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On Parameter Estimation for Semi-linear Errors-in-Variables Models*

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This paper studies a semi-linear errors-in-variables model of the form $Y_i = x_i'\beta + g(T_i) + e_i$, $X_i = x_i + u_i$ $(1 \leqslant i \leqslant n)$. The estimators of parameters β, σ^2 and of the smooth function g are derived by using the nearest neighbor-generalized least

vergence.

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1. INTRODUCTION

Consider the semi-linear errors-in-variables model as

$$\begin{cases} Y = x'\beta + g(T) + e \\ X = x + u, \end{cases} \tag{1.1}$$

where X and x are $p \times 1$ random vectors in R^p; Y and T are random real-valued variables such that T ranges over a nondegerate compact interval of one-dimension which, without loss of generality, can be the unit interval $[0, 1]$. e is an unobservable error variable and u is a $p \times 1$ unobservable error vector with

$$E[(e, u')'] = 0, \qquad \text{Cov}[(e, u')'] = \sigma^2 I_{p+1},$$

where $\sigma^2 > 0$ is an unknown parameter, β is a $p \times 1$ vector of unknown parameters, and g is an unknown smooth function of T.

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The model (1.1) is often encountered in situations in which the true values of a set of variables satisfy the exact relationship

$$y = x'\beta + g(T). \tag{1.2}$$

In these situations we often want to make inferences on β and g through the values of y and x. However, what we often encounter is that y or even both y and x are unobservable. If y is the only unobservable variable, the well-known semiparametric model (also called the semi-linear model) is introduced:

$$Y = x'\beta + g(T) + e. \tag{1.3}$$

Many researchers, such as Engle *et al.* [7], Wahba [18], Heckman [10], Chen [3], Robinson [14], Enbank and Speckman [6], Hong and Cheng [12, 13], and Donald and Newey [5], have made the focus of their research the construction of the estimators of β, σ^2, and g and proving that these estimators can attain their optimal convergence rates $n^{-1/2}, n^{-1/2}$, and $n^{-r/(2r+1)}$ (r denotes the order of smoothness of the function g), respectively. If both x and y in (1.2) are unobservable, it is natural and necessary to consider model (1.1).

Model (1.1) can be also be regarded as the result of generalizing the following model by adding the nonlinear component $g(T)$,

$$\begin{cases} Y = x'\beta + e \\ X = x + u, \end{cases} \tag{1.4}$$

where Y and X are the observable variable and the $p \times 1$ random vector, respectively. $(e, u')'$ is a measurement error vector, and β is a vector of unknown parameter. Models (1.1) and (1.4) belong to a kind of model called the errors-in-variables model. The errors-in-variables model may be applied to many fields such as economics, biology, and forestry (see Sprent [15]). Some authors have given their attention to model (1.4) and the literature includes the work Anderson [2], Glessor [9], Fuller [8], and Amemiya and Fuller [1].

The importance of adding the nonlinear component to model (1.4) in order for it to become (1.1) may be the same as that of adding the non-linear component to the linear model to allow it to become (1.3). The objective of this paper is to discuss model (1.1) on weak conditions. The estimators of β, σ^2, and g are obtained by using the nearest neighbor-generalized least square method. It is shown that the estimators of β and σ^2 are strongly consistent and asymptotically normal. The estimator of g also achieves an optimal convergence rate of $n^{-1/3}$.

2. THE CONSTRUCTION OF THE ESTIMATORS AND MAIN RESULTS

Suppose that $\{X_i=(X_{i1}, X_{i2}, ..., X_{ip})', T_i, Y_i, 1\leqslant i\leqslant n\}$ is a sample of size n from the model

$$\begin{cases} Y_i = x_i'\beta + g(T_i) + e_i \\ X_i = x_i + u_i \end{cases} \qquad (1\leqslant i\leqslant n). \tag{2.1}$$

The estimators of β, σ^2, and g are obtained by the following process. For any $t\in[0, 1]$, we arrange $|T_1-t|, |T_2-t|, ..., |T_n-t|$ in increasing order.

$$|T_{R(1,t)}-t|\leqslant|T_{R(2,t)}-t|\leqslant\cdots\leqslant|T_{R(n,t)}-t| \tag{2.2}$$

(ties are broken by comparing indices). Obviously, $R(1, t), R(2, t), ..., R(n, t)$ is a permutation of $\{1, 2, ..., n\}$. Choose a group of fixed nonnegative numbers $\{v_{ni}: 1\leqslant i\leqslant n\}$ and let $k \triangleq k_n$ be a natural number dependent solely on n. Suppose $\{v_{ni}: 1\leqslant i\leqslant n\}$ and k satisfy

(a) $\dfrac{k}{\sqrt{n}(\log n)^2}\to\infty, \qquad \dfrac{k}{n^{3/4}}\to 0 \qquad (n\to\infty),$

(b) $\sum_{i=1}^{n} v_{ni}=1, \qquad \max_{1\leqslant i\leqslant k} v_{ni}=O\left(\dfrac{1}{k}\right), \quad \sum_{i>k} v_{ni}=o(n^{-1/2}).$

Now we can define a probability weight vector $\{w_{ni}(t)=w_{ni}(t; T_1, T_2, ..., T_n), 1\leqslant i\leqslant n\}$ which satisfies $w_{nR(i,t)}(t)=v_{ni}$, $1\leqslant i\leqslant n$. Obviously, $0\leqslant v_{ni}\leqslant 1$, $0\leqslant w_{ni}(t)\leqslant 1$, for any $1\leqslant i\leqslant n$, $t\in[0, 1]$.

It follows from (2.1) that

$$Y_i - x_i'\beta = g(T_i)+e_i, \qquad 1\leqslant i\leqslant n.$$

We may define the nearest neighbor pseudo-estimator of g as

$$\begin{aligned}\hat{g}_n(t) &= \sum_{i=1}^{n} w_{ni}(t)(Y_i - x_i'\beta) = \sum_{i=1}^{n} w_{ni}(t)\, Y_i - \left(\sum_{i=1}^{n} w_{ni}(t)\, x_i\right)'\beta \\ &= \hat{g}_{1n}(t) - \hat{g}_{2n}(t)'\,\beta. \end{aligned} \tag{2.3}$$

However, since β is an unknown vector, we have to estimate β at first. Since x_i's are unobservable, the least square method may be invalid. But we can obtain $\hat{\beta}_n$, the estimator of β, by using the generalized least square method, that is, we can define $\hat{\beta}_n$ as one of the solutions of

$$\sum_{i=1}^{n} \left| \frac{\widetilde{Y}_i - \widetilde{X}_i' \hat{\beta}_n}{\sqrt{1+\|\hat{\beta}\|^2}} \right|^2 = \min_{a \in R^p} \sum_{i=1}^{n} \left| \frac{\widetilde{Y}_i - \widetilde{X}_i' a}{\sqrt{1+\|a\|^2}} \right|^2 \tag{2.4}$$

where $\widetilde{X}_i = X_i - \sum_{s=1}^n w_{ns}(T_i)\, X_s$, $\widetilde{Y}_i = Y_i - \sum_{s=1}^n w_{ns}(T_i)\, Y_s$ for $1 \leqslant i \leqslant n$.

Denote

$$X = (X_1, X_2, ..., X_n)',\ Y = (Y_1, Y_2, ..., Y_n)',\ \widetilde{X} = (\widetilde{X}_1, \widetilde{X}_2, ..., \widetilde{X}_n)',$$
$$\widetilde{Y} = (\widetilde{Y}_1, \widetilde{Y}_2, ..., \widetilde{Y}_n)'.$$

It follows from (2.4) that $\hat{\beta}_n$ satisfies

$$(1+\|\hat{\beta}_n\|^2)\left(\frac{1}{n}\widetilde{X}'\widetilde{Y} - \frac{1}{n}\widetilde{X}'\widetilde{X}\hat{\beta}_n\right) + \left[\begin{pmatrix} 1 \\ -\hat{\beta}_n \end{pmatrix}' \begin{pmatrix} \frac{1}{n}\widetilde{Y}'\widetilde{Y} & \frac{1}{n}\widetilde{Y}'\widetilde{X} \\ \frac{1}{n}\widetilde{X}'\widetilde{Y} & \frac{1}{n}\widetilde{X}'\widetilde{X} \end{pmatrix} \begin{pmatrix} 1 \\ -\hat{\beta}_n \end{pmatrix}\right] \hat{\beta}_n = 0 \tag{2.5}$$

Remark 1. If $p = 1$, from (2.5) we obtain

$$\hat{\beta}_n = \frac{(2/n)\,\widetilde{X}'\widetilde{Y}}{\sqrt{((1/n)\,\widetilde{Y}'\widetilde{Y} - (1/n)\,\widetilde{X}'\widetilde{X})^2 + 4((1/n)\,\widetilde{X}'\widetilde{Y})^2} - ((1/n)\,\widetilde{Y}'\widetilde{Y} - (1/n)\,\widetilde{X}'\widetilde{X})}.$$

If $p \geqslant 2$, $\hat{\beta}_n$ has no explicit expression.

We define the estimators of g and σ^2 respectively as

$$g_n^*(t) = \sum_{i=1}^n w_{ni}(t)\, Y_i - \left(\sum_{i=1}^n w_{ni}(t)\, X_i\right)' \hat{\beta}_n \tag{2.6}$$

$$\widehat{\sigma_n^2} = \frac{1}{n} \sum_{i=1}^n \left(\frac{\widetilde{Y}_i - \widetilde{X}_i' \hat{\beta}_n}{1+\|\hat{\beta}_n\|^2} \right)^2. \tag{2.7}$$

The following conditions are sufficient for the statement of our main results.

Condition 1. The distribution of T_1 is absolutely continuous and its density $r(t)$ satisfies

$$0 < \inf_{0 \leqslant t \leqslant 1} r(t) \leqslant \sup_{0 \leqslant t \leqslant 1} r(t) < \infty.$$

Condition 2. $\Sigma = \mathrm{Cov}(x - E(x \mid T))$ is a positive definite matrix.

Condition 3. $E(|e|^2 + \|x\|^2 + \|u_1\|^2) < \infty$; g and g_{2j} are continuous functions on interval $[0, 1]$, where $g_{2j} = E(x_{1j} \mid T_1 = t)$ is the jth component of $g_2(t) = E(x_1 \mid T_1 = t)$ for $(1 \leqslant j \leqslant p)$.

Condition 4. $E(|e_1|^4 + \|x_1\|^4 + \|u_1\|^4) < +\infty$; g and g_{2j} satisfy the Lipschitz condition and $E(x_{1j}^2 \mid T_1 = t)$ is a bounded function of t for $1 \leqslant j \leqslant p$.

Condition 4′. $E(|e_1|^3 + \|x_1\|^3 + \|u_1\|^3) < +\infty$; g and g_{2j} satisfy the Lipschitz condition and $E(x_{1j}^2 \mid T_1 = t)$ is a bounded function of t for $1 \leqslant j \leqslant p$.

Remark 2. Conditions 1–3 are necessary for studying the optimal convergence rate of the nonparametric regression estimates. See Stone [17] and Cheng [3]. Condition 4 guarantees the asymptotic normality of $\sqrt{n}(\hat{\beta}_n - \beta)$. Condition 4′ guarantees that the estimator $\hat{g}_n$ of g can reach its optimal convergence rate $n^{-1/3}$.

The main results are stated as following:

THEOREM 1. *Suppose that Condition* 1–3 *and* (a), (b) *hold. Then*

$$\hat{\beta}_n \to \beta \ a.s., \qquad \widehat{\sigma_n^2} \to \sigma^2 \ a.s.$$

THEOREM 2. *Suppose that Conditions* 1, 2, 4 *and* (a), (b) *hold and that* $\Omega_1 = \mathrm{Cov}[(e_1 - u_1'\beta)(x_1 - E(x_1 \mid T_1) + u_1) + ((e_1 - u_1'\beta)^2/(1 + \|\beta\|^2))\,\beta]$ *is a* $p \times p$ *positive definite matrix. Then*

$$\sqrt{n}\,\Omega_1^{-1/2}\Sigma(\hat{\beta}_n - \beta) \to^{\mathrm{d}} N(0, I_p).$$

where $\to^{\mathrm{d}}$ *stands for convergence in distribution.*

THEOREM 3. *Suppose that Conditions* 1, 2, 4′ *and* (a), (b) *hold and take* $k = [cn^{2/3}]$ *for some positive constant* c ([a] *denotes the largest integer no larger than* a). *Then*

$$g_n^*(t) - g(t) = O_p(n^{-1/3}) \qquad \text{for} \quad t \in [0, 1].$$

THEOREM 4. *Suppose that Conditions* 1, 2, 4 *and* (a), (b) *hold and that* $\Omega_2 = \mathrm{Cov}[(e_1 - u_1'\beta)^2)/(1 + \|\beta\|^2)] > 0$. *Then*

$$\sqrt{n}\,\Omega_2^{-1/2}(\widehat{\sigma_n^2} - \sigma^2) \to^{\mathrm{d}} N(0, 1).$$

Remark 3. If we construct the estimators of β, σ^2 and g by using the kernel-type probability weight $\{w_{ni}(t) = K((T_i - t)/h)/(\sum_{j=1}^n K((T_j - t)/h))$: $1 \leqslant i \leqslant n\}$, then, Theorems 1–4 above hold under suitable conditions (may be the window size $h \sim (k/n)$).

3. PROOFS OF MAIN RESULTS

We first give some notations

$$\widetilde{x_i} = x_i - \sum_{s=1}^{n} w_{ni}(T_i)\, x_s,$$

$$\widetilde{X_i} = X_i - \sum_{s=1}^{n} w_{ni}(T_i)\, X_s,$$

$$\widetilde{Y}_i = Y_i - \sum_{s=1}^{n} w_{ni}(T_i)\, Y_s,$$

$$\widetilde{e_i} = e_i - \sum_{s=1}^{n} w_{ni}(T_i)\, e_s,$$

$$\widetilde{u_i} = u_i - \sum_{s=1}^{n} w_{ni}(T_i)\, u_s,$$

$$h_i = (h_{i1}, h_{i2}, ..., h_{ip})' = x_i - E(x_i \mid T_i), \qquad (1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant p)$$

$$g_1(t) = E(Y_1 \mid T_1 = t),$$

$$\tilde{x} = (\widetilde{x_1}, \widetilde{x_2}, ..., \widetilde{x_n})'$$

$$\widetilde{X} = (\widetilde{X_1}, \widetilde{X_2}, ..., \widetilde{X_n})',$$

$$\widetilde{Y} = (\widetilde{Y_1}, \widetilde{Y_2}, ..., \widetilde{Y_n})'.$$

Now we give some lemmas.

LEMMA 1. (i) *Suppose* (a), (b) *and Condition* 3 *are satisfied. Then*

$$\max_{1 \leqslant i \leqslant n} \left| \sum_{s=1}^{n} w_{ns}(T_i)\, e_s \right| = o(1) \text{ a.s.} \tag{3.1}$$

$$\max_{1 \leqslant i \leqslant n} \left| \sum_{s=1}^{n} w_{ns}(T_i)\, h_{sj} \right| = o(1) \text{ a.s.} \tag{3.2}$$

$$\max_{1 \leqslant i \leqslant n} \left| \sum_{s=1}^{n} w_{ns}(T_i)\, u_{sj} \right| = o(1) \text{ a.s.} \qquad (1 \leqslant j \leqslant p) \tag{3.3}$$

(ii) *Suppose* (a), (b) *hold,* $E(|e|^l + \|x\|^l + \|u_1\|^l) < \infty$, *and* g, g_{2j} $(1 \leqslant j \leqslant p)$ *satisfy the Lipschitz condition. Then*

$$\max_{1 \leqslant i \leqslant n} \left| \sum_{s=1}^{n} w_{ns}(T_i)\, e_s \right| = o(n^{1/l - 1/2}) \text{ a.s.} \tag{3.4}$$

$$\max_{1 \leqslant i \leqslant n} \left| \sum_{s=1}^{n} w_{ns}(T_i)\, h_{sj} \right| = o(n^{1/l - 1/2}) \text{ a.s.} \qquad (1 \leqslant j \leqslant p) \tag{3.5}$$

$$\max_{1 \leqslant i \leqslant n} \left| \sum_{s=1}^{n} w_{ns}(T_i)\, u_{sj} \right| = o(n^{1/l - 1/2}) \text{ a.s.} \qquad (1 \leqslant j \leqslant p) \tag{3.6}$$

for $l = 3$ *or* 4.

Proof. We are going to prove (3.2) and (3.5) (for $l = 4$) only; (3.1) and (3.3) can be proved as (3.2), (3.4), and (3.6) can be proved as (3.5).

For any $\varepsilon > 0$, set

$$x_{sj}^{(1)} = x_{sj} I_{\{|x_{sj}| \leqslant \varepsilon^2 s^{1/2}\}}, \qquad x_{sj}^{(2)} = x_{sj} I_{\{|x_{sj}| > \varepsilon^2 s^{1/2}\}}$$
$$g_{2j}^{(1)}(T_s) = E(x_{sj}^{(1)} \mid T_s), \qquad g_{2j}^{(2)}(T_s) = E(x_{sj}^{(2)} \mid T_s),$$

where I_B denotes the indicator function of set B. Since $E\|x_1\|^2 < \infty$, by the three-series theorem we obtain $\sum_{s=1}^{+\infty} |x_{sj}^{(2)}| < \infty$.

Observe that

$$\sum_{s=1}^{n} E\,|g_{2j}^{(2)}(T_s)| \leqslant \sum_{s=1}^{n} E\,|x_{sj}|\, I_{\{|x_{sj}| > \varepsilon^2 s^{1/2}\}} \leqslant \varepsilon^{-2}$$
$$\times \sum_{s=1}^{n} s^{-1/2} E x_{sj}^2 I_{\{|x_{sj}| \geqslant \varepsilon^2 s^{1/2}\}} \leqslant 2\varepsilon^{-2} \sqrt{n}\, E x_{1j}^2,$$

$$\max_{1 \leqslant s \leqslant n} |g_{2j}^{(2)}(T_s)| \leqslant \max_{1 \leqslant s \leqslant n} (|g_{2j}(T_s)| + |g_{2j}^{(1)}(T_s)|) \leqslant 2\varepsilon^2 n^{1/2}$$
$$\text{(for large enough } n)$$

and that $\sum_{s=1}^{n} E(g_{2j}^{(2)}(T_s))^2 \leqslant n E x_{1j}^2$. By Bernstein inequality (see Hoffding [11]), we have

$$P\left\{ \sum_{s=1}^{n} |g_{2j}^{(2)}(T_s)| \geqslant \varepsilon \sqrt{n} \log n \right\}$$
$$\leqslant P\left\{ \left| \sum_{s=1}^{n} [\,|g_{2j}^{(2)}(T_s)| - E\,|g_{2j}^{(2)}(T_s)|\,] \right| \geqslant \frac{1}{2} \varepsilon \sqrt{n} \log n \right\}$$
$$\leqslant 2 \exp\left\{ -\frac{\varepsilon^2 n (\log n)^2}{8[\sum_{s=1}^{n} E(g_{2j}^{(2)}(T_s))^2 + \varepsilon^3 n \log n]} \right\}$$
$$\leqslant 2 \exp\left\{ -\frac{\log n}{16\varepsilon} \right\} = 2n^{-1/16\varepsilon}$$

for n large enough, and then by the Borel–Cantelli lemma, $\sum_{s=1}^{n} |g_{2j}^{(2)}(T_s)| \leqslant \varepsilon \sqrt{n} \log n$ a.s. for $0 < \varepsilon < 1/16$.

Let $b_n \triangleq \max_{1 \leqslant i, s \leqslant n} w_{ns}(T_i) = \max_{1 \leqslant i \leqslant n} v_{ni} \leqslant (c/k)$ for some $c > 0$. We get

$$\max_{1 \leqslant i \leqslant n} \left| \sum_{s=1}^{n} w_{ns}(T_i)(x_{sj}^{(2)} - g_{2j}^{(2)}(T_s)) \right| = o(1) \qquad \text{a.s.}$$

If we can prove

$$\max_{1 \leqslant i \leqslant n} \left| \sum_{s=1}^{n} w_{ns}(T_i)(x_{sj}^{(1)} - g_{2j}^{(1)}(T_s)) \right| \leqslant \varepsilon \qquad \text{a.s.} \quad (n \to \infty) \tag{3.7}$$

then (3.2) will hold.

For each $i (1 \leqslant i \leqslant n)$, let $Z_{ns} = w_{ns}(T_i)(x_{sj}^{(1)} - g_{2j}^{(1)}(T_s))$ for $1 \leqslant s \leqslant n$. Then, given $\triangle_n = \{T_1, T_2, ..., T_n\}$, $Z_{n1}, Z_{n2}, ..., Z_{nn}$ are conditionally independent variables. Moreover,

$$E(Z_{ns} \mid \triangle_n) = 0, \qquad \max_{1 \leqslant s \leqslant n} |Z_{ns}| \leqslant \varepsilon^2 n^{1/2} b_n,$$

$$\text{and} \qquad E(Z_{ns}^2 \mid \triangle_n) \leqslant b_n^2 E(x_{sj}^2 \mid T_s).$$

Set $\theta_n = I_{\{(1/n) \sum_{s=1}^{n} E(x_{sj}^2 \mid T_s) \leqslant E x_{1j}^2 + 1\}}$. By the Bernstein inequality and condition (a) we have

$$\begin{aligned}
p_m &= P\left\{ \bigcup_{n \geqslant m} \left[\max_{1 \leqslant i \leqslant n} \left| \sum_{s=1}^{n} Z_{ns} \right| \geqslant \varepsilon, \frac{1}{n} \sum_{s=1}^{n} E(x_{sj}^2 \mid T_s) \leqslant E x_{1j}^2 + 1 \right] \right\} \\
&\leqslant \sum_{n \geqslant m} E\left[\theta_n \sum_{i=1}^{n} P\left\{ \left| \sum_{s=1}^{n} Z_{ns} \right| \geqslant \varepsilon \mid \triangle_n) \right\} \right] \\
&\leqslant 2 \sum_{n \geqslant m} \sum_{i=1}^{n} E\left[\theta_n \exp\left\{ -\frac{n(\varepsilon/n)^2}{(2/n) \sum_{i=1}^{n} E(Z_{ns}^2 \mid \triangle_n) + \varepsilon^2 n^{1/2} b_n(\varepsilon/n)} \right\} \right] \\
&\leqslant 2 \sum_{n \geqslant m} \sum_{i=1}^{n} E\left[\theta_n \exp\left\{ -\frac{n(\varepsilon/n)^2}{2(b_n^2/n) \sum_{i=1}^{n} E(x_{sj}^2 \mid T_s) + \varepsilon^2 n^{1/2} b_n(\varepsilon/n)} \right\} \right] \\
&\leqslant 2 \sum_{n \geqslant m} \sum_{i=1}^{n} E\left[\theta_n \exp\left\{ -\frac{\varepsilon^2}{2\varepsilon^3 n^{1/2} b_n} \right\} \right] \leqslant 2 \sum_{n \geqslant m} n^{-2} \to 0
\end{aligned} \tag{3.8}$$

as $m \to \infty$. It follows from (3.8) and the strong law of large numbers that

$$P\left\{\bigcup_{n\geqslant m}\left[\max_{1\leqslant i\leqslant n}\left|\sum_{s=1}^{n} Z_{ns}\right|\geqslant \varepsilon\right]\right\}\to 0 \qquad \text{as} \quad m\to\infty.$$

Hence (3.7) is true.

Next, we prove (3.5), and denote

$$x_{sj}^{(1)} = x_{sj} I_{\{|x_{sj}|\leqslant \varepsilon^2 s^{1/4}\}}, \qquad x_{sj}^{(2)} = x_{sj} I_{\{|x_{sj}| > \varepsilon^2 s^{1/4}\}},$$

$$g_{2j}^{(1)}(T_s) = E(x_{sj}^{(1)} \mid T_s), \qquad g_{2j}^{(2)}(T_s) = E(x_{sj}^{(2)} \mid T_s)$$

$$\text{for} \quad 1\leqslant s\leqslant n, \qquad 1\leqslant j\leqslant p.$$

Note that

$$\sum_{s=1}^{n} E\,|g_{2j}^{(2)}(T_s)|^2 \leqslant \sum_{s=1}^{n} E x_{sj}^{(2)2} = \sum_{s=1}^{n} E x_{sj}^2 I_{\{|x_{sj}|>\varepsilon^2 s^{1/4}\}}$$

$$\leqslant \varepsilon^{-4}\sum_{s=1}^{n} E x_{sj}^4 \cdot s^{-1/2} I_{\{|x_{sj}|>\varepsilon^2 s^{1/4}\}} \leqslant 2\varepsilon^{-4}\sqrt{n}\, E x_{1j}^4.$$

Using the similar argument that was used to derive (3.6), we obtain

$$\max_{1\leqslant i\leqslant n}\left|\sum_{s=1}^{n} w_{ns}(T_i)(x_{sj}^{(2)} - g_{2j}^{(2)}(T_s))\right| = o(n^{-1/4}) \qquad \text{a.s.} \tag{3.9}$$

for any $0<\varepsilon<1/16$. If we can prove

$$\max_{1\leqslant i\leqslant n}\left|\sum_{s=1}^{n} w_{ns}(T_i)(x_{sj}^{(1)} - g_{2j}^{(1)}(T_s))\right| \leqslant \varepsilon n^{-1/4} \qquad \text{a.s.} \quad (n\to\infty), \tag{3.10}$$

then (3.4) will hold. Since $E(x_{1j}^2 \mid T_1 = t)$ is a bounded function of t, there exists a positive constant $M>0$, such that

$$\frac{1}{n}\sum_{s=1}^{n} E(Z_{ns}^2 \mid \triangle_n) \leqslant \frac{1}{n} M b_n. \tag{3.11}$$

It follows from (3.11) and the Bernstein inequality that

$$P\left\{\bigcup_{n\geqslant m}\left(\max_{1\leqslant i\leqslant n}\left|\sum_{s=1}^{n} w_{ns}(T_i)(x_{sj}^{(1)} - g_{2j}^{(1)}(T_s))\right| \geqslant \varepsilon n^{-1/4}\right)\right\}\to 0 \qquad \text{as} \quad m\to\infty,$$

which establishes (3.10).

Remark 4. For any $0 < \varepsilon < 1/16$, from (3.5) and (3.9), we have

$$\max_{1 \leqslant i \leqslant n} \left| \sum_{s=1}^{n} w_{ns}(T_i)(x_{sj}^{(1)} - g_{2j}^{(1)}(T_s)) \right| = o(n^{-1/4}) \qquad \text{a.s.}$$

LEMMA 2. (i) *Assume that Condition* 1 *holds and that f is a continuous function on interval* $[0, 1]$, *and* $(k/\log n) \to \infty$, $(k/n) \to 0 (n \to \infty)$. *Then*

$$\sup_{0 \leqslant t \leqslant 1} |T_{R(k,\, t)} - t| = o(1) \qquad \text{a.s.} \tag{3.12}$$

$$\max_{1 \leqslant i \leqslant n} \left| f(T_i) - \sum_{s=1}^{n} w_{ns}(T_i)\, f(T_s) \right| = o(1) \qquad \text{a.s.} \tag{3.13}$$

(ii) *Assume that Condition* 1 *holds and that f satisfies Lipschitz condition and* $(k/\log n) \to \infty$, $(k/n^{3/4}) \to 0$ $(n \to \infty)$. *Then*

$$\sup_{0 \leqslant t \leqslant 1} |T_{R(k,\, t)} - t| = o(n^{-1/4}) \qquad \text{a.s.} \tag{3.14}$$

$$\max_{1 \leqslant i \leqslant n} | f(T_i) - \sum_{s=1}^{n} w_{ns}(T_i)\, f(T_s)| = o(n^{-1/4}) \qquad \text{a.s.} \tag{3.15}$$

Proof. Equation (3.12) is due to arguments of Hong [12]. Next we are going to prove (3.13). Since f is continuous on interval $[0, 1]$, then it is uniformly continuous, and for $\varepsilon > 0$, there exists a positive number $\delta(\varepsilon)$ such that if $|f(t_1) - f(t_2)| \geqslant (\varepsilon/2)$, then $|t_1 - t_2| \geqslant \delta(\varepsilon)$. Therefore, we have

$$\begin{aligned} &\left\{ \max_{1 \leqslant i \leqslant n} \left| f(T_i) - \sum_{s=1}^{n} w_{ns}(T_i)\, f(T_s) \right| \geqslant \varepsilon \right\} \\ &\quad = \bigcup_{i=1}^{n} \left\{ \left| \sum_{s=1}^{n} v_{ns}(f(T_i) - f(T_{R(s,\, T_i)}) \right| \geqslant \varepsilon \right\} \\ &\quad \subseteq \bigcup_{i=1}^{n} \left\{ \sum_{s=1}^{k} v_{ns} \left| (f(T_i) - f(T_{R(s,\, T_i)}) \right| \geqslant \frac{\varepsilon}{2} \right\} \subseteq \{ \sup_{0 \leqslant t \leqslant 1} |T_{R(k,\, t)} - t| \geqslant \delta(\varepsilon) \} \end{aligned}$$

for n large enough, and

$$\begin{aligned} &P\left\{ \bigcup_{n \geqslant m} \max_{1 \leqslant i \leqslant n} \left| f(T_i) - \sum_{s=1}^{n} w_{ns}(T_i)\, f(T_s) \right| \geqslant \varepsilon \right\} \\ &\quad \leqslant P\left\{ \bigcup_{n \geqslant m} (\sup_{0 \leqslant t \leqslant 1} |T_{R(k,\, t)} - t| \geqslant \delta(\varepsilon)) \right\} \to 0 \end{aligned}$$

as $m \to \infty$, which implies that (3.13) is true. We will prove (3.14). It follows from the arguments of Hong [12] that

$$\sup_{0 \leqslant t \leqslant 1} |T_{R(k,\, t_i)} - t| \leqslant \max_{0 \leqslant i \leqslant i_n} |T_{R(k,\, t_i)} - t_i| + \frac{k}{n}$$

where $t_i = (ik/n)$, $1 \leqslant i \leqslant i_n = [n/k]$. Therefore, to prove (3.13), it suffices to prove

$$\max_{0 \leqslant i \leqslant i_n} |T_{R(k,\, t_i)} - t_i| = o(n^{-1/4}) \qquad \text{a.s.} \tag{3.16}$$

In fact, by Condition 1 we know that there is a positive constant c_1 such that

$$p_i = P\left\{T_1 \in \left[t_i - \frac{\sqrt{M}\,k}{n},\, t_i + \frac{\sqrt{M}\,k}{n}\right]\right\} \geqslant c_1 \frac{\sqrt{M}\,k}{n}$$

for any $M > 0$, if n is large enough. Take $M > (2/c_1)^2$ and $Q_{ni} = \#(\{T_1, T_2, ..., T_n\} \cap [t_i - (\sqrt{M}\,k/n), t_i + (\sqrt{M}\,k/n)])$, where $\#B$ denotes the number of elements in set B; then

$$P\left\{|T_{R(k,\, t_i)} - t_i| \geqslant \frac{\sqrt{M}\,k}{n}\right\} \leqslant P\left\{\frac{Q_{ni}}{n} - p_i \leqslant -\frac{p_i}{2}\right\}$$

$$\leqslant \exp\left\{-\frac{n(p_i/2)^2}{2p_i + (p_i/2)}\right\} \leqslant \exp\left\{-\frac{k}{5}\right\}$$

by the Hoffding inequality. Since $(k/n^{3/4}) \to 0$, for any $\varepsilon > 0$ there exists a natural number N such that if $n \geqslant N$ then $(\sqrt{M}\,k/n) \leqslant n^{-1/4}\varepsilon$. Therefore, by the fact that $(k/\log n) \to +\infty$, we obtain

$$\sum_{n=N}^{+\infty} P\{\max_{0 \leqslant i \leqslant i_n} |T_{R(k,\, t_i)} - t_i| \geqslant n^{-1/4}\varepsilon\}$$

$$\leqslant \sum_{n=N}^{+\infty} P\left\{\max_{0 \leqslant i \leqslant i_n} |T_{R(k,\, t_i)} - t_i| \geqslant \frac{\sqrt{M}\,k}{n}\right\}$$

$$\leqslant \sum_{n=N}^{+\infty} \sum_{i=1}^{i_n} P\left\{|T_{R(k,\, t_i)} - t_i| \geqslant \frac{\sqrt{M}\,k}{n}\right\} \leqslant \sum_{n=N}^{+\infty} \sum_{i=1}^{i_n} \exp\left\{-\frac{k}{5}\right\} < +\infty.$$

It follows from the Borel–Cantelli Lemma that (3.16) is true and then (3.14) holds.

For each i $(1 \leqslant i \leqslant n)$, by the fact that f satisfies the Lipschitz condition we obtain

$$\begin{aligned}\left|f(T_i) - \sum_{s=1}^{n} w_{ns}(T_i)\, f(T_s)\right| &= \left|f(T_i) - \sum_{s=1}^{n} w_{nR(s,\,T_i)}(T_i)\, f(T_{R(s,\,T_i)})\right| \\ &= \left|f(T_i) - \sum_{s=1}^{n} v_{ns} f(T_{R(s,\,T_i)})\right| \\ &= \left|\sum_{s=1}^{n} v_{ns}(f(T_i) - f(T_{R(s,\,T_i)}))\right| \\ &\leqslant c \sum_{s=1}^{n} v_{ns}\, |T_i - T_{R(s,\,T_i)}| \\ &\leqslant c \left(\sum_{s>k} v_{ns} + \frac{1}{k} \sum_{s=1}^{k} |T_i - T_{R(s,\,T_i)}|\right) \\ &\leqslant c \left(\sum_{s>k} v_{ns} + \sup_{0 \leqslant t \leqslant 1} |T_{R(k,\,t)} - t|\right). \qquad (3.17)\end{aligned}$$

Therefore, (3.15) holds in view of (3.14), (3.17), and condition (b).

LEMMA 3. *Under the condition of Theorem* 1, *we have*

$$\frac{1}{n} \tilde{X}' \tilde{X} \to \Sigma + \sigma^2 I_p \quad \text{a.s.}, \qquad \frac{1}{n} \tilde{X}' \tilde{Y} \to \Sigma\beta \quad \text{a.s.},$$

$$\frac{1}{n} \tilde{Y}' \tilde{Y} \to \beta' \Sigma \beta + \sigma^2 \quad \text{a.s.}, \qquad A_n \to A \quad \text{a.s.}$$

where

$$A_n = \begin{pmatrix} \frac{1}{n} \tilde{Y}' \tilde{Y} & \frac{1}{n} \tilde{Y}' \tilde{X} \\ \frac{1}{n} \tilde{X}' \tilde{Y} & \frac{1}{n} \tilde{X}' \tilde{X} \end{pmatrix}, \qquad A = \begin{pmatrix} \beta^\tau \Sigma \beta + \sigma^2 & \beta^\tau \Sigma \\ \Sigma\beta & \Sigma + \sigma^2 I_p \end{pmatrix}.$$

Proof. First we prove $(1/n)\, \tilde{x}' \tilde{x} \to \Sigma$ a.s. It suffices to check the convergence of the (s, m) element of $(1/n)\, \tilde{x}' \tilde{x}$ for $1 \leqslant s, m \leqslant p$.

Observe that $x_i = h_i + g(T_i)(1 \leqslant i \leqslant n)$, and we have

$$\begin{aligned}\left(\frac{1}{n}\tilde{x}'\tilde{x}\right)_{s,m} &= \left(\frac{1}{n}\sum_{i=1}^{n}\tilde{x}_i\tilde{x}_i'\right)_{s,m} \\ &= \frac{1}{n}\sum_{i=1}^{n}\left[h_{is} - \sum_{j=1}^{n} w_{nj}(T_i)\,h_{js} + \left(g_{2s}(T_i) - \sum_{j=1}^{n} w_{nj}(T_i)\,g_{2s}(T_j)\right)\right] \\ &\quad \times\left[h_{im} - \sum_{j=1}^{n} w_{nj}(T_i)\,h_{jm} + \left(g_{2m}(T_i) - \sum_{j=1}^{n} w_{nj}(T_i)\,g_{2m}(T_j)\right)\right] \\ &= \frac{1}{n}\sum_{i=1}^{n} h_{is}h_{im} + R_{1n}(s,m).\end{aligned}$$

By virtue of Lemmas 1, 2 and the strong law of large numbers, we have

$$\frac{1}{n}\sum_{i=1}^{n} h_{is}h_{im} \to Eh_{1s}h_{1m} \quad \text{a.s.} \qquad R_{1n}(s,m) \to 0 \quad \text{a.s.}$$

Therefore, $((1/n)\,\tilde{x}'\tilde{x})_{s,m} \to Eh_{1s}h_{1m}$ a.s. and so $(1/n)\,\tilde{x}'\tilde{x} \to Eh_1h_1' = \Sigma$ a.s. Note that $X_i = h_i + u_i + g_2(T_i)$ $(1 \leqslant i \leqslant n)$. It follows from the similar argument that $(1/n)\,\tilde{X}'\tilde{X} \to \Sigma + \sigma^2 I_p$ a.s.; the others can be proved in a similar way.

LEMMA 4. *Suppose that* $\{\lambda_n(a)\colon n \geqslant 1\}$ *and* $\lambda(a)$ *are a sequence of random continuous functions and nonrandom continuous functions, respectively*; $\lambda(a)$ *has the sole minimum point* a_0: *i.e.*, $\inf_{\{\|a = a_0\| \geqslant d\}} \lambda(a) > \lambda(a_0)$, *for any* $d > 0$, *where* a_n *is a minimum point of* $\lambda_n(a)$. *If* $\sup_{a \in R^p} |\lambda_n(a) - \lambda(a)| \to 0$ a.s., *then* $a_n \to a_0$ a.s.

Proof of Theorem 1. We have defined the estimates of β and σ^2, i.e., $\hat{\beta}_n$ and $\widehat{\sigma_n^2}$. Let

$$\lambda_n(a) = \frac{(1, -a')\,A_n(1, -a')'}{1 + \|a\|^2}, \qquad \lambda(a) = \frac{(1, -a')\,A(1, -a')'}{1 + \|a\|^2}.$$

Note that $\lambda(a)$ has the sole point β, and

$$\sup_a |\lambda_n(a) - \lambda(a)| \leqslant \|A_n - A\| \to 0 \qquad \text{a.s.}$$

by Lemma 3; then we have $\hat{\beta}_n \to \beta$ a.s. by Lemma 5, and it follows that $\widehat{\sigma_n^2} \to \sigma^2$ a.s. by the definition of $\widehat{\sigma_n^2}$.

To prove Theorems 2–4, we need more lemmas as follows:

LEMMA 5. *Suppose that* (a), (b) *and Condition* 1 *hold, and that* g, g_{2j} $(1 \leqslant j \leqslant p)$ *satisfy the Lipschitz condition,* $E(\|x_1\|^3 + \|u_1\|^3 + |e_1|^3) < \infty$. *Then, for any* s, m $(1 \leqslant s, m \leqslant p)$, *we have*

$$\frac{1}{n} \sum_{i=1}^{n} \xi_i \left(f(T_i) - \sum_{j=1}^{n} w_{nj}(T_i)\, f(T_j) \right) = o(n^{-3/4} \log n) \qquad \text{a.s.} \qquad (3.18)$$

$$\frac{1}{n} \sum_{i=1}^{n} \xi_i \left(\sum_{j=1}^{n} w_{nj}(T_i) \eta_j \right) = o(n^{-2/3} \log n) \qquad \text{a.s.} \qquad (3.19)$$

where $f = g$ *or* g_{2s}, *the sequences* $\{\xi_1, \xi_2, ..., \xi_n\}$ *and* $\{\eta_1, \eta_2, ..., \eta_n\}$ *can be any two different sequences among* $\{h_{1s}, h_{2s}, ..., h_{ns}\}$, $\{u_{1m}, u_{2m}, ..., u_{nm}\}$, *and* $\{e_1, e_2, ..., e_n\}$.

Proof. These conclusions can be proved in a similar way as that used for Lemma 1, so we omit it.

LEMMA 6. *Suppose that* (a), (b), *and Condition* 1 *hold, that* g, g_{2j} *satisfy the Lipschitz condition, that* $E(x_{1j}^2 \mid T_1 = t)$ *is a bounded function of* $t (1 \leqslant j \leqslant p)$, *and that* $E(\|x_1\|^l + \|u_1\|^l + |e_1|^l) < \infty$. *Then, we have*

$$\frac{1}{n} \sum_{i=1}^{n} h_{is} \left(\sum_{j=1}^{n} w_{nj}(T_i)\, h_{jm} \right) = o(n^{-(l-1)/6}) \qquad \text{a.s.} \qquad (3.20)$$

$$\frac{1}{n} \sum_{i=1}^{n} e_i \left(\sum_{j=1}^{n} w_{nj}(T_i)\, e_j \right) = o(n^{-(l-1)/6}) \qquad \text{a.s.} \qquad (3.21)$$

$$\frac{1}{n} \sum_{i=1}^{n} u_{is} \left(\sum_{j=1}^{n} w_{nj}(T_i)\, u_{jm} \right) = o(n^{-(l-1)/6}) \qquad \text{a.s.} \qquad (3.22)$$

for $1 \leqslant s, m \leqslant p$, $l = 3$ *or* 4.

Proof. These conclusions can be proved in a similar way. We are going to prove that (3.20) holds for the case of $l = 4$, i.e.,

$$\frac{1}{n} \sum_{i=1}^{n} h_{is} \left(\sum_{j=1}^{n} w_{nj}(T_i)\, h_{jm} \right) = o(n^{-1/2}) \qquad \text{a.s.} \qquad (3.23)$$

Observe that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} h_{is}\left(\sum_{j=1}^{n} w_{nj}(T_i)\, h_{jm}\right)=\frac{1}{\sqrt{n}}\sum_{i=1}^{n} w_{ni}(T_i)\, h_{is}h_{im}$$
$$+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{j\neq i} w_{nj}(T_i)\, h_{is}h_{jm}=J_1+J_2.$$

Since

$$|J_1|\leqslant\frac{b_n}{\sqrt{n}}\sum_{i=1}^{n}|h_{is}|\,|h_{im}|\leqslant\frac{c}{k\sqrt{n}}\sum_{i=1}^{n}|h_{is}|\,|h_{im}|\to 0\qquad\text{a.s.},$$

where $b_n=\max_{1\leqslant i\leqslant n} v_{ni}$. If we can prove $J_2=o(1)$, (3.23) will hold. Take ε_0, such that $0<\varepsilon_0<1/16$, and set

$$x_{is}^{(1)}=x_{is}I_{\{|x_{is}|\leqslant\varepsilon_0^2 i^{1/4}\}},\qquad x_{is}^{(2)}=x_{is}I_{\{|x_{is}|>\varepsilon_0^2 i^{1/4}\}},$$
$$g_{2s}^{(1)}(T_i)=E(x_{is}^{(1)}\mid T_i),\qquad g_{2s}^{(2)}(T_i)=E(x_{is}^{(2)}\mid T_i),$$
$$h_{is}^{(1)}=x_{is}^{(1)}-E(x_{is}^{(1)}\mid T_i),\qquad h_{is}^{(2)}=x_{is}^{(2)}-g_{2s}^{(2)}(T_i)$$
$$(1\leqslant i\leqslant n,\ 1\leqslant s\leqslant p),$$

$$U_n=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{j\neq i} w_{nj}(T_i)\, h_{is}^{(1)}h_{jm}^{(1)}.$$

It can be shown that $\sum_{i=1}^{n}|g_{2j}^{(2)}(T_i)|\leqslant\varepsilon_0 n^{1/4}\log n$ a.s. $(n\to\infty)$ and $\sum_{i=1}^{n}|x_{is}^{(2)}|<+\infty$. Thus

$$\sum_{i=1}^{n}|h_{is}^{(2)}|\leqslant 2\varepsilon_0 n^{1/4}\log n\qquad\text{a.s.}\quad(n\to\infty).\tag{3.24}$$

Similar to the proof of (3.4)–(3.6), we have

$$\max_{1\leqslant i\leqslant n}\left|\sum_{j\neq i} w_{nj}(T_i)\, h_{jm}\right|=o(n^{-1/4})\qquad\text{a.s.}\tag{3.25}$$

$$\max_{1\leqslant j\leqslant n}\left|\sum_{i\neq j} w_{nj}(T_i)(x_{is}^{(1)}-g_{2s}^{(1)}(T_i))\right|=o(n^{-1/4})\qquad\text{a.s.}\tag{3.26}$$

Note that

$$
\begin{aligned}
|J_n - U_n| &= \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sum_{j \neq i} w_{nj}(T_i)\, h_{is} h_{jm} - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sum_{j \neq i} w_{nj}(T_i)\, h_{is}^{(1)} h_{jm}^{(1)} \right| \\
&\leqslant \frac{1}{\sqrt{n}} \left| \sum_{i=1}^{n} \sum_{j \neq i} w_{nj}(T_i)\, h_{is}^{(2)} h_{jm} \right| + \frac{1}{\sqrt{n}} \left| \sum_{i=1}^{n} \sum_{j \neq i} w_{nj}(T_i)\, h_{is}^{(1)} h_{jm}^{(1)} \right| \\
&\leqslant \frac{1}{\sqrt{n}} \left(\max_{1 \leqslant i \leqslant n} \left| \sum_{j \neq i} w_{nj}(T_i)\, h_{jm} \right| \right) \sum_{i=1}^{n} |h_{is}^{(2)}| \\
&\quad + \frac{1}{\sqrt{n}} \left(\max_{1 \leqslant j \leqslant n} \left| \sum_{i \neq j} w_{nj}(T_i)\, h_{is}^{(1)} \right| \right) \sum_{j=1}^{n} |h_{jm}^{(1)}|). \qquad (3.27)
\end{aligned}
$$

Combining (3.24)–(3.27), we obtain $|J_2 - U_n| = o(1)$ a.s. Therefore, in order to prove $J_2 = o(1)$, it suffices to prove $U_n = o(1)$ a.s. For any $\varepsilon > 0$, let $j_n = [\varepsilon^3 \sqrt{n/(\log n)}]$ and

$$
\begin{aligned}
B_q &= \left\{ \left[\frac{(q-1)\, n}{j_n} \right] + 1, \left[\frac{(q-1)\, n}{j_n} \right] + 2, ..., \left[\frac{qn}{j_n} \right] \right\}, \\
B_q^c &= \{1, 2, ..., n\} - B_q, \qquad B_{qi} = B_q - \{i\}
\end{aligned}
$$

for $1 \leqslant q \leqslant j_n$. Write

$$
\begin{aligned}
U_n &= \frac{1}{\sqrt{n}} \sum_{q=1}^{j_n} \sum_{i \in B_q} \sum_{j \in B_{qi}} w_{nj}(T_i)\, h_{is}^{(1)} h_{jm}^{(1)} + \frac{1}{\sqrt{n}} \sum_{q=1}^{j_n} \sum_{i \in B_q} \sum_{j \in B_q^c} w_{nj}(T_i)\, h_{is}^{(1)} h_{jm}^{(1)} \\
&= \frac{1}{\sqrt{n}} \sum_{q=1}^{j_n} \xi_{nq} + \frac{1}{\sqrt{n}} \sum_{q=1}^{j_n} \delta_{nq} \triangleq U_{n1} + U_{n2} \qquad (3.28)
\end{aligned}
$$

Let $\delta_{nq} = \sum_{i \in B_q} \gamma_{niq} = \sum_{i \in B_q} d_{niq} h_{is}^{(1)}$, where $d_{niq} = \sum_{j \in B_q^c} w_{nj}(T_i)\, h_{is}^{(1)} h_{jm}$. It is clear that given $\Delta_{nq} = \{(T_i, x_j)\colon 1 \leqslant i \leqslant n,\ j \in B_q^c\}$, $\{\gamma_{niq}\colon i \in B_q\}$ are conditionally independent variables. Since

$$
E(\gamma_{niq} \mid \triangle_{nq}) = 0, \qquad E(\gamma_{niq}^2 \mid \triangle_{nq}) \leqslant M(\max_{1 \leqslant i \leqslant n} |d_{niq}|)^2 = M d_{nq}^2,
$$

for each $i \in B_q$, some constant $M > 0$, and $\max_{i \in B_q} |\gamma_{niq}| \leqslant 2\varepsilon_0^2 n^{1/4}\, d_{nq}$.

Therefore, similar to the Proof of Lemma 1, by condition (a) we may show that

$$
d_n = \max_{1 \leqslant i \leqslant j_n} d_{nq} = \max_{1 \leqslant q \leqslant j_n} \max_{1 \leqslant i \leqslant n} \left| \sum_{j \in B_q^c} w_{nj}(T_i)\, h'_{jm} \right| = o(n^{-1/4} (\log n)^{-1/2})
$$

when $d_n \leqslant \varepsilon n^{-1/4}(\log n)^{-1/2}$, then by the Bernstein inequality we get

$$P\{|\delta_{nq}| \geqslant \varepsilon\sqrt{n}\, j_n^{-1} \mid \triangle_{nq}\} \leqslant 2\exp\left\{-\frac{\varepsilon^2 n j_n^{-2}}{2Md_n^2(\# B_q) + 2\varepsilon\sqrt{n}\, j_n^{-1} n^{1/4} d_n}\right\}$$

$$\leqslant 2\exp\{-c\varepsilon^2(n\log n)^{1/2} j_n^{-1}\}$$

$$\leqslant 2\exp\left\{-\frac{c}{\varepsilon}\log n\right\}$$

Therefore, for any $\varepsilon > 0$ small enough,

$$P\left\{\bigcup_{n\geqslant m}(|U_{n2}| \geqslant \varepsilon, d_n \leqslant \varepsilon n^{-1/4}(\log n)^{-1/2})\right\}$$

$$\leqslant \sum_{n\geqslant m}\sum_{q=1}^{j_n} E[I_{\{d_n \leqslant \varepsilon n^{-1/4}(\log n)^{-1/2}\}} P\{|\delta_{nq}| \geqslant \varepsilon\sqrt{n} \mid \delta_{nq}\}] \to 0$$

as $m \to \infty$, which together with the fact that $d_n = o(n^{-1/4}(\log n)^{-1/2})$ establishes $U_{n2} = o(1)$ a.s. Therefore, to prove $U_n = o(1)$ a.s., we need only show that $U_{n1} = o(1)$ a.s. Under Condition 1, we have

$$Ew_{nj}^2(T_i) \leqslant \frac{c}{nk}, \qquad E[w_{nj}(T_i)\, w_{ni}(T_j)] \leqslant \frac{c}{nk} \tag{3.29}$$

for $1 \leqslant i \neq j \leqslant p$, some $c > 0$, and

$$E\xi_{nq} = E\left(\sum_{i\in B_q}\sum_{j\in B_{qi}} w_{nj}(T_i)\, h_{is}^{(1)} h_{jm}^{(1)}\right)$$

$$= E\left[E\left(\sum_{i\in B_q}\sum_{j\in B_{qi}} w_{nj}(T_i)\, h_{is}^{(1)} h_{jm}^{(1)} \mid T_1, T_2, ..., T_n\right)\right] = 0$$

$$E(\xi_{nq_1}\xi_{nq_2})$$

$$= E\left[\left(\sum_{i_1\in B_{q_1}}\sum_{j_1\in B_{q_1 i_1}} w_{nj_1}(T_{i_1})\, h_{i_1 s}^{(1)} h_{j_1 m}^{(1)}\right)\left(\sum_{i_2\in B_{q_2}}\sum_{j_2\in B_{q_2 i_2}} w_{nj_2}(T_{i_2})\, h_{i_2 s}^{(1)} h_{j_2 m}^{(1)}\right)\right]$$

$$= E\left[\sum_{i_1\in B_{q_1}}\sum_{j_1\in B_{q_1 i_1}}\sum_{i_2\in B_{q_2}}\sum_{j_2\in B_{q_2 i_2}} E(h_{i_1 s}^{(1)} h_{i_2 s}^{(1)} h_{j_1 m}^{(1)} h_{j_2 m}^{(1)} \mid T_1, T_2, ..., T_n)\right]$$

$$= 0$$

for $q_1 \neq q_2$, therefore, we obtain

$$
\begin{aligned}
P\{|U_{n1}| \geqslant \varepsilon\} &\leqslant \frac{1}{n\varepsilon^2} E\left(\sum_{q=1}^{j_n} \xi_{nq}\right)^2 = \frac{1}{n\varepsilon^2} \sum_{q=1}^{j_n} E(\xi_{nq}^2) \\
&= \frac{1}{n\varepsilon^2} \sum_{q=1}^{j_n} E\left(\sum_{i \in B_q} \sum_{j \in B_{qi}} w_{nj}(T_i)\, h_{is}^{(1)} h_{jm}^{(1)}\right)^2 \\
&= \frac{1}{n\varepsilon^2} \sum_{q=1}^{j_n} \sum_{i \in B_q} \left(\sum_{j \in B_{qi}} E[w_{nj}^2(T_i)\, h^{(1)2}_{jm}\, h^{(1)2}_{is}]\right. \\
&\quad + \frac{1}{n\varepsilon^2} \sum_{q=1}^{j_n} E\left[\sum_{i_1, i_2 \in B_q, i_1 \neq i_2} \left(\sum_{j_1 \in B_{qi_1}} w_{nj_1}(T_{i_1})\, h_{j_1 m}^{(1)} h_{i_1 s}^{(1)}\right)\right. \\
&\quad \times \left.\left(\sum_{j_2 \in B_{qi_2}} w_{nj_2}(T_{i_2})\, h_{j_2 m}^{(1)} h_{i_2 s}^{(1)}\right)\right] \triangleq R_1 + R_2.
\end{aligned}
$$

Since $E(x_{1m}^2 \mid T_1 = t) \leqslant M$, M is a positive constant independent of m and t for $1 \leqslant m \leqslant p$. Thus, by (3.29) we have

$$
\begin{aligned}
R_1 &= \frac{1}{n\varepsilon^2} \sum_{q=1}^{j_n} E\left[E\left(\sum_{i \in B_q} \sum_{j \in B_{qi}} w_{nj}(T_i)\, h'^2_{jm} h'^2_{is} \mid T_1, T_2, ..., T_n\right)\right] \\
&\leqslant \frac{1}{n\varepsilon^2} \sum_{q=1}^{j_n} \sum_{i \in B_q} \sum_{j \in B_{qi}} E[w_{nj}^2(T_i)\, E(x_{jm}^2 \mid T_j)\, E(x_{is}^2 \mid T_i)] \\
&\leqslant \frac{M^2}{n\varepsilon^2} \sum_{q=1}^{j_n} \sum_{i \in B_q} \sum_{j \in B_{qi}} E[w_{nj}^2(T_i)] \leqslant \frac{M^2}{n\varepsilon^2} j_n(\# B_q)^2 \frac{c}{nk}
\end{aligned} \tag{3.30}
$$

and by the Cauchy inequality and (3.29) we have

$$
\begin{aligned}
R_2 &= \frac{1}{n\varepsilon^2} \sum_{q=1}^{j_n} \sum_{i_1, i_2 \in B_q, i_1 \neq i_2} \sum_{j_1 \in B_{qi_1}} \sum_{j_2 \in B_{qi_2}} E[w_{nj_1}(T_{i_1})\, w_{nj_2}(T_{i_2}) \\
&\quad E(h_{j_1 m}^{(1)} h_{i_1 s}^{(1)} h_{j_2 m}^{(1)} h_{i_2 s}^{(1)} \mid T_1, T_2, ..., T_n)] \\
&\leqslant \frac{1}{n\varepsilon^2} \sum_{q=1}^{j_n} \sum_{i_1, i_2 \in B_q, i_1 \neq i_2} E\{w_{ni_1}(T_{i_2})\, w_{ni_2}(T_{i_1})[E(h^{(1)2}_{i_1 m} \mid T_{i_1}) \\
&\quad \times E(h^{(1)2}_{i_1 s} \mid T_{i_1})]^{1/2} [E(h^{(1)2}_{i_2 m} \mid T_{i_2})\, E(h^{(1)2}_{i_2 s} \mid T_{i_2})]^{1/2}\} \\
&\leqslant \frac{1}{n\varepsilon^2} \sum_{q=1}^{j_n} \sum_{i_1, i_2 \in B_q, i_1 \neq i_2} E\{w_{ni_1}(T_{i_2})\, w_{ni_2}(T_{i_1})[E(x_{i_1 m}^2 \mid T_{i_1})\, E(x_{i_1 s}^2 \mid T_{i_1})]^{1/2} \\
&\quad [E(x_{i_2 m}^2 \mid T_{i_2})\, E(x_{i_2 s}^2 \mid T_{i_2})]^{1/2}\} \\
&\leqslant \frac{M^2}{n\varepsilon^2} j_n(\# B_q)^2\, E(w_{ni_1}(T_{i_2})\, w_{ni_2}(T_{i_1})) \leqslant \frac{M^2}{n\varepsilon^2} j_n(\# B_q)^2 \frac{c}{nk}.
\end{aligned} \tag{3.31}
$$

It follows from (3.30) and (3.31) that

$$P\{|U_{n1}| \geqslant \varepsilon\} = R_1 + R_2 \leqslant \frac{2M^2}{n\varepsilon^2} j_n(\#B_q)^2 \frac{c}{nk} \leqslant \frac{c^*}{n(\log n)^{3/2}}$$

for some constants c, $c^* > 0$. Therefore, by the Borel–Cantelli lemma we get $U_{n1} = o(1)$ a.s., which ends the proof of (3.20).

Now, we can prove the following lemma which is essential to the proof of Theorems 2–4.

LEMMA 7. *Suppose that Condition* 1 *and* (a), (b) *hold, and that* g, g_{2j} *satisfy the Lipschitz condition,* $E(x_{1j}^2 \mid T_1 = t)$ *is a bounded function of* t $(1 \leqslant j \leqslant p)$, *and* $E(\|x_1\|^l + \|u_1\|^l + |e_1|^l) < \infty$. *Then*

$$\frac{1}{n} \tilde{X}' \tilde{X} = \frac{1}{n} \sum_{i=1}^{n} (h_i + u_i)(h_i + u_i)' + o(n^{-(l-1)/6}) \qquad \text{a.s.,} \tag{3.32}$$

$$\frac{1}{n} \tilde{X}' \tilde{Y} = \frac{1}{n} \sum_{i=1}^{n} (h_i + u_i)(h_i' \beta + e_i) + o(n^{-(l-1)/6}) \qquad \text{a.s.,} \tag{3.33}$$

$$\frac{1}{n} \tilde{Y}' \tilde{Y} = \frac{1}{n} \sum_{i=1}^{n} (h_i' \beta + e_i)^2 + o(n^{-(l-1)/6}) \qquad \text{a.s.,} \tag{3.34}$$

for $l = 3$ *or* 4.

Proof. These conclusions can be proved in similar ways, so we are going to prove (3.32) only. Observe that $X_i = h_i + u_i + g_2(T_i)$ $(1 \leqslant i \leqslant n)$ and write

$$\left(\frac{1}{n} \tilde{X}' \tilde{X}\right)_{s,m} = \left(\frac{1}{n} \left[X_i - \sum_{j=1}^{n} w_{nj}(T_i) X_j\right]\left[X_i - \sum_{j=1}^{n} w_{nj}(T_i) X_j\right]\right)_{s,m}$$
$$= \frac{1}{n} \sum_{i=1}^{n} (h_{is} + u_{is})(h_{im} + u_{im}) + R_{2n}(s, m).$$

By virtue of Lemmas 1 and 6, we have $R_{2n}(s, m) = o(n^{-(l-1)/6})$ a.s. $(1 \leqslant s, m \leqslant p)$. Then

$$\frac{1}{n} \tilde{X}' \tilde{X} = \frac{1}{n} \sum_{i=1}^{n} (h_i + u_i)(h_i + u_i)' + o(n^{-(l-1)/6}) \qquad \text{a.s.}$$

LEMMA 8. *Suppose that* (b) *holds and that* g_{2j} *satisfies Lipschitz condition,* $E(x_{1j}^2 \mid T_1 = t)$ *is a bounded function of* t, *and* $k/(\sqrt{n} \log n) \to \infty$, $(k/n) \to 0$ $(n \to \infty)$. *Then*

$$\sup_{0 \leqslant t \leqslant 1} \left| g_{2j}(t) - \sum_{j=1}^{n} w_{nj}(t)\, x_{ij} \right| = o(1) \qquad \text{a.s.}$$

for $1 \leqslant j \leqslant p$.

Proof. This result is due to Cheng [4].

LEMMA 9. *Under the condition of Theorem* 3,

$$\hat{\beta}_n - \beta = o(n^{-1/3}) \qquad \text{a.s.}$$

Proof. Define the vector function

$$\begin{aligned} f(a) &= (f_1(a), f_2(a), ..., f_p(a))' \\ &= (1 + \|a\|^2) \left(\frac{1}{n} \tilde{X}' \tilde{Y} - \frac{1}{n} \tilde{X}' \tilde{X} a \right) + \left[\frac{1}{n} \tilde{Y}' \tilde{Y} - \frac{2}{n} \tilde{Y}' \tilde{X} a + a' \left(\frac{1}{n} \tilde{X}' \tilde{X} \right) a \right] a \end{aligned}$$

where $a \in R^p$. Thus we write (2.8) as $f(\hat{\beta}_n) = 0$. On the other hand, using Taylor's formula, we have

$$f(\hat{\beta}_n) = f(\beta) + C_n(\hat{\beta}_n - \beta) \tag{3.35}$$

where

$$C_n = \begin{pmatrix} \left. \dfrac{\partial f_1}{\partial a_1}, \dfrac{\partial f_1}{\partial a_2}, ..., \dfrac{\partial f_1}{\partial a_p} \right|_{a = \beta + \tau_1(\hat{\beta}_n - \beta)} \\ \left. \dfrac{\partial f_2}{\partial a_1}, \dfrac{\partial f_2}{\partial a_2}, ..., \dfrac{\partial f_2}{\partial a_p} \right|_{a = \beta + \tau_2(\hat{\beta}_n - \beta)} \\ \left. \dfrac{\partial f_p}{\partial a_1}, \dfrac{\partial f_p}{\partial a_2}, ..., \dfrac{\partial f_p}{\partial a_p} \right|_{a = \beta + \tau_p(\hat{\beta}_n - \beta)} \end{pmatrix}$$

for some $\tau_1, \tau_2, ..., \tau_p \in [0, 1]$. By a simple calculation, we get

$$\begin{aligned} \frac{\partial f}{\partial a} = &-(1 + \|a\|^2) \frac{1}{n} \tilde{X}' \tilde{X} + 2a \left(\frac{1}{n} \tilde{X}' \tilde{Y} - \frac{1}{n} \tilde{X}' \tilde{X} a \right)' \\ &+ \left(\frac{1}{n} \tilde{Y}' \tilde{Y} - \frac{2}{n} \tilde{Y}' \tilde{X} a \right) I_p + \left(-\frac{2}{n} \tilde{X}' \tilde{Y} + \frac{2}{n} \tilde{X}' \tilde{X} a \right) a', \end{aligned}$$

and by virtue of Theorem 1 and Lemma 3 we get $C_n \to -(1+\|\beta\|^2)\,\Sigma$ a.s. It follows from (3.35) and the fact $f(\hat{\beta}_n)=0$ that

$$\begin{aligned} C_n(\hat{\beta}_n-\beta) &= -f(\beta) \\ &= -\left\{(1+\|\beta\|^2)\left(\frac{1}{n}\tilde{X}'\tilde{Y}-\frac{1}{n}\tilde{X}'\tilde{X}\beta\right)\right. \\ &\quad \left. +\left[\frac{1}{n}\tilde{Y}'\tilde{Y}-\frac{2}{n}\tilde{Y}'\tilde{X}\beta+\beta'\left(\frac{1}{n}\tilde{X}'\tilde{X}\right)\beta\right]\beta\right\}. \end{aligned} \tag{3.36}$$

Applying Lemma 7, we have

$$\begin{aligned} n^{1/3}(\hat{\beta}_n-\beta) = -n^{-2/3}C_n^{-1}\sum_{i=1}^{n}&[(1+\|\beta\|^2)(e_i-u_i\beta)(h_i+u_i) \\ &+(e_i-u_i'\beta)^2\beta]+o(1) \qquad \text{a.s.} \end{aligned}$$

Hence, using $E(\|x_1\|^3+\|u_1\|^3+|e_1|^3)<\infty$ and the Marcinkiewicz strong law of large numbers, we obtain $n^{1/3}(\hat{\beta}_n-\beta)=o(1)$ a.s.

Proof of Theorem 2. From (3.36) and Lemma 7, we get

$$\begin{aligned} C_n(\hat{\beta}_n-\beta) = -\frac{1}{n}\sum_{i=1}^{n}&[(1+\|\beta\|^2)(e_i-u_i\beta)(h_i+u_i) \\ &+(e_i-u_i'\beta)^2\beta]+o(n^{-1/2}) \qquad \text{a.s.} \end{aligned}$$

Thus, by the Central Limit Theorem, we have

$$\sqrt{n}\,\Omega_1^{-1/2}(\hat{\beta}_n-\beta)\to^{\mathrm{d}} N(0,I_p).$$

Remark 5. Particularly, if $(e,u')'\sim N(0,\sigma^2 I_{p+1})$, the asymptotic covariance of $\sqrt{n}(\hat{\beta}_n-\beta)$ is

$$(1+\|\beta\|^2)\left[\sigma^2\Sigma+\sigma^4\left(I_p+\frac{\beta\beta'}{1+\|\beta\|^2}\right)\right].$$

Proof of Theorem 3. Take $k=[cn^{2/3}]$, where c is a positive constant. In Section 2, we have defined the estimator of g as

$$g^*(t)=\sum_{j=1}^{n}w_{nj}(t)Y_j-\left(\sum_{j=1}^{n}w_{nj}(t)\,X_j\right)'\hat{\beta}_n.$$

Let

$$\tilde{g}_n(t) = \sum_{j=1}^{n} w_{nj}(t)\, Y_j - \left(\sum_{j=1}^{n} w_{nj}(t)\, X_j\right)' \beta,$$

$$\hat{g}_n(t) = \sum_{j=1}^{n} w_{nj}(t)\, Y_j - \left(\sum_{j=1}^{n} w_{nj}(t)\, x_j\right)' \beta = \hat{g}_{1n} - \hat{g}'_{2n}\beta.$$

We have

$$g^*(t) - g(t) = (g^*(t) - \tilde{g}_n(t)) + (\tilde{g}_n(t) - \hat{g}_n(t)) + (\hat{g}_n(t) - g(t)) = I_1 + I_2 + I_3. \tag{3.37}$$

We consider I_1 first. Let $\hat{\beta}_n = (\hat{\beta}_{n1}, \hat{\beta}_{n2}, ..., \hat{\beta}_{np})'$, $\beta = (\beta_1, \beta_2, ..., \beta_p)'$. Note that

$$\begin{aligned}
I_1 = g^*(t) - \tilde{g}_n(t) &= \left(\sum_{j=1}^{n} w_{nj}(t)\, X_j\right)' (\hat{\beta}_n - \beta) \\
&= \left(\sum_{j=1}^{n} w_{nj}(t)\, x_j\right)' (\hat{\beta}_n - \beta) + \left(\sum_{j=1}^{n} w_{nj}(t)\, u_j\right)' (\hat{\beta}_n - \beta) \\
&= \sum_{s=1}^{p} \left(\sum_{j=1}^{n} w_{nj}(t)\, x_{js} - g_{2s}(t)\right) (\hat{\beta}_{ns} - \beta_s) + \sum_{s=1}^{p} g_{2s}(t)(\hat{\beta}_{ns} - \beta_s) \\
&\quad + \left(\sum_{j=1}^{n} w_{nj}(t)\, u_j\right)' (\hat{\beta}_n - \beta).
\end{aligned}$$

By virtue of Lemma 9 and Lemma 1, we obtain

$$\sum_{s=1}^{p} \left(\sum_{j=1}^{n} w_{nj}(t)\, x_{js} - g_{2s}(t)\right) (\hat{\beta}_{ns} - \beta_s) = o(n^{-1/2}) \qquad \text{a.s.}$$

Since $g_{2s}(t)$ is a bounded function of t, we have $g_{2s}(t)(\hat{\beta}_{ns} - \beta_s) = o(n^{-1/3})$ a.s. Furthermore, note that

$$\begin{aligned}
E\left(\sum_{j=1}^{n} w_{nj}(t)\, u_{js}\right)^2 &= E\left(\sum_{j=1}^{n} w_{nj}^2(t)\, u_{js}^2\right) \\
&= \sigma^2 \sum_{j=1}^{n} v_{nj}^2 \leqslant \sigma^2 \left[\sum_{j=1}^{k} v_{nj}^2 + \left(\sum_{j=k+1}^{n} v_{nj}\right)^2\right] = O(n^{-2/3}).
\end{aligned}$$

We obtain $\sum_{j=1}^{n} w_{nj}(t)\, u_{js} = O_p(n^{-1/3})$. Therefore,

$$I_1 = O_p(n^{-1/3}). \tag{3.38}$$

Note that

$$I_2 = \tilde{g}_n(t) - \hat{g}_n(t) = \left(\sum_{j=1}^{n} w_{nj}(t)\, u_j \right)' \beta = \sum_{s=1}^{p} \left(\sum_{j=1}^{n} w_{nj}(t)\, u_{js} \right) \beta_s.$$

Thus, if follows from (3.38) that $I_2 = O_p(n^{-1/3})$. Let $g_1(t) = E(Y_1 \mid T_1 = t)$. Taking conditional expectation on the two sides of $Y_1 = x_1' \beta + g(T_1) + e_1$, we get $g_1(t) = g_2'(t)\, \beta + g(t)$. It follows from Wei and Su [19] that

$$E(\hat{g}_{1n}(t) - g_1(t))^2 = O(n^{-2/3}), \qquad E(\hat{g}_{2nj}(t) - g_{2j}(t))^2 = O(n^{-2/3}).$$

Then we have $\hat{g}_{1n}(t) - g_1(t) = O_p(n^{-1/3})$ and $\hat{g}_{2nj}(t) - g_{2j}(t) = O_p(n^{-1/3})$. Hence

$$\begin{aligned} I_3 &= \hat{g}_n(t) - g(t) = (\hat{g}_{1n}(t) - \hat{g}_{2n}(t))'\, \beta - [\, g_1(t) - g_2'(t) \beta \,] \\ &= (\hat{g}_{1n}(t) - g_1(t)) - (\hat{g}_{2n}(t) - g_2(t))'\, \beta \\ &= (\hat{g}_{1n}(t) - g_1(t)) - \sum_{s=1}^{p} (\hat{g}_{2ns}(t) - g_{2s}(t))\, \beta_s = O_p(n^{-1/3}). \qquad (3.39) \end{aligned}$$

Combining (3.37), (3.38), and (2.39), we complete the proof of Theorem 3.

Proof of Theorem 4. Let

$$\tilde{A}_n = \begin{pmatrix} \frac{1}{n} \sum_{i=1}^{n} (h_i' \beta + e_i)^2 & \frac{1}{n} \sum_{i=1}^{n} (h_i' \beta + e_i)(h_i + u_i)' \\ \frac{1}{n} \sum_{i=1}^{n} (h_i' \beta + e_i)(h_i + u_i) & \frac{1}{n} \sum_{i=1}^{n} (h_i + u_i)(h_i + u_i)' \end{pmatrix}.$$

By the definition of $\widehat{\sigma_n^2}$, we get

$$\sqrt{n}\, (\widehat{\sigma_n^2} - \sigma^2) = \frac{1}{1 + \|\hat{\beta}_n\|^2} \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [(e_i - u_i' \beta)^2 - (1 + \|\beta\|^2)\, \sigma^2] \right.$$
$$\left. + \sqrt{n} (D_1 + D_2 + D_3 + D_4 - D_5) \right\},$$

where

$$D_1 = (1, -\hat{\beta}_n')(A_n - \tilde{A}_n)(1, -\hat{\beta}_n')', \qquad D_2 = (1, -\hat{\beta}_n')(\tilde{A}_n - A)(0, (\beta - \hat{\beta}_n)')'$$
$$D_3 = (0, (\beta - \hat{\beta}_n)')\, A (0, (\beta - \hat{\beta}_n)')', \qquad D_4 = (0, (\beta - \hat{\beta}_n)')(\tilde{A}_n - A)(1, -\beta')'$$
$$D_5 = \|\hat{\beta}_n - \beta\|^2.$$

It follows from Lemma 7 and Theorem 2 that $\sqrt{n}\,(D_1+D_2+D_3+D_4-D_5) \to^{\mathrm{P}} 0$. Therefore, applying the Central Limit Theorem, we obtain

$$\sqrt{n}\,(\widehat{\sigma_n^2}-\sigma^2) \to^{\mathrm{d}} N(0, \Omega_2).$$

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REFERENCES

1. Amemiya, Y., and Fuller, W. A. (1984). Estimation for the multivariate errors-in-variables model with estimated error covariance matrix. *Ann. Statist.* **12** 497–509.
2. Anderson, T. W. (1984). Estimating linear statistical relationships. *Ann. Statist.* **12** 1–45.
3. Chen, H. (1988). Convergence rates for parametric components in a partly linear model. *Ann. Statist.* **16** 136–146.
4. Cheng, P. E. (1985). Strong consistency of nearest neighbor regression function estimators. *J. Multivariate Anal.* **15** 63–72.
5. Donald, S. G., and Newey, W. K. (1994). Series estimation of semilinear models. *J. Multivariate Anal.* **50** 30–40.
6. Enbank, R. L., and Speckman, P. (1990). Curve fitting by polynomial trigonometric regression. *Biometrika* **77** 1–9.
7. Engle, R. F., Granger, C. W. J., Rice, J., and Weiss, A. (1984). Semiparametric estimates of the relation between weather and electricity sales. *J. Amer. Statist. Assoc.* **81** 310–320.
8. Fuller, W. A. (1987). *Measurement Error Models.* Wiley, New York.
9. Glessor, L. J. (1990). Improvements of the naive approach to estimation in nonlinear errors-in-variables regression models. *Contemp. Math.* **112** 99–114.
10. Heckman, N. E. (1986). Spline smoothing in a partly linear model. *J. Roy. Statist. Soc. Ser. (B)* **48** 244–248.
11. Hoeffding, W. (1963). Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.* **58** 13–30.
12. Hong, S. Y. (1991). The theory of estimation for a kind of partial linear regression model. *Chinese Sci. Ser. A* 1258–1272.
13. Hong, S. Y., and Cheng, P. (1994). The convergence rate of estimation for parameter in a semiparametric model. *Chinese J. Appl. Probab. Statist.* **1** 62–71.
14. Robinson, P. (1988). Root-*N*-consistent semiparametric regression. *Econometrica* **56** 931–954.
15. Speckman, P. (1988). Kernel smoothing in partial linear models. *J. Roy. Statist. Soc. Ser. B* **50** 413–436.
16. Sprent, D. (1990). Some history of functional and structural relationships. *Contemp. Math.* **112** 3–15.
17. Stone, C. J. (1982). Optimal global rates of convergence for nonparametric regression. *Ann. Statist.* **10** 1040–1053.
18. Wahba, G. (1984). In *Statistics: An Appraisal, Proc. 50th Anniversary Conference of the Iowa State Statistical Laboratory*, pp. 205–235. Iowa State Univ. Press, Ames, IA.
19. Wei, L. S., and Su, C. (1986). On the pointwise L_p convergence rates of nearest neighbor estimate of nonparametric regression function. *J. Math. Res. Exposition* **2** 117–123.