JOURNAL OF COMBINATORIAL TEORY, Series A **49**, 218–232 (1988)

Almost Resolvable Decompositions of $2K_n$ into Cycles of Odd Length

KATHERINE HEINRICH*

Department of Mathematics, Simon Fraser University, Burnaby, British Columbia, Canada V5A 1S6

AND

C. C. LINDNER[†] AND C. A. RODGER

Department of Algebra, Combinatorics and Analysis, Division of Mathematics, Auburn University, Auburn, Alabama 36849

Communicated by the Managing Editors

Received May 4, 1987

We prove that there exists an almost resolvable m-cycle decomposition of $2K_{ms+1}$ for all odd $m \geq 3$ and all $s \geq 1$. © 1988 Academic Press, Inc.

1. Introduction and Notation

The *complete graph* on n vertices is denoted by K_n and the graph on n vertices in which each pair of vertices is joined by exactly 2 edges is denoted by $2K_n$. An *m-cycle* is a sequence of m distinct vertices $(u_1, u_2, ..., u_m)$ such that u_i is adjacent to u_{i+1} and u_m is adjacent to u_1. A *spanning* subgraph H of G is a subgraph for which $V(H) = V(G)$. An *i-factor* of a graph G is a spanning subgraph of G that is regular of degree i (so each component of a 2-factor is a cycle).

An *m-cycle decomposition* of a graph G is defined to be an ordered pair $(G, C(m))$, where $C(m)$ is a collection of edge-disjoint m-cycles which partition the edge set $E(G)$ of G. An m-cycle decomposition is *resolvable* if the m-cycles in $C(m)$ can be partitioned into 2-factors of G.

The Oberwolfach problem was first formulated by Ringel and was first mentioned in [8]. The part of this problem which has attracted the most

* Research supported by NSERC Grant A7829.

[†] Research supported by NSF Grant DMS-8401263.

0097-3165/88 $3.00

attention so far, asks whether it is possible to find resolvable m-cycle decompositions of K_n when n is odd or of $K_n - F$ (see [11]) when n is even, where F is a 1-factor of K_n. Clearly a necessary condition for a resolvable m-cycle decomposition to exist is that m divides n.

When $m = 3$ and n is odd then the solution of the Oberwolfach problem is equivalent to finding a Kirkman triple system of order n for all $n \equiv 3 \pmod 6$, which was settled by Ray-Chadhuri and Wilson [17]. When $m = 3$ and n is even the solution of the Oberwolfach problem is equivalent to finding a nearly Kirkman triple system for all $n \equiv 0 \pmod 6$. Such systems do not exist when n is 6 or 12, but otherwise do exist [12].

In [1], this problem has now been solved for all even $m \geqslant 4$ and in [2] the problem has been solved for all odd $m \geqslant 5$ except possibly when $n = 4m$. For early results, see [9, 11, 13] and see [10] for a history of early results with some improvements. The more general formulation of the Oberwolfach problem in which not all cycles need have the same length has yet to be solved.

Define a subgraph H of G to be an *almost parallel class* if for some vertex v, H is a 2-factor of $G - \{v\}$. In this case v is called the *deficiency* of the almost parallel class. An m-cycle decomposition $(G, C(m))$ is *almost resolvable* if $C(m)$ can be partitioned into almost parallel classes.

A natural extension of the Oberwolfach problem is to ask whether or not it is possible to find an almost resolvable m-cycle decomposition of K_n. A simple counting argument shows that there is *never* such a decomposition when $n > 1$ (since each almost parallel class accounts for $n - 1$ edges, so $n - 1$ must divide $n(n-1)/2$, but n is odd).

However, almost resolvable m-cycle decompositions of $2K_n$ do exist.

EXAMPLE 1.1. $n = 7$ and $m = 3$;

$$C(3) = \{(i, i+1, i+3), (i+2, i+4, i+5) \mid 0 \leqslant i \leqslant 6\},$$

where each term is reduced modulo 7. Each almost parallel class consists of the 3-cycles shown and has deficiency $i + 6$.

EXAMPLE 1.2. $n = 6$ and $m = 5$:

Almost parallel class	Deficiency
$(0, 1, 2, 3, 4)$	∞
$(\infty, 0, 1, 4, 2)$	3
$(\infty, 1, 2, 0, 3)$	4
$(\infty, 2, 3, 1, 4)$	0
$(\infty, 3, 4, 2, 0)$	1
$(\infty, 4, 0, 3, 1)$	2

EXAMPLE 1.3. $n=11$ and $m=5$;

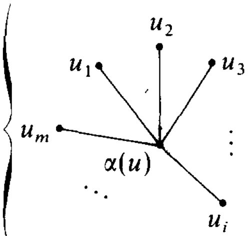
$$C(5)=\{(i, i+1, i+4, i+8, i+2),\\ (i+7, i+6, i+3, i+10, i+5) \mid 0 \leqslant i \leqslant 10\},$$

where each term is reduced modulo 11. Each almost parallel class consists of the two 5-cycles shown, and has deficiency $i+9$.

An obvious necessary condition for $(2K_n, C(m))$ to be almost resolvable is that $n \equiv 1 \pmod{m}$. Bennett and Sotteau [5] have shown that if $m=3$ then this condition is also sufficient. That is, they constructed what they called an almost resolvable 2-fold triple system of order n for all $n \equiv 1$ (mod 3). Lindner and Rodger [14] have shown that with at most 9 exceptions, there is an almost resolvable 5-cycle decomposition (pentagon system) $(2K_n, C(5))$ for each $n \equiv 1 \pmod{5}$. Several people [3, 4] have also considered the problem of finding almost resolutions of the complete *directed* graph D_n into directed cycles of length m with the additional property that for each pair of vertices i and j and for each k, i and j are distance k from each other in exactly one cycle. The cases where $m=3$ or 4 and where n is sufficiently large have been settled.

The purpose of this paper is to prove that for ANY odd cycle length m and *all* $n=ms+1$, $n \equiv 1 \pmod{m}$ is also a sufficient condition for the existence of an almost resolvable m-cycle decomposition of $2K_n$. So in particular, we completely settle the problem when $m=5$, clearing up the 9 exceptions in [14] (for the case when m is even, see [6]).

This problem is of additional interest for the following reason. A *nesting* of an m-cycle decomposition $(G, C(m))$ is a mapping $\alpha: C(m) \rightarrow \{1, ..., |V(G)|\}$ ($V(G)$ is the vertex set of G) such that P^* is an edge-disjoint decomposition of $E(G)$, where

$P^* = \Bigg\{$ 

$\Bigg| \; u=(u_1, ..., u_m) \in C(m). \Bigg\}$

A simple counting argument shows that a necessary condition for $(K_n, C(m))$ to be nested is that $n \equiv 1 \pmod{2m}$. One problem then is to show that for all $n \equiv 1 \pmod{2m}$ there exists an m-cycle decomposition of K_n which can be nested. If $m=3$ then this is precisely the nesting problem for Steiner triple systems which has been completely settled [7, 16, 18].

When $m=5$ this has also been completely settled [14, 15, 20] and for each odd $m \geqslant 7$ this has been settled with at most 13 exceptions [15].

A necessary condition for $(2K_n, C(m))$ to be nested is that $n \equiv 1 \pmod{m}$. This has been shown to be sufficient [14] when $m=5$. However, whenever there exists an almost resolvable m-cycle decomposition $(2K_n, C(m))$, this decomposition can also be nested simply by mapping each m-cycle occurring in an almost parallel class to the deficiency of the almost parallel class.

EXAMPLE 1.4. A nesting of the 3-cycle decomposition $(2K_7, C(3))$ in Example 1.1 is

$$\alpha((i, i+1, i+3)) = i+6,$$
$$\alpha((i+2, i+4, i+5)) = i+6$$

for $0 \leqslant i \leqslant 7$, reducing each term modulo 7.

EXAMPLE 1.5. A nesting of the 5-cycle decomposition $(2K_{11}, C(5))$ in Example 1.3 is given by

$$\alpha((i, i+1, i+4, i+8, i+2)) = i+9,$$
$$\alpha((i+7, i+6, i+3, i+10, i+5)) = i+9$$

for $0 \leqslant i \leqslant 11$, reducing each term modulo 11.

Finally we remark that the nesting of $(K_n, C(m))$ has an interesting graph theoretical interpretation since it is equivalent to partitioning the edge-set $E(2K_n)$ into wheels, each with m spokes, so that for each pair of vertices u and v, one of the edges between u and v is the spoke of a wheel and the other is on the rim of a wheel.

In the following sections we will always assume that m is odd.

2. $n \equiv m+1 \pmod{2m}$

The main ingredients in our construction of almost resolvable m-cycle decompositions of $2K_n$ in this case are a *skew Room square* and a pair of *compatible m-nesting sequences.*

Let s be *odd* and set $N=\{0, 1, ..., s\}$. A *Room square* of order s is an $s \times s$ array S with the $\binom{s+1}{2}$ 2-element subsets of N arranged in the cells of S, at most one pair in each cell and each pair used exactly once, so that the rows *and* the columns form a 1-factorization of K_{s+1} (based on N). A Room square is *standardized* if cell (i, i) is occupied by $\{0, i\}$ for $1 \leqslant i \leqslant s$. A *skew* Room square is a standardized Room square with the additional property that for all $i \neq j \in \{1, ..., s\}$, exactly one of the cells (i, j) and (j, i) is occupied. A skew Room square of order s exists for all odd $s \geqslant 7$ (see [21], for example).

EXAMPLE 2.1. A skew Room square of order 7:

0 1	3 7	5 6		2 4		
	0 2	4 1	6 7		3 5	
		0 3	5 2	7 1		4 6
5 7			0 4	6 3	1 2	
	6 1			0 5	7 4	2 3
3 4		7 2			0 6	1 5
2 6	4 5		1 3			0 7

Let $[x]$ be the greatest integer less than or equal to x. Define $|i-j|_m$ to be x, where $x \leqslant [m/2]$ and $x = i - j \pmod m$ or $x = j - i \pmod m$. An *m-nesting sequence* is a sequence $(d_0, ..., d_{[m/2]})$ which for $0 \leqslant d_i \leqslant m-1$ and for $0 \leqslant i \leqslant [m/2]$ satisfies the properties:

(1) $\{|d_i - d_{i-1}|_m \mid 1 < i < [m/2]\} = \{i \mid 1 < i < [m/2]\}$, and

(2) $\{|d_{[m/2]} - d_i|_m \mid 0 \leqslant i \leqslant [m/2]-1\} = \{i \mid 1 \leqslant i \leqslant [m/2]\}$.

Two m-nesting sequences $(e_0, ..., e_{[m/2]})$ and $(d_0, ..., d_{[m/2]})$ are *compatible* if

(1) $e_{[m/2]} = d_{[m/2]}$, and

(2) $\{e_0, ..., e_{[m/2]}, d_0, ..., d_{[m/2]}\} = \{i \mid 0 \leqslant i \leqslant m-1\}$.

EXAMPLE 2.2. (0, 1, 4) and (3, 2, 4) are compatible 5-nesting sequences. (0, 1, 6, 2) and (4, 3, 5, 2) are compatible 7-nesting sequences.

LEMMA 2.3. *There exists a pair of compatible m-nesting sequences for all odd $m \geqslant 3$.*

Proof. For $0 \leqslant i \leqslant [m/2]$, define

$$e_i = (-1)^{i+1} [(i+1)/2] \pmod m,$$

$$d_i = [m/2] + 1 + (-1)^i [(i+1)/2] \pmod m. \quad \blacksquare$$

(The 5- and 7-nesting sequences in Example 2.2 are constructed in this way.)

The final ingredient we need before we present our first construction is an almost resolvable m-cycle decomposition of $2K_{m+1}$. Of course each almost parallel class then consists of exactly one m-cycle, so any m-cycle decomposition of $2K_{m+1}$ is almost resolvable.

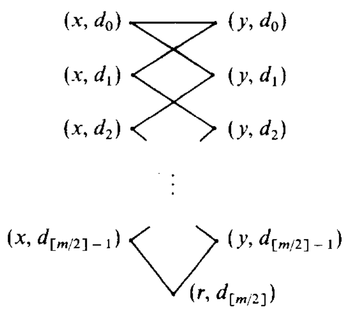
LEMMA 2.4. *Let m be odd. There exists an almost resolvable m-cycle decomposition of $2K_{m+1}$.*

Proof. Let the vertex set $V(K_{m+1}) = \{\infty, 0, 1, ..., m-1\}$. Define the set C of m-cycles by

$$C = \{(0, 1, ..., m-1)\}$$
$$\cup \{(\infty, x, x+1, x-1, x+2, x-2, ..., x+[m/2]) \mid 0 \leqslant x \leqslant m-1\}. \quad \blacksquare$$

(The 5-cycle decomposition of Example 1.2 is constructed in this way.)

We are now ready to present our first construction. Most of the m-cycles in the construction are of the form depicted in the following figure, where the m-cycle is denoted by $(x, y, r; d_0, d_1, ..., d_{[m/2]})$ (the vertices are ordered pairs of integers):



To prevent the notation (or the pictures!) from becoming too complicated, we denote the m-cycle in the above figure by $(x, y, r; d_0, d_1, ..., d_{[m/2]})$.

The skew Room square construction. Let $s \geqslant 7$ be *odd*, $X = \{1, 2, ..., s\}$, and let $2K_{ms+1}$ be based on the symbols $\{\infty\} \cup (X \times \{0, ..., m-1\})$. Further, let S be a skew Room square of order s (based on the symbols in $\{\infty\} \cup X$) and let $(e_0, ..., e_{[m/2]})$ and $(d_0, ..., d_{[m/2]})$ be compatible m-nesting sequences. In this constructon, all additions are performed modulo m.

Define a collection of m-cycles C as follows:

(1) for each $x \in X$ define an m-cycle decomposition of $2K_{m+1}$ on the $m+1$ vertices in $\{\infty\} \cup (\{x\} \times \{0, 1, ..., m-1\})$ (see Lemma 2.4) and place these $m+1$ cycles in C;

(2) for each pair $x \neq y \in X$ and for each i, $0 \leqslant i \leqslant m-1$, place the m-cycle $(x, y, r; d_0+i, d_1+i, ..., d_{[m/2]}+i)$ in C, where r is the *row* of S containing the pair $\{x, y\}$; and

(3) for each pair $x \neq y \in X$ and for each i, $0 \leqslant i \leqslant m-1$, place the m-cycle $(x, y, c; e_0+i, e_1+i, ..., e_{[m/2]}+i)$ in C, where c is the *column* of S containing the pair $\{x, y\}$.

It is straightforward to see that $(2K_n, C)$ is an m-cycle decomposition. We shall now show that it is almost resolvable. For each $y \in \{\infty\} \cup (X \times \{0, 1, ..., m-1\})$ define the almost parallel class $\pi(y)$ with deficiency y as follows:

(1) $\pi(\infty) = \{((x, 0), (x, 1), ..., (x, m-1)) \mid x \in X\}$; and

(2) for each $(x, i) \in X \times \{0, 1, ..., m-1\}$,

$$\begin{aligned}\pi(x, i) = \{&(a, b, r; d_0+i, d_1+i, ..., d_{[m/2]}+i) \mid \\ &\text{all } \{a, b\} \text{ in } \textit{column } x \text{ of } S\} \\ \cup \{&(a, b, c; e_0+i, e_1+i, ..., e_{[m/2]}+i) \mid \\ &\text{all } \{a, b\} \text{ in } \textit{row } x \text{ of } S\} \\ \cup\ & p(x, i),\end{aligned}$$

where $p(x, i)$ is the m-cycle in the m-cycle decomposition of $2K_{m+1}$ on the $m+1$ vertices in $\{\infty\} \cup (\{x\} \times \{0, ..., m-1\})$ that has deficiency (x, i).

THEOREM 2.6. *If m is odd and if $n = sm+1$ for some* odd *integer s then there exists an almost resolvable m-cycle decomposition of $2K_n$.*

Proof. Skew Room squares of every odd order $s \geqslant 7$ exist (see [21], for example), so if $s \geqslant 7$ then the theorem follows by the skew Room square construction. If $s \in \{3, 5\}$ then almost resolvable m-cycle decompositions of $2K_n$ are constructed in Lemma 5.1. ∎

3. $n \equiv 1 \pmod{2m}$

The main difference in our construction of almost resolvable m-cycle decompositions of $2K_n$ in this case is that we use skew Room frames instead of skew Room squares. We shall also require almost resolvable m-cycle decompositions of $2K_{2m+1}$ and $2K_{4m+1}$. These, along with some other decompositions, are constructed in Section 4, but it is convenient to state these results now.

LEMMA 3.1. *If m is odd and if $n = sm + 1$ where $s \in \{2, 4, 6, 8, 12\}$ then there exists an almost resolvable m-cycle decomposition of $2K_n$.*

We now proceed to the skew Room frame construction, but first, the definition of a skew Room frame! A *skew Room frame* of order $2s$ is defined as follows. Let $X = \{1, 2, ..., 2s\}$ and let $H = \{h_1, ..., h_l\}$ be a partition of X (for $1 \leqslant i \leqslant l$, h_i is called a *hole*). Denote by $T(X)$ the $\binom{2s}{2}$ 2-element subsets of X. For our purpose, it will be sufficient to assume that each hole contains either 2 or 4 elements of X. Let F be a $2s \times 2s$ array and fill in (a subset of) the cells of F as follows:

(1) For each hole $h_i \in H$, fill the cells of $h_i \times h_i$ with

$x_1 x_2$	
	$x_1 x_2$

if $h_i = \{x_1, x_2\}$

$x_1 x_2$	$x_3 x_4$		
	$x_1 x_2$	$x_3 x_4$	
		$x_1 x_2$	$x_3 x_4$
$x_3 x_4$			$x_1 x_2$

if $h_i = \{x_1, x_2, x_3, x_4\}$.

(In what follows the cells $h_i \times h_i$, $h_i \in H$, will be called a *square* hole);

(2) distribute the pairs in $T(X) \backslash H$ among the cells not belonging to a square hole (each pair used exactly once) so that each row and column of F is a 1-factor of K_{2s}; and

(3) if $\{a, b\} \in T(X) \backslash H$, *exactly one* of the cells (a, b) and (b, a) is occupied.

The resulting array is called a *skew Room frame* of order $2s$.

EXAMPLE 3.2. A skew frame of order 10 with each hole containing 2 elements.

1 2			6 9		8 10		3 5	4 7	
	1 2	6 10		7 9		4 5			3 8
5 10		3 4			2 7		1 9	6 8	
	5 9		3 4	1 8		2 10			6 7
8 9		1 7		5 6			4 10	2 3	
	7 10		2 8		5 6	3 9			1 4
4 6		2 9		3 10		7 8		1 5	
	3 6		1 10		4 9		7 8		2 5
	4 8		5 7		1 3		2 6	9 10	
3 7		5 8		2 4		1 6			9 10

The Skew Room Frame Construction. Let $X=\{1, ..., 2s\}$ and let $2K_{2ms+1}$ be based on the vertices $\{\infty\} \cup (X \times \{0, ..., m-1\})$. Further, let $(d_0, ..., d_{[m/2]})$ and $(e_0, ..., e_{[m/2]})$ be compatible m-nesting sequences and let F be a skew Room frame of order $2s$ (based on X). In this construction, all additions are defined modulo m. Define a collection of m-cycles C as follows:

(1) for each hole $h \in H$ define an almost resolvable m-cycle decomposition of $2K_{|h|m+1}$ on the vertex set $\{\infty\} \cup (h \times \{0, ..., m-1\})$ and place these m-cycles in C;

(2) for each x and y belonging to *different* holes and for each i, $0 \leqslant i \leqslant m-1$, place in C the m-cycle $(x, y, r; d_0+i, ..., d_{[m/2]}+i)$, where r is the *row* of F containing the pair $\{x, y\}$; and

(3) for each x and y belonging to different holes and for each i, $0 \leqslant i \leqslant m-1$, place in C the m-cycle $(x, y, c; e_0+i, ..., e_{[m/2]}+i)$, where c is the *column* of F containing the pair $\{x, y\}$.

It is apparent that $(2K_{2ms+1}, C)$ is an m-cycle decomposition. We now show that it is also almost resolvable.

For each hole $h \in H$, denote by $\pi(\infty, h)$ the almost parallel class that has deficiency ∞ and by $\pi((x, i), h)$ the almost parallel class with deficiency (x, i) in the resolution of $2K_{|h|m+1}$ on the vertex set $\{\infty\} \cup (h \times \{0, ..., m-1\})$.

For each $y \in \{\infty\} \cup (X \times \{0, ..., m-1\})$ define the almost parallel class $\pi(y)$ with deficiency y as follows:

(1) $\pi(\infty) = \bigcup_{h \in H} \pi(\infty, h)$; and

(2) for each $(x, i) \in X \times \{0, ..., m-1\}$ with $x \in h$,

$$\begin{aligned}\pi((x, i)) = \pi((x, i), h)\\ &\cup \{(a, b, r; d_0 + i, ..., d_{[m/2]} + i) \mid \text{all } \{a, b\} \text{ in column } x \text{ of } F\}\\ &\cup \{(a, b, c; e_0 + i, ..., e_{[m/2]} + i) \mid \text{all } \{a, b\} \text{ in row } x \text{ of } F\}.\end{aligned}$$

THEOREM 3.3. *If m is odd, s is even, and $n = sm + 1$, then there exists an almost resolvable m-cycle decomposition of $2K_n$.*

Proof. Lemma 3.1 takes care of the cases where $s \in \{2, 4, 6, 8, 12\}$. The combined work in [15, 19] gives a skew Room frame of every order $s \notin \{2, 4, 6, 8, 12\}$ with holes of size 2 or 4, and so the skew Room frame construction handles the remaining cases. ∎

THEOREM 3.4. *For any odd $m \geqslant 3$ and for any s there exists an almost resolvable m-cycle decomposition of $2K_{ms+1}$.*

Proof. This follows from Theorems 2.6 and 3.3. ∎

4. PROOF OF LEMMA 3.1

We begin with an almost resolvable m-cycle decomposition of $2K_{2m+1}$. This construction is based on a solution to a nesting problem by Dean Hoffman.

LEMMA 4.1. *For any odd $m \geqslant 3$ there exists an almost resolvable m-cycle decomposition of $2K_{2m+1}$.*

Proof. Let the vertex set of $2K_{2m+1}$ be $\{0, 1, ..., 2m\}$ and let $m = 2x + 1$ (so $x \geqslant 1$). Let

$$c = (-1, 2, -3, ..., (-1)^x x, (-1)^x (x+1), (-1)^{x+1} (x+2), ..., (-1)^{2x} (2x+1)),$$

where each component of c is reduced modulo $2m+1$. Let $-c$ and $c + i$ be formed by replacing each component c_j (for $1 \leqslant j \leqslant m$) of c by

$-c_j \pmod{2m+1}$ and $c_j+i \pmod{2m+1}$, respectively. Then c and $-c$ form an almost parallel class with deficiency 0 and $C=\{c+i, -c+i \mid 0 \leqslant i \leqslant 2m\}$ is an almost resolvable m-cycle decomposition of $2K_{2m+1}$. ∎

EXAMPLE 4.2. For $m=3$, $c=(6,5,3)$, $-c=(1,2,4)$ and $C=\{(6+i, 5+i, 3+i), (1+i, 2+i, 4+i) \mid 0 \leqslant i \leqslant 6\}$. For $m=5$, $c=(10,2,3,7,5)$, $-c=(1,9,8,4,6)$, and $C=\{(10+i, 2+i, 3+i, 7+i, 5+i), (1+i, 9+i, 8+i, 4+i, 6+i) \mid 0 \leqslant i \leqslant 10\}$.

To prove the rest of Lemma 3.1, we need to introduce some new terms. The *wreath product* $G \cdot H$ of two graphs G and H is formed by replacing each vertex of G with a copy of H and joining 2 vertices in different copies of H with an edge if and only if the corresponding vertices of G are adjacent. Let G^c denote the complement of G, so K_m^c is the graph with m vertices and no edges.

LEMMA 4.3 [2]. *Let $M \leqslant m$ and both m and M be odd. Let C_M be a cycle of length M. Then there exists a* resolvable *m-cycle decomposition of $C_M \cdot K_m^c$.*

Remark. As was pointed out by the authors of [2], the proof given in Theorem 5 of [2] works even if t is odd, providing the restriction that t be a prime in Lemma 4 is removed.

Lemma 4.3 enables us to prove the following lemma which we later generalize. We prove this result first to avoid the complications which arise in the generalization which may cloud the idea.

LEMMA 4.4. *Let $M \leqslant m$ with both m and M odd. Let $p \equiv 1 \pmod M$. If there exists an almost resolvable M-cycle decomposition of $2K_p$ then there exists an almost resolvable m-cycle decomposition of $2K_{mp+1}$.*

Proof. Let $2K_p$ be defined on the vertex set $\{1, ..., p\}$. Consider an almost resolution of $2K_p$ into cycles of length M; for $1 \leqslant j \leqslant p$ let $\pi(j)$ be the almost parallel class with deficiency j in this almost resolution and $G(j)$ be the subgraph of $2K_p$ corresponding to $\pi(j)$ (so $G(j)$ is a 2-factor of $K_p - \{j\}$).

For $1 \leqslant i \leqslant (p-1)/M$ and $1 \leqslant j \leqslant p$, by Lemma 4.3 there exists a resolvable m-cycle decomposition of $G(j) \cdot K_m^c$, which (of course) consists of m 2-factors. We will need to be specific, so for $1 \leqslant k \leqslant m$ let $f(j,k)$ be the m 2-factors of $G(j) \cdot K_m^c$ in a fixed 2-factorization.

Finally, for $1 \leqslant j \leqslant p$ define G_j to be a copy of $2K_{m+1}$ defined on the vertex set $\{\infty\} \cup (j \times \{1, ..., m\})$. By Lemma 2.4 there exists an almost resolvable m-cycle decomposition of G_j; let the almost parallel class with deficiency ∞ or (j, k) (for $1 \leqslant k \leqslant m$) in such a decomposition of G_j be denoted by $\pi(j, \infty)$ or $\pi(j, k)$, respectively.

We can now define an almost resolvable m-cycle decomposition of $2K_{mp+1}$ on the vertex set $X = \{\infty\} \cup (\{1, ..., p\} \times \{1, ..., m\})$ as follows. Define the almost parallel class $P(i)$ with deficiency i for each $i \in X$ as follows:

(1) $P(\infty) = \{\pi(j, \infty) \mid 1 \leqslant j \leqslant p\}$, and

(2) $P((j, k)) = \pi(j, k) \cup f(j, k)$ for $1 \leqslant j \leqslant p$ and $1 \leqslant k \leqslant m$. ∎

Lemmas 4.1 and 4.4 together with a construction that can be used when n is a prime power [14] are enough to prove Lemma 3.1 (as we shall see) except in the case where $m = 7$ and $n = 85$. The following lemma is more general than we need to remove this exception, but is also of interest in its own right.

LEMMA 4.5. *Let $M \leqslant m$ and both m and M be odd. Let r be an order for which there exist a pair of orthogonal latin squares. Then there exists a* resolvable *m-cycle decomposition of $C_M \cdot K^c_{mr}$, where C_M is an M-cycle.*

Proof. $C_M \cdot K^c_r$ defined on the vertex set $\{1, ..., r\} \times \{1, ..., M\}$ has a resolvable M-cycle decomposition, defined as follows. Let L_1 and L_2 be a pair of orthogonal latin squares of order r.

For $1 \leqslant l \leqslant r$ and for each cell (i, j) in L_2 that contains the symbol l, if the corresponding cell (i, j) in L_1 contains the symbol k then define the M-cycle $(c_1, ..., c_M)$, where $c_1 = (i, 1)$, $c_{2x} = (j, 2x)$, and $c_{2x+1} = (k, 2x+1)$. The r M-cycles arising from a given symbol l in L_2 form a parallel class. Now apply Lemma 4.3 to each M-cycle in this decomposition in the obvious way. ∎

LEMMA 4.6. *Let $M \leqslant m$ and both m and M are odd. Let r be an order for which there exists a pair of orthogonal latin squares and let $p \equiv 1 \pmod{M}$. If there exists an almost resolvable M-cycle decomposition of $2K_p$ and if there exists an almost resolvable m-cycle decomposition of $2K_{mr+1}$ then there exists an almost resolvable m-cycle decomposition of $2K_{pmr+1}$.*

Proof. The proof is the same as the proof of Lemma 4.4 except that Lemma 4.5 is used instead of Lemma 4.3. ∎

Finally, we are ready to prove Lemma 3.1.

LEMMA 3.1. *If m is odd and if $n = sm + 1$ where $s \in \{2, 4, 6, 8, 12\}$ then there exists an almost resolvable m-cycle decomposition of $2K_n$.*

Proof. Recall that Lemma 2.4 shows that there exists an almost resolvable m-cycle decomposition of $2K_{m+1}$. Now consider the following cases:

1. If $s=2$ then the result is proved in Lemma 4.1.
2. If $s=4$ then $s\equiv 1 \pmod 3$, so apply Lemma 4.4 with $M=3$ and $p=4$.
3. If $s=6$ then $s\equiv 1 \pmod 5$, so apply Lemma 4.4 with $M=5$ and $p=6$ when $m\geqslant 5$.
4. If $s=8$ then $s\equiv 1 \pmod 7$, so apply Lemma 4.4 with $M=7$ and $p=8$ when $m\geqslant 7$.
5. If $s=12$ then $s\equiv 1 \pmod{11}$, so apply Lemma 4.4 with $M=11$ and $p=12$ when $m\geqslant 11$.
6. If $s=12$ then apply Lemma 4.6 with $M=3$, $p=4$, and $r=3$.
7. If $m=3$ then the problem has been solved [5].
8. If $m=5$ and $s\in\{8, 12\}$ then the problem has been solved [14]. ∎

5. The Cases $s=3$ and $s=5$

Lemma 5.1. *If $s\in\{3,5\}$ and if $n=ms+1$ then there exists an almost resolvable m-cycle decomposition of $2K_n$.*

Proof. We consider the cases $s=3$ and $s=5$ in turn.

Let $s=3$. If $m\in\{3,5\}$ then an almost resolvable m-cycle decomposition of $2K_n$ has been found [5, 14], so we can assume that $m\geqslant 7$. Let the vertex set of $2K_n$ be $\{\infty\}\cup(\{0,1,...,m-1\}\times\{0,1,2\})$. Define the following parameters:

$$
\begin{aligned}
&y_i=\lfloor m/4\rfloor+\lceil i/2\rceil(-1)^{i+1} && \text{for } 0\leqslant i\leqslant\lfloor m/2\rfloor,\\
&z_i=-\lceil m/4\rceil+\lceil i/2\rceil(-1)^{i+1} && \text{for } 0\leqslant i\leqslant\lfloor m/2\rfloor,\\
&a_i=\lfloor m/4\rfloor+\lceil i/2\rceil(-1)^{i} && \text{for } 0\leqslant i\leqslant\lfloor m/2\rfloor-1,\\
&b_i=-\lfloor m/4\rfloor+\lceil i/2\rceil(-1)^{i} && \text{for } 2\leqslant i\leqslant\lfloor m/2\rfloor-1,\\
&b_1=-\lfloor m/4\rfloor \text{ and } b_0=-\lceil m/4\rceil && \\
&c_i=-\lceil m/4\rceil+\lceil i/2\rceil(-1)^{i} && \text{for } 0\leqslant i\leqslant\lfloor m/2\rfloor-1,\\
&d_i=\lceil m/4\rceil+\lceil i/2\rceil(-1)^{i} && \text{for } 2\leqslant i\leqslant\lfloor m/2\rfloor-1,\\
&d_1=\lceil m/4\rceil \text{ and } d_0=\lfloor m/4\rfloor. &&
\end{aligned}
$$

If $m=4x+1$ (so $x\geqslant 2$) then define an almost parallel class of $2K_n$ by

$$C=\{(0,1,0;y_0,y_1,...,y_{[m/2]}),(1,2,2;z_0,z_1,...,z_{[m/2]}),\\ ((a_0,2),(a_1,2),...,(a_{[m/2]-1},2),(b_{[m/2]-1},0),\\ (b_{[m/2]-2},0),...,(b_1,0),\infty,(b_0,0))\}.$$

The deficiency in this case is (0, 1). If $m=4x+3$ (so $x\geqslant 1$) then define an almost parallel class of $2K_n$ by

$$C=\{(0,1,0;y_0,y_1,...,y_{[m/2]}),(1,2,2;z_0,z_1,...,z_{[m/2]}),\\ ((c_0,0),(c_1,0),...,(c_{[m/2]-1},0),(d_{[m/2]-1},2),\\ (d_{[m/2]-2},2),...,(d_1,2),\infty,(d_0,2))\}.$$

The deficiency in this case is $(\lfloor m/2\rfloor, 1)$. Now let $C+(i,j)$ be the almost parallel class formed by adding (i,j) to each vertex in each cycle in C (where $\infty+(i,j)=\infty$), reducing the first component modulo m and the second modulo 3. Then

$$\bigcup_{\substack{0\leqslant i\leqslant m-1\\ 0\leqslant j\leqslant 2}} (C+(i,j))\cup\{((0,i),(2,i),...,(2(m-1),i))\,|\,0\leqslant i\leqslant 2\}$$

is an almost resolvable m-cycle decomposition of $2K_{3m+1}$.

Let $m=5$. Let the vertex set of $2K_n$ be $\{\infty\}\cup(\{0,1,...,m-1\}\times\{0,1,2,3,4\})$. For each j, $0\leqslant j\leqslant 4$ let $\bigcup_{i=0}^{m-1}\pi(i,j)\cup\pi(\infty,j)$ be an almost resolvable m-cycle decomposition of $2K_{m+1}$ on the vertex set $\{\infty\}\cup(\{0,1,...,m-1\}\times\{j\})$, where $\pi(i,j)$ has deficiency (i,j) (see Lemma 2.4). Define an almost parallel class of $2K_n$ by

$$C(i,j)=\pi(i,j)\cup(\{0,1,3;a_0,...,a_{[m/2]}),(2,3,0;a_0,...,a_{[m/2]}),\\ (0,2,1;b_0,...,b_{[m/2]}),(1,3,2;b_0,...,b_{[m/2]})\}+(i,j)),$$

where the parameters are defined as

$$a_i=\lfloor(m+1)/4\rfloor+\lceil i/2\rceil(-1)^i \qquad \text{for} \quad 0\leqslant i\leqslant\lfloor m/2\rfloor$$
$$b_i=-\lceil m/4\rceil+\lceil i/2\rceil(-1)^{i+1} \qquad \text{for} \quad 0\leqslant i\leqslant\lfloor m/2\rfloor.$$

The deficiency of $C(i,j)$ is of course (i,j). Then $\bigcup_{0\leqslant i\leqslant m-1,0\leqslant j\leqslant 4} C(i,j)\cup(\bigcup_{0\leqslant j\leqslant 4}\pi(\infty,j))$ is an almost resolvable m-cycle decomposition of $2K_{5m+1}$.

References

1. B. Alspach and R. Haggkvist, Some observations on the Oberwolfach problem, *J. Graph Theory* **9** (1985), 177–187.
2. B. Alspach, P. Schellenberg, D. R. Stinson, and D. Wagner, The Oberwolfach problem and factors of uniform odd length cycles, submitted.
3. F. E. Bennett, Conjugate orthogonal latin squares and Mendelsohn designs, *Ars Combin.* **19** (1985), 51–62.
4. F. E. Bennett, E. Mendelsohn, and N. Mendelsohn, Resolvable perfect cyclic designs, *J. Combin. Theory Ser. B* **29** (1980), 142–150.
5. F. E. Bennett and D. Sotteau, Almost resolvable decompositions of K_n^*, *J. Combin. Theory Ser. B* **30** (1981), 228–232.
6. J. Burling and K. Heinrich, Near 2-factorizations of $2K_n$: Cycles of even length, submitted.
7. C. J. Colbourn and M. J. Colbourn, Nested triple systems, *Ars Combin.* **16** (1983), 27–34.
8. R. K. Guy, Unsolved combinatorial problems, *in* "Combinatorial Mathematics and Its Applications, Proceedings, Conf. Oxford 1967" (D. J. A. Welsh, Ed.), p. 121, Academic Press, New York, 1971.
9. P. Hell, A. Kotzig, and A. Rosa, Some results on the Oberwolfach problem, *Aequationes Math.* **12** (1975), 1–5.
10. J. D. Horton, B. K. Roy, P. J. Schellenberg, and D. R. Stinson, On decomposing graphs into isomorphic uniform 2-factors, *Ann. Discrete Math.* **27** (1985), 297–319.
11. C. Huang, A. Kotzig, and A. Rosa, On a variation of the Oberwolfach problem, *Discrete Math.* **27** (1979), 261–277.
12. C. Huang, E. Mendelsohn, and A. Rosa, On partially resolvable t-partitions, *Ann. Discrete Math.* **12** (1982), 169–183.
13. E. Kohler, Uber der Oberwolfacher problem, *in* "Beitr. Geom. Algebra, Basel, 1977," pp. 189–201.
14. C. C. Lindner and C. A. Rodger, Nesting and almost resolvability of pentagon systems, *Europ. J. Combin.*, to appear.
15. C. C. Lindner, C. A. Rodger, and D. R. Stinson, Nesting of m-cycle decompositions, *Ann. Discrete Math.*, to appear.
16. C. C. Lindner and D. R. Stinson, The spectrum for the conjugate invariant subgroups of perpendicular arrays, *Ars Combin.* **18** (1983), 51–60.
17. D. K. Ray-Chadhuri and R. M. Wilson, Solution of Kirkman's schoolgirl problem, *in* Proc. Sympos. Pure Math. Vol. 19, pp. 187–204, Amer. Math. Soc., Providence, RI, 1971.
18. D. R. Stinson, The spectrum of nested Steiner triple systems, *Graphs Combin.* **1** (1985), 189–191.
19. D. R. Stinson, On the existence of skew Room frames of type 2^n, *Ars Combin.* **24** (1987), 115–128.
20. D. R. Sinson, University of Manitoba, private communication.
21. D. R. Stinson, The spectrum of skew Room squares, *J. Austral. Math. Soc. Ser. A* **31** (1981), 475–480.