This paper examines the question of whether a given pattern
\[ x, x + a_1, \ldots, x + a_{m-1} \]
of \( k \)th power residues of length \( m \) can be postponed indefinitely. This is the case when there exists a prime \( q \), called a delay prime, which does not contain this pattern even if \( q \) itself is considered as a \( k \)th power residue. It is conjectured that if there exists no delay prime then there exists a finite limit
\[ \Lambda = \Lambda (k, m; a_1, \ldots, a_{m-1}) \]
for which the corresponding pattern will occur before \( \Lambda \) in every sufficiently large prime of the form \( kn + 1 \).

In this paper we will inquire into the conditions under which the first appearance of a given pattern of \( k \)th power residues can be postponed indefinitely. The patterns which were previously considered were strings of \( m \) consecutive power residues.

In [8] we let \( r(k, m, p) \) be the least member of the first appearance of this pattern and defined for a sufficiently large prime \( p \)
\[ \Lambda (k, m) = \limsup_{p \to \infty} r(k, m, p). \]

We called a prime \( q = kn + 1 \) exceptional if it did not contain such a pattern of residues. We showed that if there exists a prime \( q \) which remains exceptional no matter what character is assigned to \( q \), then \( q \) can be used to construct an infinite sequence of primes \( P_i \) for which \( r(k, m, P_i) > N \), where \( N \) is an arbitrarily large positive integer, and therefore that \( \Lambda (k, m) = \infty \). Such primes \( q \) will be called delay primes. Every delay prime \( q \) leads to \( k \) purely multiplicative functions \( F_k \) for which a given pattern is postponed indefinitely [13].
The above discussion which was carried out for strings of residues applies equally well to any pattern of $k$th power residues,

$$x, x + a_1, x + a_2, \ldots, x + a_{m-1}. \quad (1)$$

Therefore we can define $r(k, m, p) = r(k, m, p; a_1, a_2, \ldots, a_{m-1})$ as the least member in the first appearance of the pattern (1) and

$$\land (k, m) = \land (k, m; a_1, a_2, \ldots, a_{m-1}) = \limsup_{p \to \infty} r(k, m, p). \quad (2)$$

It was proved in [8] that $\land (2, 3; 1, 2) = \infty$ by using the delay prime $q = 3$. In [14] it was proved that this choice of $q$ is unique. On the other hand it was shown in [1] that

$$\land (3, 3; 1, 2) = 23532 \quad (3)$$

and in [1, 2, 9, 12] that

$$\begin{array}{ccccccc}
  k & 2 & 3 & 4 & 5 & 6 & 7 \\
  \land (k, 2; 1) & 9 & 77 & 1224 & 7888 & 202124 & 1649375
\end{array} \quad (4)$$

Mills [13] conjectured that $\land (k, 2; 1)$ is always finite.

The Case $m = 2$

In this case we are concerned with a pair of $k$th power residues differing by a constant $a_1 = a$. If $a$ is a residue of $p$, then letting $x = ay$ we have a pair of consecutive residues of $p$ and therefore

$$\land (k, 2; a) \leq a \land (k, 2; 1) \quad \text{if} \quad \chi_k(a) = 1. \quad (5)$$

If $k = 2$ we have by (4) and (5) that $\land (2, 2; a) = 9a$ if $a$ is a quadratic residue of $p$. If $a$ is a non-residue, but $a + 1$ is a residue, we have the pair $1, 1 + a$. If both $a$ and $a + 1$ are residues, then $a(a + 1) = a^2 + a$ is a residue and we have the pair $(a^2, a^2 + a)$ so that

$$\land (2, 2; a) \leq \max(9a, a^2) \quad (6)$$

and hence is always finite.

For $k > 2$, although $\land (k, 2; a)$ is finite for $k \leq 7$ by (4) and (5) in case $a$ is a $k$th power residue of $p$, this is no longer true in general, as can be seen by the following:
THEOREM 1. $q$ is a delay prime for the pattern $x, x + a$ if $q = k + 1$ and $a \neq 0, 1 \pmod{q}$ or if $q = 2k + 1$ and $a \neq 0, \pm 1, \pm 2 \pmod{q}$. In other words,

$$\bigwedge (k, 2; a) = \infty \quad \begin{cases} 
\text{if} & a \neq 0, +1 \pmod{q = k + 1}, \\
\text{if} & a \neq 0, \pm 1, \pm 2 \pmod{2q + 1}.
\end{cases}$$ (7)

Proof. If $q = k + 1$, then 1 is the only $k$th power residue, while for $q = 2k + 1$ the only residues are 1 and $q - 1$. The assumption that $q$ or some multiple of $q$ can be a residue prevents $q$ from being a delay prime for the values of $a$ stated in the theorem.

In particular this shows that for $k = 3$,

$$\bigwedge (3, 2; a) = \infty \quad \text{for} \quad a \equiv 3, 4 \pmod{7}. \quad (8)$$

Experiments seem to indicate that in all other cases the values of $\bigwedge (3, 2; a)$ are finite. Thus for $a = 2$ we have by (4) and (5) that $\bigwedge (3, 2; 2) \leq 154$ if 2 is a cubic residue of $p$.

If 2 is not a cubic residue we can assume that the cubic character $\chi_3(2) = \omega$. Then if $\chi_3(3) = 1$ we have the pair $(1, 3)$ of cubic residues. If $\chi_3(3) = \omega^2$, we have the pair $(6, 8)$, since $\chi_3(2) = \omega$. Hence $\chi_3(3) = \omega$. We now consider three cases modulo 5. If $\chi_3(5) = 1$, we have the pair $(25, 27)$, if $\chi_3(5) = \omega$ we have $(18, 20)$ and if $\chi_3(5) = \omega^2$ we have the pair $(8, 10)$ so that in all cases

$$\bigwedge (3, 2; 2) \leq 154. \quad (9)$$

Similar arguments show that

$$\bigwedge (3, 2; a) \leq 77a \quad \text{for} \quad a = 1, 2, 5, 6, 7, 8, 9, 12. \quad (10)$$

We conjecture that (10) holds for all values of $a \not\equiv \pm 3 \pmod{7}$.

Similarly we have for $k = 4$ by Theorem 1 that

$$\bigwedge (4, 2; a) = \infty \quad \text{if} \quad a \equiv 2, 3 \pmod{5}. \quad (11)$$

We conjecture that $\bigwedge (4, 2; a)$ is finite in all other cases.

For example, for $a = 4$, since $\bigwedge (4, 2; 1) = 1224$, we have

$$\bigwedge (4, 2; 4) \leq 4896 \quad \text{if} \quad \chi_4(a) = 1. \quad (12)$$

If $\chi_4(4) = -1$ we can let $\chi_4(2) = i$ and consider this pattern $x, x + 4$ modulo $q = 3, 5, 7, 11, 13$ as follows:
Hence if \( \chi_4(4) = -1 \) we have \( \wedge (4; 4) \leq 896 \), and hence \( \wedge (4; 4) \leq 4896 \) in all cases.

**The Case \( m = 3 \)**

We have already noted in the introduction that the delay prime \( q = 3 \) was used to prove that \( \wedge (2, 3; 1, 2) = \infty \). Similarly, it can easily be seen that \( q = 5 \) and \( q = 7 \) are delay primes for the following patterns of quadratic residues:

\[
(a_1, a_2) \equiv (1, 3), (2, 3), (2, 4) \pmod{5}
\]

\[
(a_1, a_2) \equiv (1, 5), (2, 3), (4, 6) \pmod{7}
\]

and therefore the corresponding \( \wedge (2, 3; a_1, a_2) = \infty \).

It has recently been proved independently by Hudson [5] and by Peter
Montgomery that the two purely multiplicative functions which correspond to the pattern $x, x+1, x+3$ are unique. We note that $q = 5$ and $7$ are both delay primes for $(2, 3)$.

Walum [15] proved that $\wedge (2, 3; 3, 4)$ is finite. Guy [4] found that for $a_1 < a_2 \leq 25$ all patterns not listed in (11) have a finite $\wedge (2, 3; a_1, a_2) \leq 185$ and that in particular $\wedge (2, 3; 3, 4) = 174$ and conjectures that all patterns not listed in (11) have a finite limit.

In order that $q$ be a delay prime for the pattern $x, x + a_1, x + a_2$ it must be a delay prime for the patterns $x + a$ with $a = a_1, a_2$ and $a_2 - a_1$. By Theorem 1 the only patterns $(a_1, a_2)$ for which $\wedge (k, 3; a_1, a_2)$ can be finite are

\[ (a_1, a_2) \equiv (0, 0), (0, \pm 1), (1, 1) \pmod{q = k + 1} \]
\[ (a_1, a_2) \equiv (0, 0), (0, \pm 1), (0, \pm 2), (1, \pm 1) \pmod{q = 2k + 1} \]

On the other hand, there exists no delay prime for

\[ (a_1, a_2) = (w^k \mid w^k \mid v^k). \quad (13) \]

The case $(0, 0)$ can always be reduced to one of the other cases by dividing out the power of $q$ common to $a_1$ and $a_2$.

For $k = 3, q = 7$ we have by (12) the following cases for $a_1 < a_2 < 12$.

\[ (a_1, a_2) = (1, 2), (1, 6), (1, 7), (1, 8), (1, 9), (2, 7), (2, 9) \]
\[ (5, 6), (5, 7), (6, 7), (6, 8), (7, 8), (7, 9), (8, 9). \]

We know that $\wedge (3, 3; 1, 2) = 23532$ and that

\[ (1, 8), (2, 9), (7, 8), (7, 9), (8, 9) \]

cannot be delayed by any prime $q$ by (13). Of the remaining eight patterns $(1, 7), (1, 9), (2, 7), (6, 7)$ and $(6, 8)$ are delayed by $q = 13$, while $(5, 6)$ and $(5, 7)$ are delayed by $q = 19$, leaving only the five patterns in (15) for which there is no delay prime. We conjecture that $\wedge (3, 3; a_1, a_2)$ is finite for these patterns.

For $k = 4$ the restriction with $q = 5$ in (12) is even more drastic and the two additional delay moduli $q = 13$ and $q = 17$ delay all patterns with $a_1 < a_2 < 24$, except for $(4, 5)$ and $(1, 16)$, neither of which can be delayed by any $q$ by (13) because $-4 \equiv (1 + i)^4 \pmod{q}$.

If $k$ is a prime and $2k + 1$ is not a prime we can look for the first prime $q = kn + 1$ and try to use it for a delay prime. In doing this we will confine ourselves to the patterns with either $a_1 = 1$, or $a_2 - a_1 = 1$ and write our
patterns as either \( x, x + 1, x + a \) or \( x, x + a - 1, x + a \), so that all our patterns will contain a pair of consecutive \( k \)th prime residues for \( k \) a prime. For this purpose we will use an idea of Krasner [7], which he used in connection with the first case of Fermat’s last theorem.

**Theorem 2.** A prime \( q = kn + 1 \), with \( k \) an odd prime, does not possess a pair of consecutive \( k \)th power residues if

\[
\begin{align*}
 n &\equiv 0 \pmod{3}, \\
 2^n &\equiv 1 \pmod{q} \\
 q &> 3^{n/4}.
\end{align*}
\] (16)

*Proof.* Suppose, on the contrary, that \( r \) is any residue and that

\[
r^i - r^j \equiv \pm 1 \pmod{q}
\] (17)

has a solution for some \( i \) and \( j \). Since \( n \) is even

\[
r^{n/2} \equiv -1, \quad r^i \equiv -r^{i + n/2} \pmod{q}
\] (18)

so that (17) can be replaced by

\[
\alpha_i r^c + \alpha_j r^d + \alpha_1 \equiv 0 \pmod{q}, \quad 0 \leq c < d < n/2.
\]

where \( \alpha_i = \pm 1 \). The resultant \( R \) of the two polynomials

\[
x^{n/2} + 1 \quad \text{and} \quad \alpha_1 x^c + \alpha_2 x^d + \alpha_3
\]

is

\[
R = \prod_{i=1}^{n/2} \left( \alpha_1 e^{(2i-1)c} + \alpha_2 e^{(2i-1)d} + \alpha_3 \right) \equiv 0 \pmod{q},
\] (19)

where \( \varepsilon \) is a primitive \( n \)th root of unity. Since \( R \neq 0 \) because \( n \neq 0 \pmod{3} \), and since \( c \neq d \) because 2 is not a \( k \)th power residue by (16), the resultant \( R \) is the determinant of the circulant matrix of order \( n/2 \) having three units 1 or \(-1\) in each row. By Hadamard’s theorem \( R \) does not exceed \( 3^{n/4} \), and since it is divisible by \( q \) by (19) we have arrived at a contradiction if \( 3^{n/4} > q \). Hence the theorem follows from the last condition in (16).

**Corollary.** If there exists a prime \( q = kn + 1 \) satisfying Theorem 2, then \( q \) is a delay prime for the patterns

\[
x, x + 1, x + a \quad \text{and} \quad x, x + a - 1, x + a
\] (20)

if neither \( a \) nor \( a - 1 \) is zero or a \( k \)th power residue of \( q \).

In other words, under these conditions

\[
\cap (k, 3; 1, a) = \cap (k, 3; a - 1, a) = \infty.
\] (21)
If \( n = 2 \), the \( k \)th power residues are 1 and \( q - 1 \), hence (21) holds for all \( a \) with \( 2 < a < q - 1 \).

We let \( q_0 \) be the least \( q \) for which \( kn + 1 \) is a prime with \( n \not\equiv 0 \pmod{3} \) and let \( r_0 \) be the least \( k \)th power residue greater than 1, then (21) holds for all patterns (20) with \( 2 < a < r_0 \).

We tabulate \( q_0 \) and the corresponding \( r_0 \) for all prime \( k < 200 \).

<table>
<thead>
<tr>
<th>( n = 2 )</th>
<th>( n = 4 )</th>
<th>( n = 8 )</th>
<th>( n = 10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k ) ( q_0 ) ( r_0 )</td>
<td>( k ) ( q_0 ) ( r_0 )</td>
<td>( k ) ( q_0 ) ( r_0 )</td>
<td>( k ) ( q_0 ) ( r_0 )</td>
</tr>
<tr>
<td>3 7 6</td>
<td>17 137 10</td>
<td>19 191 7</td>
<td></td>
</tr>
<tr>
<td>5 11 10</td>
<td>71 569 86</td>
<td>31 311 6</td>
<td></td>
</tr>
<tr>
<td>11 23 22</td>
<td>101 809 44</td>
<td>103 1031 264</td>
<td></td>
</tr>
<tr>
<td>23 47 46</td>
<td>107 857 188</td>
<td>109 1091 79</td>
<td></td>
</tr>
<tr>
<td>29 59 58</td>
<td>137 1097 79</td>
<td>151 1511 423</td>
<td></td>
</tr>
<tr>
<td>11 23 22</td>
<td>73 293 138</td>
<td>149 1193 186</td>
<td></td>
</tr>
<tr>
<td>53 107 106</td>
<td>79 317 114</td>
<td></td>
<td>181 1811 433</td>
</tr>
<tr>
<td>83 167 166</td>
<td>97 389 115</td>
<td></td>
<td></td>
</tr>
<tr>
<td>89 179 178</td>
<td>127 509 208</td>
<td></td>
<td></td>
</tr>
<tr>
<td>113 227 226</td>
<td>139 557 118</td>
<td></td>
<td></td>
</tr>
<tr>
<td>131 263 262</td>
<td>163 653 149</td>
<td></td>
<td></td>
</tr>
<tr>
<td>173 347 346</td>
<td>193 773 317</td>
<td></td>
<td></td>
</tr>
<tr>
<td>179 359 358</td>
<td>199 797 215</td>
<td></td>
<td></td>
</tr>
<tr>
<td>191 383 382</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\( n = 14 \)

| \( k \) \( q_0 \) \( r_0 \) |
|---|---|---|
| 47 | 659 | 12 |
| 59 | 827 | 20 |
| 167 | 2339 | 190 |

\( n = 16 \)

| \( k \) \( q_0 \) \( r_0 \) |
|---|---|---|
| 61 | 977 | 52 |

\( n = 38 \)

| \( k \) \( q_0 \) \( r_0 \) |
|---|---|---|
| 197 | 7487 | 634 |

The table shows that (21) holds for \( k < 200 \) for \( a = 3, 4, \) and 5. This table can be also used for patterns with \( m > 3 \), since every pattern of length \( m \), contains patterns of length 3, and if for any of these \( \land = \infty \), then the same is true for the pattern of length \( m > 3 \). All the values in the table satisfy Theorem 2, except for \( k = 197, n = 38 \), \( q_0 = 7487 \), since \( 3^{n/4} = \)
34091.96 > q_0. Nevertheless, q = 7487 does not have a pair of consecutive 197th power residues, so that
\[ \bigwedge (197, 3; 1, a) = \bigwedge (197, 3; a - 1, a) = \infty \quad \text{for} \quad 2 < a < 634. \]

The Case \( m = 4 \)

It follows immediately that any prime \( q \) which satisfies Theorem 2 is a delay prime for the pattern
\[ x, x + 1, x + a - 1, x + a \quad \text{if} \quad a \not\equiv 0, 1 \pmod{q}, \]

since such a prime \( q \) does not have a pair of consecutive residues, except \((-1, 0)\) and \((0, 1)\). Therefore if \( q \) satisfies Theorem 2, then
\[ \bigwedge (k, 4; 1, a - 1, a) = \infty, \quad a \not\equiv 0, 1 \pmod{q}. \quad (22) \]

Graham [3] proved that
\[ A(k, 4; 1, 2, 3) = \infty \quad (23) \]

by constructing a sequence of primes \( P_t \) which have all odd primes \( p_2 = 3, p_3 = 5, \ldots, p_t \) as residues and 2 as a non-residue. Such primes \( P_t \) have oddly-even numbers for non-residues and therefore cannot have a quadruplet of \( k \)-th power residues. This shows more generally that the pattern \( x, x + a_1, x + a_2, x + a_3 \), where the \( a_i \) are incongruent and different from zero modulo four, can be delayed indefinitely, or
\[ \bigwedge (k, 4; a_1, a_2, a_3) = \infty \quad \text{if} \quad 0 \not\equiv a_i \not\equiv a_j \pmod{4}. \quad (24) \]

For \( k \) odd, the first pattern which we cannot delay with (22) is \( x, x + 1, x + q - 1, x + q \), but this can be delayed by (24) if \( q \equiv 3 \pmod{4} \). Thus
\[ (k, 4; 1, x + q - 1, x + q) = \infty \quad \text{if} \quad q \equiv 3 \pmod{4}. \quad (25) \]

For \( k = 3 \), the first pattern of this type which has a chance of having a finite limit is \( x, x + 1, x + 13, x + 14 \). Of course the pattern \((1, 8, 27)\) cannot be delayed with any modulus.

Some time ago we constructed the delaying sequence of primes \( P_t \) used by Graham up to \( t = 13 \) for cubic residues. We give a few values to indicate the rapid growth of \( P_t \).

<table>
<thead>
<tr>
<th>( t )</th>
<th>( p_t )</th>
<th>( P_t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>13</td>
<td>6883</td>
</tr>
<tr>
<td>10</td>
<td>29</td>
<td>28 08703</td>
</tr>
<tr>
<td>13</td>
<td>41</td>
<td>1182 90001</td>
</tr>
</tbody>
</table>
The corresponding sequence for \( k = 2 \) such that the Legendre symbols

\[
\left( \frac{2}{P_i} \right) = -1, \quad \left( \frac{p_i}{P_i} \right) = 1 \quad \text{for} \quad i = 2, 3, \ldots, t,
\]

will appear in the sequel to the paper [11]. We give here an except from the table:

<table>
<thead>
<tr>
<th>( t )</th>
<th>( P_t )</th>
<th>( P_t = 3 \mod 8 )</th>
<th>( P_t = 5 \mod 8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>29</td>
<td>66491</td>
<td>82021</td>
</tr>
<tr>
<td>20</td>
<td>71</td>
<td>421 43219</td>
<td>430 30381</td>
</tr>
<tr>
<td>30</td>
<td>113</td>
<td>5 16371 56931</td>
<td>8 18194 44589</td>
</tr>
<tr>
<td>38</td>
<td>163</td>
<td>5931 00821 51459</td>
<td>2534 02395 22789</td>
</tr>
</tbody>
</table>

The last two values of \( P_t \) have no pattern of incongruent modulo four quadratic residues with least element less than 167.

Hudson [6] found that a string of four consecutive quadratic residues has delay primes \( q = 5, 7, 11, 13 \) and 53, which together with the Graham function give 13 purely multiplicative functions having the property that \( \land (2, 4; 1, 2, 3) = \infty \). He conjectures that there are no others.

**References**


