Dynamic susceptibility computations for thin magnetic films

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Abstract
This paper deals with the dynamic modelling of thin ferromagnetic layers, based on the coupling of Maxwell’s equations with the nonlinear Landau–Lifschitz–Gilbert law. A 2-D micromagnetic model is described which involves a FDTD code to determine equilibrium configurations and a finite element method to compute magnetostatic fields. Finally, after linearization, the susceptibility spectra of films supporting a weak-stripe-domain structure are computed and successfully compared to existing measurements without introducing any fitting parameter.

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1. Introduction
Magnetic materials are attractive for high frequency devices (nonreciprocal components like isolators or circulators, write heads, radar absorbing materials). For these applications, a precise knowledge of the frequency response of such magnetic materials in terms of dynamic susceptibility spectra is of primary interest. This turns out to be delicate in nontrivial cases. Indeed, except for homogeneous materials, the main features of spectra (number of resonances, resonance frequencies, intensities and linewidth) depend strongly on the equilibrium configuration of magnetization and on the exciting field orientation.

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Recently, the high-frequency susceptibility spectra of thin magnetic films have been computed and successfully compared [13] to existing measurements. In the same paper, the physical origin of the resonances was discussed, but the numerical tools developed for this purpose were not described. In the present article we propose a comprehensive presentation of the numerical method.

1.1. Magnetic susceptibility

The magnetic medium is represented as a macroscopic continuum characterized by the magnetization $M$. In the time domain, the normalized magnetization $m = M/|M|$ is assumed to obey the so-called Landau–Lifschitz–Gilbert (LLG) law:

$$\frac{\dot{m}}{\dot{t}} = |\gamma| H_T(M) \times m + \alpha m \times \frac{\dot{m}}{\dot{t}},$$

(1)

where $|\gamma|$ is the gyromagnetic ratio, $\alpha$ is the Gilbert damping constant and $H_T(M)$ is the total magnetic field. This field arises from a number of different contributions:

$$H_T(M) = H + H_e(M) + H_a(M).$$

The magnetic field $H$ is the external field. In the magnetostatic limit, this field is called the demagnetizing field, denoted by $H_m(M)$. The two other fields are respectively the exchange field $H_e(M)$ and the anisotropy field $H_a(M)$, as defined further in Section 2.1.

Assuming a given magnetic medium at equilibrium, and as far as weak harmonic excitations $h(e^{i\omega t})$ are considered, a frequency domain model can be derived from the above evolutionary problem. This leads to the introduction of a linear relationship between $M$ and $H$, with a magnetic susceptibility tensor $\chi$ defined by

$$M = \chi(\omega)H \ (B = (1 + \chi(\omega))H).$$

The motivation for susceptibility computation is related to the fact that $\chi''(\omega)$, the imaginary part of $\chi(\omega)$, contains potential information on the damping behavior of the material in the frequency domain. It also provides input data for 3-D electromagnetic codes, in which the magnetic coating can be modelled using an impedance boundary condition.

Nevertheless, major numerical difficulties may arise, mainly due to the feature of the equilibrium magnetization configuration.

1.2. Domain structure

In this work, submicronic magnetic layers are considered, and among them, perpendicular anisotropy layers, as illustrated in last sections (see Figs. 4, 5 and 7).

From the geometrical point of view, such layers can be assumed to be periodic along the $x$-axis (with period $p_o$), and constant along the $z$-axis. Plane $(Oxz)$ is considered as the plane of the layer. Along the $y$-axis, the thickness is finite and denoted $t$ (see Fig. 1).

From the magnetic point of view, several parameters are necessary to describe the layer; they will be recalled later. Strongly depending on these parameters, the equilibrium configurations is far from being “idealistic” (homogeneous, or “up-down” alternation of spins). It then seems impossible to derive an analytical theory to predict dynamic properties, as this was done in [14] or [12].
Moreover, if there exist large areas for which the magnetization distribution is rather uniform, there also exist “magnetic walls” where the variations of $M$ are sharp. Unfortunately the wall width can be much smaller than the period $p_0$, or the thickness $t$. It is easy to understand that a suitable discretization of the layer, that is to say a discretization fine enough to correctly catch the sharp variations of $M$, will generally lead to huge discrete problems.

This explains our choice for an explicit FDTD scheme, as described in [4], to discretize the Maxwell–LLG system. Other methods could have been used (see for instance [7] or [8]), but FDTD is the easiest one to implement and is believed to be more robust for long-time computations. Furthermore, FDTD is straightforward for taking into account the required boundary conditions.

Concerning susceptibility spectra computations, as far as we know, the only existing reference is [5], where a 3-D numerical tool is described. This tool is however expensive and actually not well adapted to the geometrical feature of thin films as introduced above. Specific developments for instance would be necessary in order to take into account periodicity along the $x$-axis or invariancy along the $z$-axis.

Although the final linear system that we build to compute the susceptibility tensor is highly dimensioned, it can still be solved using a direct method. Thus, we do not have to rely on iterative methods like in [5] for which fast products are often necessary.

The outline of the paper is as follows. The physical modelling is presented in Section 2 where all the different equations and mathematical definitions are introduced. A nonlinear Cauchy problem is defined and briefly analyzed. Concerning equilibrium, stationary states are presented as limit states of the evolutionary system. In Section 3, we show how to use a classical FDTD scheme to compute the stationary states. The speed-up of convergence is achieved by increasing the gyromagnetic ratio. The problem is then linearized and an harmonic approach is adopted in the remainder of the article. This requires the computation of a demagnetizing tensor; this can be done using a P1 finite element method as shown in Section 4.2. It is then easy in Section 4.3 to build the susceptibility tensor. Some results are finally presented in order to show the excellent agreement, without introducing any fitting parameters, between computed and measured spectra.

2. The physical modelling

All the necessary equations are discussed in this section, as well as some elementary mathematical properties.
2.1. Landau–Lifschitz–Gilbert equation

The time domain model is based on the LLG equation as repeated below

\[
\frac{\dot{m}}{\dot{t}} = |\gamma| H_T(M) \times m + \alpha m \times \frac{\dot{m}}{\dot{t}}.
\]

As mentioned in our introduction, \( M(x, t) = M_s m(x, t) \), where \( M_s \) is the saturation magnetization. It is well known that \( m \) remains normalized for all values of \( t \). Indeed, we have

\[
\frac{1}{2} \frac{d}{dt}(|m|^2) = m \cdot \frac{\dot{m}}{\dot{t}} = 0.
\]

(2)

This identity holds whatever \( H_T(M) \). Usually this total magnetic field is defined as \( H_T(M) = H + H_I(M) \), where \( H_I(M) \) is the linear local field

\[
H_I(M) = H_s + H_a(M) + H_e(M),
\]

which derives from an internal energy in the following sense:

\[
H_I(M) = -\nabla_M \mathcal{E}_I(M) = -\nabla_M [\mathcal{E}_s(M) + \mathcal{E}_e(M) + \mathcal{E}_a(M)].
\]

In more detail we can have:

1. A static field \( H_s \); the associated energy is: \( \mathcal{E}_s(M) = \mu_0 \|H_s - M\|^2 \). Note that all the examples presented in this paper are computed with \( H_s = 0 \).

2. The exchange field \( H_e(M) \) is proportional to the Laplacian of \( M \), the exchange constant \( A \) being positive:

\[
H_e(M) = -\frac{2A}{\mu_0 M_s^2} \Delta M = -\frac{2A}{\mu_0 M_s} \Delta m = A'M.\]

(3)

The exchange energy is then given by \( \mathcal{E}_e(M) = A\|\nabla m\|^2 \). As the definition of \( H_e \) requires second order space derivatives, it is necessary to introduce a specific condition on the boundary of the ferromagnet. Commonly, it is assumed that

\[
\frac{\partial M}{\partial n} = 0.
\]

(4)

3. Concerning the anisotropy field \( H_a(M) \), several definitions are possible. For the sake of simplicity, we shall retain a two-term definition. Let \( \mathbf{p} \) be a unit vector along the anisotropy direction, and \( P(m) \) be the projection on the plane perpendicular to \( \mathbf{p} \). Then we have:

\[
H_a(M) = -\frac{2K_u}{\mu_0 M_s} P(M) - \frac{2K_p}{\mu_0 M_s} (\mathbf{p} \cdot m)\mathbf{p}
= -K'_u P(M) - K'_p (\mathbf{p} \cdot m)\mathbf{p},
\]

(5)

where the uniaxial perpendicular anisotropy constant \( K_u \) and the planar anisotropy constant \( K_p \) are both positive. The related energy is

\[
\mathcal{E}_a(M) = K_u \|P(M)\|^2 + K_p \|\mathbf{p} \cdot m\|^2.
\]
2.2. A nonlinear Cauchy problem

The coupling between the LLG equation and Maxwell’s equations leads to, \( \forall t > 0, \)
\[
\begin{cases}
\frac{\partial E}{\partial t} - \text{curl} \ H + \sigma E = 0, \\
\frac{\partial H}{\partial t} + \text{curl} \ E = -\mu_0 M_s \frac{\partial m}{\partial t}, \\
\frac{\partial m}{\partial t} = |\gamma| H_T(M) \times m + \alpha m \times \frac{\partial m}{\partial t},
\end{cases}
\]
\( (6) \)

where \( E \) is the electric field, \( H \) the magnetic field, \( \varepsilon_0 \) and \( \mu_0 \) the vacuum electric permittivity and magnetic permeability, and \( \sigma \) the electrical conductivity. The magnetic coupling then reads
\[
B = \mu_0(H + M_s m).
\]

When associated with initial data \((E_0, H_0, M_0)\), these equations define a nonlinear Cauchy problem whose mathematical analysis is nontrivial. In this paper we take advantage of two mathematical properties.

Firstly, if the initial data are divergence free, these conditions hold for all time:
\[
(\text{div}(H_0 + M_0) = 0) \Rightarrow (\text{div}(H + M) = 0, \ \forall t \geq 0),
\]
\( (7) \)
and, if \( \sigma \) is constant everywhere (both within and outside the magnetic layer)
\[
(\text{div}E_0 = 0) \Rightarrow (\text{div}E = 0, \ \forall t \geq 0).
\]
\( (8) \)
The proof is straightforward and unchanged compared to the linear case.

Secondly, any solution \((E, H, M)\) to system \((6)\) is such that the total energy
\[
\delta(E, H, M) = \frac{\varepsilon_0}{2} ||E||^2 + \frac{\mu_0}{2} ||H||^2 + \delta_f(M)
\]
\( (9) \)
is decreasing with time. More precisely, it can be shown (see [4] or [8]) that
\[
\frac{d\delta(E, H, M)}{dt} = -\int_{\Omega} \frac{\gamma}{1 + \frac{\alpha}{\gamma} M_s^2} \left| \frac{\partial m}{\partial t} \right|^2 dx - \int_{\Omega} \sigma |E|^2 dx,
\]
\[
= -\int_{\Omega} \frac{\gamma}{1 + \frac{\alpha}{\gamma} M_s^2} M_s |H_T \times m|^2 dx - \int_{\Omega} \sigma |E|^2 dx.
\]
This energetic identity will be useful in Section 3.

2.3. Stationary states

Because of the energy decay, solutions to the above time problem quickly converge to stationary states, for which all the time derivatives vanish:
\[
\frac{\partial E}{\partial t} = 0, \quad \frac{\partial H}{\partial t} = 0, \quad \frac{\partial m}{\partial t} = 0.
\]
Plugging these identities in (6) leads to the following set of equations:

\[
\begin{align*}
\text{curl } \mathbf{H}_{eq} + \sigma \mathbf{E}_{eq} &= 0, \\
\text{curl } \mathbf{E}_{eq} &= 0, \\
\mathbf{H}^T(\mathbf{M}_{eq}) \times \mathbf{m}_{eq} &= 0.
\end{align*}
\]

Concerning fields at equilibrium \( \mathbf{E}_{eq}, \mathbf{H}_{eq} \) and \( \mathbf{M}_{eq} \), several conclusions are easy to draw.

(i) \( \mathbf{E}_{eq} = 0 \): the electric field vanishes at equilibrium, because then \( \mathbf{E}_{eq} \in L^2 \) is both curl-free (10) and divergence-free (8). As a major consequence, this shows, at least as long as stationary states are looked for, that time computations can be performed with an homogeneous Dirichlet boundary condition:

\[ \mathbf{E} \times \mathbf{n} = 0 \]

on the boundary of the calculus domain. Luckily, this is simpler and more robust than any other boundary condition. Besides, it provides a first criterion to test the convergence of the solution to the limit state (see Fig. 6).

(ii) \( \mathbf{H}_{eq} = \mathbf{H}_m(\mathbf{M}_{eq}) \): the magnetic field reduces to the so-called demagnetizing field, which is a solution to (see (10) and (7))

\[
\begin{align*}
\text{div } \mathbf{H}_m(\mathbf{M}_{eq}) &= -\text{div } \mathbf{M}_{eq} \\
\text{curl } \mathbf{H}_m(\mathbf{M}_{eq}) &= 0.
\end{align*}
\]

A direct computation of this field is described in Section 4.

(iii) \( \mathbf{H}_{eq} \times \mathbf{m}_{eq} = \mathbf{H}_m(\mathbf{M}_{eq}) \times \mathbf{m}_{eq} = 0 \): this identity provides a second criterion to test the convergence of the solution (see Fig. 6).

Remark 1. In the above definitions, it must be underlined that fields at equilibrium depend neither on the gyromagnetic ratio \( \gamma \), nor on the conductivity \( \sigma \). However, damping is governed by these two constants. Consequently, they can be slowly “increased” during the computation in order to speed-up the convergence to the stationary states. If it is rather classical to add some artificial conductivity, it is more “odd” to modify a physical constant like \( \gamma \). Actually, as explained in Section 3.3.2, the product \( \gamma \Delta t \) is increased, so that the method appears as an adaptative time-step algorithm.

3. Stationary state computation

3.1. Space discretization

The FDTD scheme defined in [4] can be used in any space dimension, but due to the geometric particularities of thin magnetic layers, the presentation is here restricted to the 2-D case. Let \( \Delta x \) and \( \Delta y \) be the space steps along the \( x \)- and the \( y \)-axis, respectively. The calculus domain \( \Omega \) consists of a magnetic layer \( \Omega_F \) (unbounded along the \( z \)-axis) surrounded by a vacuum. The number of rectangular cells is \( N_x \times N_y \) in \( \Omega_F \), and \( N_x' \times N_y' \) (\( N_y' > N_y \)) in the whole domain \( \Omega \) (see Fig. 1).

Let \( \mathbf{E}_h = (E_x, E_y, E_z), \mathbf{H}_h = (H_x, H_y, H_z) \) and \( \mathbf{M}_h = (M_x, M_y, M_z) \) be the discrete fields.
3.1.1. Maxwell’s equations discretization

The most popular FDTD scheme devoted to Maxwell’s equations is certainly the Yee’s scheme. Nevertheless, it is not adapted to our case where $H_h$ and $M_h$ must share the same degrees of freedom. For this reason a modification was proposed in [4]. In summary, it can be seen as a change of the discrete space for $H_h$.

In the Yee’s scheme, the electric field is piecewise linear: $E_h \in U_h = Q_{0,1} \times Q_{1,0} \times Q_{0,1}$. As shown in Fig. 2, this space, and consequently the degrees of freedom, remain unchanged in the new scheme. For the magnetic field the idea is to work with piecewise constant elements: $H_h \in V_h = (P_0)^3$ (instead of $H_h \in Q_{1,0} \times Q_{0,1} \times P_0$).

Degrees of freedom are then all taken at the center of the cells, as illustrated in Fig. 2. The new scheme is still a second order scheme; it is more dispersive, but this is a rather good point since propagating waves are of no interest in our study.

The semi-discrete equations are then unchanged for the transverse electric mode $(E_x, E_y, H_z)$:

$$
\begin{align*}
\frac{\partial E_x(i+\frac{1}{2}, j)}{\partial t} &= \frac{H_z(i+\frac{1}{2}, j+\frac{1}{2}) - H_z(i+\frac{1}{2}, j-\frac{1}{2})}{\Delta y}, \\
\frac{\partial E_y(i, j+\frac{1}{2})}{\partial t} &= \frac{H_z(i+\frac{1}{2}, j+\frac{1}{2}) - H_z(i-\frac{1}{2}, j+\frac{1}{2})}{\Delta x}, \\
\frac{\partial H_z(i)}{\partial t} &= \frac{E_x(i+\frac{1}{2}, j+\frac{1}{2}) - E_x(i+\frac{1}{2}, j)}{\Delta y} - \frac{E_x(i+1, j+\frac{1}{2}) - E_x(i, j+\frac{1}{2})}{\Delta x},
\end{align*}
$$

while the second half of the scheme, for the transverse electric mode $(H_x, H_y, E_z)$, reads:

$$
\begin{align*}
\frac{\partial H_x(i+\frac{1}{2}, j)}{\partial t} &= -\frac{E_z(i, j+1) + E_z(i+1, j+1) - E_z(i+1, j) - E_z(i, j)}{2\Delta y}, \\
\frac{\partial H_y(i, j+\frac{1}{2})}{\partial t} &= \frac{-E_z(i+1, j) + E_z(i+1, j+1) - E_z(i, j+1) - E_z(i, j)}{2\Delta x}, \\
\frac{\partial E_z(i)}{\partial t} &= \frac{H_y(i+\frac{1}{2}, j+\frac{1}{2}) + H_y(i+\frac{1}{2}, j-\frac{1}{2}) - H_y(i-\frac{1}{2}, j+\frac{1}{2}) - H_y(i-\frac{1}{2}, j-\frac{1}{2})}{2\Delta x}, \\
&\quad - \frac{H_x(i+1, j+\frac{1}{2}) + H_x(i+1, j-\frac{1}{2}) - H_x(i-\frac{1}{2}, j+\frac{1}{2}) - H_x(i-\frac{1}{2}, j-\frac{1}{2})}{2\Delta y}.
\end{align*}
$$

Remark 2. Even in the 2-D case, it is necessary to deal with all the equations which are coupled by the LLG equation.
3.1.2. Boundary conditions

Let \( \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \) be the boundary of the calculus domain \( \Omega \). According to the discrete space defined above, the boundary conditions are verified only by the electric field \( \mathbf{E}_h \).

The Dirichlet condition is assumed on \( \Gamma_2 \) and \( \Gamma_4 \); it reads

\[
E_x(i, j = 1, Ny) = 0, \quad E_z(i, j = 1, Ny) = 0, \quad \forall i.
\]

The periodic boundary condition holds on the vertical boundaries \( \Gamma_1 \) and \( \Gamma_3 \); it is computed following the image principle:

\[
E_y(i = 1, j) = E_y(i = Nx - 1, j), \quad E_y(i = Nx, j) = E_y(i = 2, j), \quad \forall j,
\]

and the same for \( E_z \).

3.1.3. LLG equation discretization

We take \( M_h \in V_h = (P_0)^3 \) because, as long as the exchange contribution is neglected, the LLG equation is an ordinary differential equation in time. So, choosing the degrees of freedom of \( M_h \) at the center of the cells, as for \( \mathbf{H}_h \), makes the “space discretization” rather straightforward.

As a consequence, we now only have to define a discrete Laplacian operator. A classical five point scheme is proposed, which yields, for \( i \in \{2, \ldots, Nx - 1\} \) and \( j \in \{2, \ldots, Ny - 1\} \),

\[
\mathbf{H}_e(i, j) = A'(\frac{M_{i+1,j} - 2M_{i,j} + M_{i-1,j}}{\Delta x^2} + \frac{M_{i,j+1} - 2M_{i,j} + M_{i,j-1}}{\Delta y^2}).
\]

We denote \( \Delta_h : V_h \rightarrow V_h \) the discrete operator such that \( \mathbf{H}_e := A\Delta_h M_h \). The definition must be completed on the four boundaries of \( \Omega_F \). On the vertical boundaries \( (i = 1 \text{ or } Nx) \), periodicity gives

\[
\mathbf{H}_e(1, j) = A'\left(\frac{M_{2,j} - 2M_{1,j} + M_{N_x,j}}{\Delta x^2} + \frac{M_{1,j+1} - 2M_{1,j} + M_{1,j-1}}{\Delta y^2}\right)
\]

and

\[
\mathbf{H}_e(Nx, j) = A'\left(\frac{M_{1,j} - 2M_{N_x,j} + M_{N_x-1,j}}{\Delta x^2} + \frac{M_{N_x,j+1} - 2M_{N_x,j} + M_{N_x,j-1}}{\Delta y^2}\right).
\]

For the Neuman condition (4) on the horizontal boundaries \( (j = 1 \text{ or } Ny) \), the image principle leads to consider fictitious points

\[
M(i, 0) = M(i, 2) \quad \text{and} \quad M(i, Ny + 1) = M(i, Ny - 1), \quad \forall i,
\]

so that

\[
\frac{M(i, 0) - M(i, 2)}{2\Delta y} = 0 = \frac{M(i, Ny + 1) - M(i, Ny - 1)}{2\Delta y}, \quad \forall i.
\]

This is clearly a second order discretization of (4), and the definition of the discrete exchange field on \( \Gamma_5 \) and \( \Gamma_6 \) is finally the same for all \( j \in \{1, \ldots, Ny\} \).
3.2. Time discretization

3.2.1. Scheme definition

We use a classical leap-frog scheme. Let \( \Delta t \geq 0 \) be the time step, the electric field is computed for each time step

\[ E^n_h = E_h(n\Delta t), \]

while the magnetic fields are computed for “half” time steps

\[ m^{n+1/2}_h \simeq m_h \left( \left( n + \frac{1}{2} \right) \Delta t \right) \quad \text{and} \quad H^{n+1/2}_h \simeq H_h \left( \left( n + \frac{1}{2} \right) \Delta t \right). \]

These fields are evaluated at time \( n\Delta t \) using half sums. The scheme is then as follows. Given \( H^{n-1/2}_h \), \( m^{n-1/2}_h \), \( B^{n-1/2}_h \) and \( E^n_h \), we successively compute:

(i) \( B^{n+1/2}_h \) from the first Maxwell’s equation:

\[ \frac{B^{n+1/2}_h - B^{n-1/2}_h}{\Delta t} = -\text{curl}_h \ E^n_h \quad (B^{n-1/2}_h = \mu_0(H^{n-1/2}_h + M_s m^{n-1/2}_h)); \] (14)

(ii) \( m^{n+1/2}_h \) from the LLG equation LLG:

\[ \frac{m^{n+1/2}_h - m^{n-1/2}_h}{\Delta t} = |\gamma| H^{n}_\text{Th} \times \frac{m^{n+1/2}_h + m^{n-1/2}_h}{2} + \gamma m^n_h \times \frac{m^{n+1/2}_h - m^{n-1/2}_h}{\Delta t}, \] (15)

with

\[ H^n_\text{Th} = \frac{1}{\mu_0} B^n_h + H_s + A \Delta_h m^n_h - K P(m^n_h) \quad \left( m^n_h = \frac{m^{n-1/2}_h + m^{n+1/2}_h}{2} \right). \] (16)

(iii) \( H^{n+1/2}_h \) from the magnetic coupling:

\[ H^{n+1/2}_h = \frac{B^{n+1/2}_h}{\mu_0} - M_s m^{n+1/2}_h, \] (17)

(iv) \( E^{n+1}_h \) from the second Maxwell’s equation:

\[ \frac{E^{n+1}_h - E^n_h}{\Delta t} = \text{curl}_h H^{n+1/2}_h. \] (18)

The main features of the scheme are (for further details, see [4] or [11]):

- second order accuracy in time (for a constant time step),
- conservation of the discrete norm of \( m_h \): \( |m^{n-1/2}_h| = |m^{n-1/2}_h| \),
- stability under a classical CFL condition: \( \frac{c_h}{\Delta t} \leq \sqrt{2} \) (where \( h = \min(\Delta x, \Delta y) \)).
This last property is ensured by a discrete energy decay: it is shown that

\[
U_{h}^{n+1/2}(E_{h}, H_{h}, M_{h}) = \frac{1}{2}(\mu_{0}|H_{h}^{n+1/2}|^{2} + \epsilon_{\omega}(E_{h}^{n}, E_{h}^{n+1})) + \mathcal{E}_{l}(m_{h}^{n+1/2})
\]

decreases at each time step, at least as long as \(\gamma\) is constant.

### 3.2.2. Computation

The scheme is implicit because of the nonlinearity of (15). An explicit computation is possible when \(A = 0\) (no exchange contribution, see [4]), but probably not in our case. For this reason we propose to use an iterative method. First we have to introduce some new notation:

\[
x = m_{h}^{n}, \quad x^{0} = m_{h}^{n-1/2}, \quad a = -\left(\frac{\gamma|\Delta t}{2}H_{\text{Th}}^{n} + \alpha m_{h}^{n-1/2}\right),
\]

so that (15) reads \(x + a \times x = x^{0}\). Then the unknown \(x\) can be computed as a function of \(a\):

\[
x = \frac{x^{0} + (a \cdot x^{0})a - a \times x^{0}}{1 + |a|^{2}}.
\]

Of course, this is only formal, since \(H_{\text{Th}}^{n}\), hence \(a\), depends on \(x\). Let us introduce \(\delta x = x - x^{0}\) to rewrite (16) as follows:

\[
H_{\text{Th}}^{n} = \left[\frac{1}{\mu_{0}}B_{h}^{n} + H_{s} + \frac{2A}{\mu_{0}M_{s}}\Delta_{h}x^{0} - \frac{2K}{\mu_{0}M_{s}}P(x^{0})\right]
\]

\[
+ \left[\frac{2A}{\mu_{0}M_{s}}\Delta_{h}\delta x - \frac{2K}{\mu_{0}M_{s}}P(\delta x)\right]
\]

\[
= H_{\text{Th}}^{n0} + \left[\frac{2A}{\mu_{0}M_{s}}\Delta_{h}\delta x - \frac{2K}{\mu_{0}M_{s}}P(\delta x)\right],
\]

where \(H_{\text{Th}}^{n0}\) is now a known quantity. Since we have \(a = a(\delta x)\), we propose the following iterative algorithm:

\[
(x)^{0} = x^{0}, \quad \delta x^{0} = 0,
\]

\[
\text{for } k \geq 1, \quad \left\{ \begin{array}{l}
x^{k} \text{ from (15) with } a(\delta x^{k-1}), \\
\delta x^{k} = x^{k} - x^{0}.
\end{array} \right.
\]

In practise, the number of iterations is controlled and kept equal to 3, as explained in the next section.

### 3.3. Static micromagnetic computations

#### 3.3.1. Weak-stripe domain structure

In this work, submicronic magnetic layers are considered, and among them, perpendicular anisotropy layers. For these layers, there exist meaningful normalized parameters (see [3]): the exchange length \(A\) and the quality factor \(Q\):

\[
A = \left(\frac{A}{2\pi M_{s}^{2}}\right)^{1/2} \quad \text{and} \quad Q = \frac{K_{u}}{2\pi M_{s}^{2}},
\]
where $M_s$ is the saturation magnetization, $A$ the exchange constant and $K_u$ the uniaxial perpendicular anisotropy constant. Once these parameters have been defined, equilibrium configurations can be studied in the plane $(Q, t/\Lambda)$, as illustrated in Fig. 3.

As explained in [3], there exists a phase transition line which divides the plane into two parts.

(1) Below the transition line, the magnetization remains in the plane layer ($M_y \simeq 0$). This is due to demagnetizing effects which are predominant as soon as the layer is thin enough. This type of distribution leads to classical dynamic properties with a single magnetic excitation in the permeability spectra.

(2) Above the transition line, the component perpendicular to the layer $M_y$ becomes significant and oscillates periodically along the $x$-axis to give rise to the so-called “stripe domains”. In that case, the dynamic susceptibility appears to be much richer (see [13]).

In the phase diagram two points located above the critical line are selected, denoted respectively A and B (Fig. 3); the reduced thickness is chosen equal respectively to $t/\Lambda = 10$ for sample A and $t/\Lambda = 60$ for sample B, which leads to a ratio $t/t_c$ (where $t_c$ is the critical thickness) nearly constant: $t/t_c = 2.2$ for sample A and $t/t_c = 2$ for sample B.

The other magnetic parameters are $4\pi M_s = 10,000$ G, $A = 1 \cdot 10^{-6}$ erg/cm, $\gamma = 1.94 \cdot 10^7$ Oe$^{-1}$ s$^{-1}$, $\alpha = 2.5 \cdot 10^{-2}$. Consequently the coordinates in the phase diagram are $Q = 0.05$ and $t/\Lambda = 60$ for sample A, and $Q = 0.5$ and $t/\Lambda = 10$ for sample B.

3.3.2. Space discretization and convergence speed-up

The computations were performed on different discretization grids. As shown in Fig. 4, the space step must be small enough to achieve a satisfactory precision. As a conclusion, using a $64 \times 32$ grid leads to reliable results for the static micromagnetic computations. The equilibrium configuration for sample B is given Fig. 5.
As explained before, $\|H \times M\|_\infty$ and $\|E\|$ are used to determine whether the distribution is converged or not. In Fig. 6 we see for instance that the cross product is decreased from more than $10^4$ to less than $5 \cdot 10^{-3}$.

These results are reached rather quickly: 3200 time steps for point A and 5600 for point B. To achieve this, the damping is controlled during the computation. As already said, it depends on conductivity $\sigma$, 
gyromagnetic ratio $\gamma$ and damping parameter $\alpha$. Furthermore, these three constants have no impact on the configuration at equilibrium (see Remark 1). Therefore they can be modified to speed-up the convergence. If $\alpha = 1$ is clearly an optimal value, it is not easy to choose the two other constants.

Concerning conductivity, the hyperbolicity of Maxwell's equations can be altered if $\sigma$ becomes predominant. In the numerical scheme, we take a moderate product $\sigma \Delta t = 0.05$ to work safely under the CFL condition. However, $\sigma$ is then huge because the time step is very small.

The case of $\gamma$, or more exactly of the product $|\gamma| \Delta t$, is a little bit different. The benefit of increasing it is balanced by the difficulty to solve (15). Indeed we have

$$x^k + \left\{ a + \frac{|\gamma| \Delta t}{2} \left[ \frac{2A}{\mu_0 M_s} \Delta_h (\delta^n m_h)^k - \frac{2K}{\mu_0 M_s} P((\delta^n m_h)^k) \right]\right\} \times x^k = x^0.$$
This system is explicitly solvable when $|\gamma| \Delta t = 0$, but requires many iterations when this product becomes large. Thus we propose to take the number of iterations $N_b$ as a criterion to determine $|\gamma| \Delta t$: when the computation is “easy” and $N_b \leq 2$, we increase the time step (in the discrete LLG equation only\(^1\)). When $N_b \geq 4$, it is decreased. This way the number of iterations is kept equal to three most of the time.

4. Susceptibility computation

In this section we explain how we compute susceptibility tensors in the frequency domain, after linearization of the LLG equation around an equilibrium state. A major part of the work is devoted to the definition of a magnetostatic or “demagnetizing” tensor. Numerical results are successfully compared to experimental measurements.

4.1. Basic ideas

Let $(m_{eq}, H_{eq})$ be an equilibrium configuration. A small perturbation $\delta h$ of this equilibrium is assumed, which yields a variation $\delta m$ of the magnetization. By definition, the susceptibility tensor is such that $\delta m = \chi \delta h$. In other words, the susceptibility computation can be achieved by determining $\delta m$ as a function of $\delta h$.

The computation is performed in the frequency domain. Plugging $m = m_{eq} + \delta m$ and $H_T = H_{eq} + \delta h + H_T(M_s \delta m)$ into the LLG equation, and assuming a harmonic time dependence $e^{i\omega t}$, we get

$$i\omega \delta m = [\gamma] [H_{eq} \times m_{eq} + H_{eq} \times \delta m + \delta h \times m_{eq} + H_T(M_s \delta m) \times m_{eq}$$
$$+ H_T(M_s \delta m) \times \delta m + \delta h \times \delta m] + iz\omega m_{eq} \times \delta m.$$  

The 0-order term vanishes ($H_{eq} \times m_{eq} = 0$ by definition) and second order terms are neglected; the linearized equation then reads:

$$i\omega \delta m + [\gamma] (H_{eq} - iz\omega m_{eq}) \times \delta m + [\gamma] m_{eq} \times H_T(M_s \delta m) = [\gamma] m_{eq} \times \delta h.$$  

To get a linear system in $\delta m$, three tensors are introduced, namely, for all $x \in \mathbb{R}^{3 N_x N_y}$,

$$D_1 x = [\gamma] m_{eq} \times x, \quad D_2 x = (|\gamma| H_{eq} - iz\omega m_{eq}) \times x \quad \text{and} \quad D_3 x = H_T(x).$$

With this notation, we get

$$(i\omega I + D_1 D_3 + D_2) \delta m = D_1 \delta h.$$  

(21)

We then have to solve this system in order to compute $\chi$; but before that, the construction of $D_3$ must be made explicit.

4.2. The demagnetizing tensor

This part is devoted to the demagnetizing contribution. It is rather straightforward to build $D_1$ and $D_2$, as well as the part of $D_3$ resulting from the exchange and anisotropy contributions (see the previous

\(^1\) As far as stationary states are concerned, we can use different time steps in the discrete Maxwell and LLG equations.
section). The work is divided into two steps: the computation of the magnetizing field created by a given magnetization distribution and the derived computation of the demagnetizing tensor.

4.2.1. The demagnetizing field

As \( \text{curl} \ H_m(M) = 0 \), the demagnetizing field can be derived from a scalar potential \( \varphi(M) \) which is a solution to a Laplace equation:

\[
H_m(M) = -\nabla \varphi(M), \quad \text{and} \quad -\Delta \varphi(M) = -\text{div} M.
\]

Here, choosing piecewise linear elements, we solve

\[
\int_{\Omega} \nabla \varphi_h \cdot \nabla \psi_h + b(\varphi_h, \psi_h) = \int_{\Omega} g_h \psi_h, \quad \forall \psi_h \in Q_{1,1}(\Omega). \tag{22}
\]

The method is well known, and it is not necessary to enter into details (see Appendix A). It only remains to derive the discrete demagnetizing field. As for \( m \), we want \( H_m \in (P_0)^2 \) (the gradient is a two component vector in the 2-D case). Unfortunately, \( \nabla \varphi_h \in Q_{1,0} \times Q_{1,0} \notin (P_0)^2 \). The potential must be projected onto \( (P_0)^2 \); using the trapezoidal formula, we get:

\[
\begin{bmatrix}
H_{mx}(i + \frac{1}{2}, j + \frac{1}{2}) = \frac{\varphi_{i+1,j+1} + \varphi_{i,j+1} - \varphi_{i,j+1} - \varphi_{i,j}}{2\Delta x}, \\
H_{my}(i + \frac{1}{2}, j + \frac{1}{2}) = \frac{\varphi_{i+1,j+1} + \varphi_{i,j+1} - \varphi_{i+1,j} - \varphi_{i,j}}{2\Delta y}.
\end{bmatrix}
\]

4.2.2. The demagnetizing tensor

We now build the complete tensor \( D_m \). We work on the same regular mesh as above. Given two points with coordinates \((i, j)\) and \((m, n)\), we introduce the \( 3 \times 3 \) elementary susceptibility tensor \( D_{m,n}^{i,j} \) which gives the demagnetizing field at the second point, created by a magnetization located at the first point:

\[
H_m(m, n) = D_{m,n}^{i,j} M(i, j). \tag{23}
\]

In the 2-D case, the last line and the last column of \( D_{m,n}^{i,j} \) are null, because the divergence does not depend on the third component and the third component of the gradient is zero.

Let now \( L_k \) and \( L_l \) be two lines of the mesh. The \( 3N_x \times 3N_x \) susceptibility tensor \( D_k^l \) between \( L_k \) and \( L_l \) is defined by:

\[
D_k^l = [D_{k,j}^l]_{(j,n) \in \{1, \ldots, N_x\}^2}, \quad \forall (k, l) \in \{1, \ldots, N_y\}^2. \tag{24}
\]

The whole tensor is then given by

\[
D_m = [D_k^l]_{(k,l) \in \{1, \ldots, N_y\}^2}. \tag{25}
\]

In practice, the construction is achieved on a \( N_x \times (2N_y - 1) \) augmented domain. Lines are numbered from \( L_{-N_y+2} \) to \( L_{N_y} \). The magnetic layer is still delimited between lines \( L_1 \) and \( L_{N_y} \). We then operate in three steps.

Step 1: Elementary tensors \( D_{1,1}^{m,n} \) are computed for \((m, n) \in \{1, \ldots, N_x\} \times \{1, \ldots, N_y\}\). The FEM code described in the previous section is used twice with null data but at point \((1, 1)\): \( M_1(0, 0) = (1, 0, 0)^t \) and \( M_2(0, 0) = (0, 1, 0)^t \).
Step 2: The tensor associated to the first line \( D^1_1 \) (for \( l \in \{ -N_y + 2, \ldots, N_y \} \)) is then derived from periodicity:

\[
D_{1, j}(l, n) = D_{1, 1}(l, (n - j)[N_x]).
\]

Step 3: It remains to compute \( D^k_l \) for all \( k \in \{ 1, \ldots, N_y \} \). This is possible since the demagnetizing action between two lines only depends on their distance, hence

\[
D^k_l = D^{l-k+1}_1, \quad (k, l) \in \{ 1, \ldots, N_y \}^2 \quad \text{(then \( l - k + 1 \in \{ -N_y + 2, \ldots, N_y \} \))}.
\]

This completes the description of the demagnetizing tensor \( D_m \).

4.3. Susceptibility spectra

For an \( N_x \times N_y \) grid, \( \delta m \) and \( \delta h \) are \( 3N \) vectors with \( N = N_x N_y \), and the linear dense system (21) is of size \( 3N \times 3N \). The solution \( \delta m \) is obtained by solving the linear system (21) using a direct method (Gauss factorization, and forward and backward elimination). The scalar dynamic susceptibility is then given by:

\[
\chi = \frac{1}{N} \sum_{i=1}^{N} \frac{\delta m_i \cdot \delta h_i}{|\delta h_i|^2}.
\]

This procedure is repeated for each frequency. The computations were performed on a parallel computer (14 processors) with the treatment of one frequency per processor. This leads to spectra of the dynamic susceptibility (more precisely, the imaginary part \( \chi'' \) of the susceptibility) for the three principle exciting directions: \( \delta h \) applied along the \( x \)-axis (\( x \)-configuration), \( y \)-axis (\( y \)-configuration) and \( z \)-axis (\( z \)-configuration). On a \( 64 \times 32 \) grid, it takes approximately 10 h to compute 300 points in the frequency domain. The susceptibility spectra in the frequency range 100 MHz to 30 GHz were first computed with a frequency step \( \Delta f = 100 \) MHz in order to locate the micromagnetic excitations. A refinement (\( \Delta f = 10 \) MHz) was then performed around the detected resonance frequencies.

The theoretical results concerning point B have been compared with experimental data obtained from microwave permeability. In the phase diagram, this film is located in the vicinity of sample B (\( Q = 0.05, \frac{t}{\Lambda} \simeq 68 \)). The in-plane microwave permeability recorded in the 10 MHz to 6 GHz frequency range along the in-plane hard axis \( x \) (\( x \)-configuration) and the easy axis \( z \) (\( z \)-configuration) are reported respectively in Figs. 4a and b. The experimental spectra are characterized by the existence of multiple well-resolved resonances. A very good agreement is found between the experimental and theoretical spectra. Other results are given in [13] (Fig. 7).

5. Conclusion

A 2D-dynamic micromagnetic code has been developed and used to analyze the magnetic excitations of thin films with stripe domains. Comparisons between theoretical and experimental spectra are very good. Static micromagnetic computations are performed very efficiently following an elementary FDTD method and increasing when necessary the gyromagnetic ratio \( \gamma \). When equilibrium configurations are given, the problem is linearized and considered in the frequency domain. The demagnetizing field is
computed using a precise FEM method. Large dense system must be then solved, and the computations are heavier. This approach would be probably too expensive in the 3-D case, where it seems necessary to work with unstructured meshes to decrease the computational size of the problem.

Appendix A. Exact boundary condition

As we said, the demagnetizing field can be derived from a scalar potential $\varphi(\mathbf{M})$ which is a solution to a Laplace equation:

\begin{equation}
\mathbf{H}_m(\mathbf{M}) = -\nabla \varphi(\mathbf{M}), \quad \text{and} \quad -\Delta \varphi(\mathbf{M}) = -\text{div} \mathbf{M}.
\end{equation}

A classical variational formulation of the Laplace equation is

\begin{equation}
\text{Find } \varphi \in H^1_p(\Omega) \text{ such that } \\
\int_{\Omega} \nabla \varphi \nabla \psi \, d\mathbf{x} - \int_{\Gamma} (\nabla \varphi \cdot \mathbf{n}) \psi \, d\mathbf{l} = \int_{\Omega} \mathbf{M} \cdot \nabla \psi \, d\mathbf{x}, \quad \forall \psi \in H^1_p(\Omega), \tag{A.1}
\end{equation}

where $H^1_p(\Omega)$ is the subspace of $H^1(\Omega)$ of functions periodic along the $x$-axis. It can be noted that the field is computed on the whole calculus domain since the Laplace equation is nonlocal in space.

On the vertical boundaries, periodicity yields

\begin{equation}
\left( \frac{\partial \varphi}{\partial x} \big|_{\Gamma_3} = - \frac{\partial \varphi}{\partial x} \big|_{\Gamma_1} \right) \Rightarrow \left( \int_{\Gamma_1} \frac{\partial \varphi}{\partial n} \psi + \int_{\Gamma_3} \frac{\partial \varphi}{\partial n} \psi = 0, \forall \psi \in H^1_p(\Omega) \right). \tag{A.2}
\end{equation}

On the horizontal boundaries we use a DtN condition

\begin{equation}
\frac{\partial \varphi}{\partial n} + T \varphi = 0, \tag{A.2}
\end{equation}
with

\[
\varphi = \sum_{n \in \mathbb{Z}} \varphi_n(y) \frac{e^{i\frac{2\pi n x}{L}}}{\sqrt{L}} \mapsto T u = \sum_{n \in \mathbb{Z}^*} \frac{2|n|\pi}{L} \varphi_n(y) \frac{e^{i\frac{2\pi n x}{L}}}{\sqrt{L}}.
\]  

(A.3)

This condition is exact. Indeed, the scalar potential \( \varphi \) is periodic along the \( s \)-axis. Denoting \( L \) the period (instead of \( p_o \) for a magnetic film), we have the Fourier expansion:

\[
\varphi(x, y) = \sum_{n \in \mathbb{Z}} \varphi_n(y) \frac{1}{\sqrt{L}} e^{i\frac{2\pi n x}{L}}, \quad \text{with} \quad \varphi_n(y) = \frac{1}{\sqrt{L}} \int_0^L \varphi(x, y)e^{-i\frac{2\pi n x}{L}} \, dx.
\]

As \( \Gamma_4 \) and \( \Gamma_2 \) are outside \( \Omega_F \), \( \varphi \) is harmonic:

\[
\Delta \varphi(x, y) = \frac{\partial^2 \varphi}{\partial x^2}(x, y) + \frac{\partial^2 \varphi}{\partial y^2}(x, y) = 0.
\]

Hence

\[
\sum_n \frac{1}{\sqrt{L}} \left( -\frac{4n^2\pi^2}{L^2} \varphi_n(y) + \frac{\partial^2 \varphi_n(y)}{\partial y^2} \right) e^{i\frac{2\pi n x}{L}} = 0,
\]

and the Fourier coefficients are solutions to ordinary differential equations:

\[
\varphi_n''(y) - \frac{4n^2\pi^2}{L^2} \varphi_n(y) = 0, \quad \forall n \in \mathbb{N}.
\]

This shows that

\[
\varphi_n(y) = C_n \exp \left( \pm \frac{2|n|\pi}{L} y \right). \quad \text{(A.4)}
\]

We keep \( L^2 \)-solutions, that is to say with \(-|n|\) when \( y \geq 0 \) and with \(|n|\) when \( y \leq 0 \). We complete the computation assuming that \( y \geq 0 \). As

\[
\varphi(x, y) = \frac{1}{\sqrt{L}} \sum_n C_n \exp \left( -\frac{2|n|\pi}{L} y \right) e^{i\frac{2\pi n x}{L}},
\]

we have on \( \Gamma_2 \)

\[
\frac{\partial \varphi}{\partial n}(x, y) = \frac{\partial \varphi}{\partial y}(x, y) = \frac{1}{\sqrt{L}} \sum_n C_n \left( -\frac{2|n|\pi}{L} \right) \exp \left( -\frac{2|n|\pi}{L} y \right) e^{i\frac{2\pi n x}{L}},
\]

or

\[
\frac{\partial \varphi}{\partial n}(x, y) = \sum_n T_n \left( \varphi_n(y) \frac{1}{\sqrt{L}} e^{i\frac{2\pi n x}{L}} \right) \quad \text{with} \quad T_n = -\frac{2|n|\pi}{L},
\]

hence the definition (A.3). Moreover, \( T_n \leq 0 \) yields the uniqueness of the solution up to a constant, which has of course no impact on the gradient.
To complete the definition of discrete Laplace problem, we see that formulation (A.1) is then equivalent to
\[
\text{Find } \varphi \in H^1_p(\Omega) \text{ so that } \\
\int_{\Omega} \nabla \varphi \nabla \psi \, dx + \int_{\Gamma} T \varphi \cdot \psi \, dl = \int_{\Omega} M \cdot \nabla \psi \, dx, \quad \forall \psi \in H^1_p(\Omega). \tag{A.5}
\]
Finally, choosing piecewise linear elements, we get (22).

References


Further Reading