppose the nonlinear function $g(t,u)$ is continuous on $[a,b] \times R$, then if $k(b-\eta) \neq b-a$, the nonlinear three-point boundary value problem*

$$\begin{cases} u'' + g(t,u) = 0, & t \in [a,b], \\ u(a) = ku(\eta), & u(b) = 0 \end{cases}$$

is equivalent to the nonlinear integral equation

$$u(t) = \int_a^b G_1(t,s) g(s,u(s)) \mathrm{d}s$$

with $G_1(t,s)$ as in (2.1).

If the endpoints of the interval are $a = 0$, $b = 1$ in the boundary condition, From Theorem 2.1, Corollaries 2.1 and 2.2 we obtain the following corollary.

Corollary 2.3. *If $k(1-\eta) \neq 1$, then the Green's function for the second-order three-point boundary value problem*

$$\begin{cases} u''(t) + f(t) = 0, & t \in [0,1], \\ u(0) = ku(\eta), & u(1) = 0 \end{cases} \tag{2.18}$$

is

$$G(t,s) = B(t,s) + \frac{k(1-t)}{1-k(1-\eta)} B(\eta,s), \tag{2.19}$$

where

$$B(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1 \\ t(1-s), & 0 \le t < s \le 1. \end{cases} \tag{2.20}$$

Hence the problem (2.18) has an unique solution

$$u(t) = \int_0^1 G(t,s) f(s) \mathrm{d}s.$$

If $g(t,u)$ is continuous in $[0,1] \times R$, then the nonlinear boundary value problem

$$\begin{cases} u'' + g(t,u) = 0, & t \in [0,1], \\ u(0) = ku(\eta), & u(1) = 0 \end{cases}$$

is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s) g(s,u(s)) \mathrm{d}s.$$

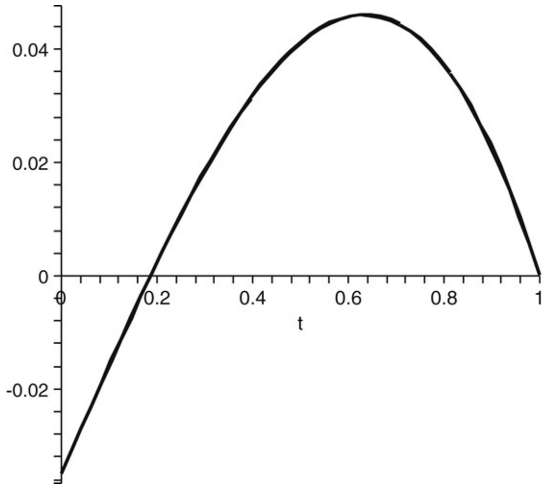

Fig. 1. Image of $u_1(t)$.

Example 1. The second-order three-point linear boundary value problem

$$\begin{cases} u''(t) + \sin(t) = 0, & t \in [0, 1], \\ u(0) = -\dfrac{3}{2}u\left(\dfrac{1}{3}\right), & u(1) = 0 \end{cases}$$

has an unique solution

$$u_1(t) = \sin(t) - \frac{5}{4}t\sin(1) - \frac{3}{4}\sin\left(\frac{1}{3}\right) + \frac{1}{4}\sin(1) + \frac{3}{4}t\sin\left(\frac{1}{3}\right), \quad 0 \le t \le 1. \tag{2.21}$$

It can be obtained by letting $\eta = \frac{1}{3}$, $k = -\frac{3}{2}$, $f(t) = \sin(t)$ in Corollary 2.3 that

$$u_1(t) = \int_0^1 B(t, s)\sin(s)\mathrm{d}s - \frac{3}{4}(1 - t)\int_0^1 B\left(\frac{1}{3}, s\right)\sin(s)\mathrm{d}s,$$

where $B(t, s)$ as in (2.20). Therefore, (2.21) is obtained by direct computation. Some properties of $u_1(t)$ are shown in the image (Fig. 1).

3. The related results for other boundary conditions

In this section, we give the Green's function and the conclusion for some boundary value problems, omitting the particular of derivation, since the proof is similar to that of Theorem 2.1.

Theorem 3.1. *If $k(\eta - a) \neq b - a$, then the Green's function for the second-order three-point boundary value problem*

$$\begin{cases} u''(t) + f(t) = 0, & t \in [a, b], \\ u(a) = 0, & u(b) = ku(\eta) \end{cases} \tag{3.1}$$

is

$$G_2(t, s) = K(t, s) + \frac{k(t - a)}{1 - k(\eta - a)}K(\eta, s),$$

hence linear boundary value problem (3.1) has an unique solution

$$u(t) = \int_a^b G_2(t, s)f(s)\mathrm{d}s,$$

where $K(t, s)$ as in (2.2).

If $g_2(t,u)$ is continuous on $[a,b]\times R$, then the nonlinear boundary value problem

$$\begin{cases} u'' + g_2(t, uv) = 0, & t \in [a,b], \\ u(a) = 0, & u(b) = ku(\eta) \end{cases}$$

is equivalent to the integral equation

$$u(t) = \int_a^b G_2(t,s) g_2(s, u(s)) \mathrm{d}s.$$

Remark 1. The expression of the Green's function in Theorem 3.1 is simpler, see [3,4,13–15], so that the unique solution of the linear problem is denoted easily, the format of the equivalent nonlinear integral equation is dapper and its property can be discussed conveniently.

Theorem 3.2. *If $b - a + k \neq 0$, then the Green's function for second-order three-point boundary value problem*

$$\begin{cases} u''(t) + f(t) = 0, & t \in [a,b], \\ u(a) = ku'(\eta), & u(b) = 0 \end{cases} \tag{3.2}$$

is

$$G_3(t,s) = K(t,s) + \frac{k(b-t)}{b-a+k} K_t(\eta, s),$$

therefore, linear boundary value problem (3.2) has an unique solution

$$u(t) = \int_a^b G_3(t,s) f(s) \mathrm{d}s,$$

where $K(t,s)$ as in (2.2) and

$$K_t(\eta, s) = \begin{cases} \dfrac{a-s}{b-a}, & a \le s < \eta, \\ \dfrac{b-s}{b-a}, & \eta < s \le b. \end{cases} \tag{3.3}$$

If $g_3(t,u)$ is continuous on $[a,b]\times R$, then the nonlinear boundary value problem

$$\begin{cases} u'' + g_3(t, u) = 0, & t \in [a,b], \\ u(a) = ku'(\eta), & u(b) = 0 \end{cases}$$

is equivalent to the integral equation

$$u(t) = \int_a^b G_3(t,s) g_3(s, u(s)) \mathrm{d}s.$$

Theorem 3.3. *If $k \neq b - a$, then the Green's function for second-order three-point boundary value problem*

$$\begin{cases} u''(t) + f(t) = 0, & t \in [a,b], \\ u(a) = 0, & u(b) = ku'(\eta) \end{cases} \tag{3.4}$$

is

$$G_4(t,s) = K(t,s) + \frac{k(t-a)}{b-a-k} K_t(\eta, s),$$

therefore, linear boundary value problem (3.4) has an unique solution

$$u(t) = \int_a^b G_4(t,s) f(s) \mathrm{d}s,$$

where $K(t,s)$ as in (2.2), $K_t(\eta,s)$ as in (3.3).

If $g_4(t,u)$ is continuous on $[a,b] \times R$, then the nonlinear boundary value problem

$$\begin{cases} u'' + g_4(t,u) = 0, & t \in [a,b], \\ u(a) = 0, & u(b) = ku'(\eta) \end{cases}$$

is equivalent to the integral equation

$$u(t) = \int_a^b G_4(t,s) g_4(s,u(s)) \mathrm{d}s.$$

Remark 2. The Green's functions in Theorems 3.2 and 3.3 are new results.

4. Applications in nonlinear singular boundary value problems

In this section, we study the iterative solutions for the following nonlinear three-point boundary value problem

$$\begin{cases} u'' + f(t,u) = 0, & t \in (0,1), \\ u(0) = ku(\eta), & u(1) = 0, \end{cases} \tag{4.1}$$

with $\eta \in (0,1), k(1-\eta) < 1$.

Let $J = (0,1)$, $I = [0,1]$, $R^+ = [0,+\infty)$,

$$D = \{x \in C(I) | \exists M_x \ge m_x > 0, \text{ such that } m_x(1-t) \le x(t) \le M_x(1-t), t \in I\}.$$

Concerning the function f we impose the following hypotheses:

$$\begin{cases} f(t,u) \text{ is nonnegative continuous on } J \times R^+, \\ f(t,u) \text{ is monotone increasing on } u, \text{ for fixed } t \in J, \\ \text{there exist } q \in (0,1) \text{ such that} \\ f(t,ru) \ge r^q f(t,u), \quad \forall 0 < r < 1, \ (t,u) \in J \times R^+. \end{cases} \tag{4.2}$$

Obviously, from (4.2) we obtain

$$f(t,\lambda u) \le \lambda^q f(t,u), \quad \forall \lambda > 1, \ (t,u) \in J \times R^+. \tag{4.3}$$

It is easy to see that if $0 < \alpha_i < 1$, $a_i(t)$ are nonnegative continuous on J, for $i = 0,1,2,\ldots,m$, then $f(t,u) = \sum_{i=1}^m a_i(t) u^{\alpha_i}$ satisfy the condition (4.2).

Concerning the boundary value problem (4.1), we have following conclusions.

Theorem 4.1. *Suppose the function $f(t,u)$ satisfy the condition (4.2), it may be singular at $t = 0$ and/or $t = 1$, and*

$$0 < \int_0^1 f(t, 1-t) \mathrm{d}t < \infty. \tag{4.4}$$

Then nonlinear singular boundary value problem (4.1) has an unique solution $w(t)$ in $C(I) \bigcap C^2(J)$. Constructing successively the sequence of functions

$$h_n(t) = \int_0^1 G(t,s) f(s, h_{n-1}(s)) \mathrm{d}s, \quad n = 1,2,\ldots, \tag{4.5}$$

for any initial function $h_0(t) \ge 0 \ (\not\equiv 0), t \in I$, then $\{h_n(t)\}$ must converge to $w(t)$ uniformly on I and the rate of convergence is

$$\max_{t \in I} |h_n(t) - w(t)| = O\left(1 - N^{q^n}\right), \tag{4.6}$$

where $0 < N < 1$, which depends on the initial function $h_0(t)$, $G(t,s)$ as in (2.19).

Proof. Let

$$P = \{x(t) | x(t) \in C(I), x(t) \geq 0\},$$

$$Fx(t) = \int_0^1 G(t,s) f(s, x(s)) \mathrm{d}s, \quad \forall x(t) \in D. \tag{4.7}$$

It is easy to see that the operator $F: D \to P$ is increasing; From Corollary 2.3 we know that if $u \in D$ satisfies

$$u(t) = Fu(t), \quad t \in I, \tag{4.8}$$

then $u \in C^1(I) \bigcap C^2(J)$ is a solution of (4.1).

For any $x \in D$, there exist positive numbers $0 < m_x < 1 < M_x$ such that

$$\begin{aligned} &m_x\,(1-s) \leq x(s) \leq M_x\,(1-s), \quad s \in I, \\ &(m_x)^q f(s, 1-s) \leq f(s, x(s)) \leq (M_x)^q f(s, 1-s), \quad s \in J. \end{aligned} \tag{4.9}$$

By (2.19), (2.20) we have

$$G(t,s) = B(t,s) + \frac{k(1-t)}{1-k(1-\eta)} B(\eta, s) \geq (1-t) \frac{k}{1-k(1-\eta)} B(\eta, s), \tag{4.10}$$

$$G(t,s) \leq t(1-t) + \frac{k(1-t)}{1-k(1-\eta)} B(\eta, s) \leq (1-t)\left(1 + \frac{k}{1-k(1-\eta)} B(\eta, s)\right). \tag{4.11}$$

Using (4.7), (4.3) and (4.9)–(4.11) and the conditions (4.2), we obtain

$$Fx(t) \geq (1-t)(m_x)^q \frac{k}{1-k(1-\eta)} \int_0^1 B(\eta, s) f(s, (1-s)) \mathrm{d}s, \quad t \in I, \tag{4.12}$$

$$\begin{aligned} Fx(t) &= \int_0^1 G(t,s) f(s, x(s)) \mathrm{d}s \\ &\leq (1-t)(M_x)^q \int_0^1 \left(1 + \frac{k}{1-k(1-\eta)} B(\eta, s)\right) f(s, 1-s) \mathrm{d}s, \quad t \in I. \end{aligned} \tag{4.13}$$

By (4.4), (4.12) and (4.13) we obtain

$$F: D \to D.$$

For any $h_0 \in D$, we let

$$\begin{aligned} l_{h_0} &= \sup\{l > 0 \mid l h_0(t) \leq (F h_0)(t), t \in I\}, \\ L_{h_0} &= \inf\{L > 0 \mid (F h_0)(t) \leq L h_0(t), t \in I\}, \end{aligned} \tag{4.14}$$

$$m = \min\left\{1, (l_{h_0})^{\frac{1}{1-q}}\right\}, \qquad M = \max\left\{1, (L_{h_0})^{\frac{1}{1-q}}\right\},$$

$$\begin{aligned} u_0(t) &= m h_0(t), \qquad u_n(t) = F u_{n-1}(t), \\ v_0(t) &= M h_0(t), \qquad v_n(t) = F v_{n-1}(t), \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{4.15}$$

Since the operator F is increasing, from (4.2), (4.14) and (4.15) we know that

$$u_0(t) \leq u_1(t) \leq \cdots \leq u_n(t) \cdots \leq v_n(t) \leq \cdots \leq v_1(t) \leq v_0(t), \quad t \in I. \tag{4.16}$$

For $t_0 = m/M$, from (4.2), (4.7) and (4.15), it can obtained by induction that

$$u_n(t) \geq (t_0)^{q^n} v_n(t), \quad t \in I, n = 0, 1, 2, \ldots. \tag{4.17}$$

From (4.16) and (4.17) we know that

$$0 \leq u_{n+p}(t) - u_n(t) \leq v_n(t) - u_n(t) \leq \left(1 - (t_0)^{q^n}\right) M h_0(t), \quad \forall n, p \tag{4.18}$$

so that there exist function $w(t) \in D$ such that

$$u_n(t) \to w(t), \qquad v_n(t) \to w(t), \quad \text{(uniformly on } I), \tag{4.19}$$

and

$$u_n(t) \le w(t) \le v_n(t), \quad t \in I, n = 0, 1, 2, \ldots. \tag{4.20}$$

From the operator F is increasing and (4.15) we have

$$u_{n+1}(t) = Fu_n(t) \le Fw(t) \le Fv_n(t) = v_{n+1}(t), \quad n = 0, 1, 2, \ldots.$$

This together with (4.19) and uniqueness of the limit imply that $w(t)$ satisfy (4.8), hence $w(t) \in C^1(I) \bigcap C^2(J)$ is a solution of (4.1).

From (4.5) and (4.15) and the operator F is increasing, we obtain

$$u_n(t) \le h_n(t) \le v_n(t), \quad t \in I, n = 0, 1, 2, \ldots, \tag{4.21}$$

thus, from (4.18), (4.20) and (4.21) we know

$$\begin{aligned} |h_n(t) - w(t)| &\le |h_n(t) - u_n(t)| + |u_n(t) - w(t)| \\ &\le 2|v_n(t) - u_n(t)| \le \left(1 - (t_0)^{q^n}\right) M |h_0(t)|, \end{aligned}$$

so that

$$\max_{t \in I} |h_n(t) - w(t)| \le \left(1 - (t_0)^{q^n}\right) M \max_{t \in I} |h_0(t)|.$$

So that (4.6) holds. □

From $h_0(t)$ which is arbitrary in D we know that $w(t)$ is the unique solution of the Eq. (4.8) in D. Suppose $w_1(t)$ is a $C^1(I) \bigcap C^2(J)$ solution of boundary value problem (4.1). Let

$$z(t) = w_1(t) - Fw(t), \quad t \in I.$$

Similar to the proof of (2.9) in Section 2 we obtain $w_1(t) = w(t)$, hence $w(t)$ is the unique solution of Eq. (4.1) in $C^1(I) \bigcap C^2(J)$.

Remark 3. If $f(t, u)$ is continuous on $I \times R^+$, then it is quite evident that the condition (4.4) holds. Hence the unique solution $w(t) \in C^2(I)$.

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