Hankel-norm estimation for fractional-order systems using the Rayleigh–Ritz method

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Abstract

In this paper, the Rayleigh–Ritz method of estimating the eigenvalues of an operator on a Hilbert space is utilized to determine the magnitude of the largest eigenvalue for the Hankel operator of fractional-order systems, the Hankel norm. This provides a measure of the possible retrievable energy from the system in the future compared to the energy that was put into the system in the past. The application of the Rayleigh–Ritz method to obtaining underestimates of the Hankel norm of a fractional-order system is described. Several examples are given, demonstrating the method.

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1. Introduction

The Riemann–Liouville definition of the fractional-order derivative operator, denoted \( -\infty d^q_t x(t) \) is

\[
-\infty d^q_t x(t) = \frac{d^n}{dt^n} \int_{-\infty}^{t} \frac{(t-\tau)^{n-q-1}}{\Gamma(n-q)} x(\tau) \, d\tau,
\]

where \( n - 1 < q < n \). Other definitions include the Grünwald–Letnikov definition, the Caputo definition, definitions based on Cauchy’s integral theorem for derivatives, convolution, infinite series can also be constructed. Under conditions that are not too restrictive, these definitions provide the same results as the Riemann–Liouville definition.

Systems that include fractional-order derivatives may be constructed as (convolution) integral operators. We consider fractional-order systems acting from \( L_2[0, \infty) \) to \( L_2[0, \infty) \). Let \( \mathcal{B}(L_2[0, \infty)) \) be the set of bounded operators on \( L_2[0, \infty) \). The norm of a bounded operator, \( K \in \mathcal{B}(L_2[0, \infty)) \), is defined to be

\[
\|K\| = \sup \left\{ \frac{\|Kh\|}{\|h\|} : h \in L_2[0, \infty), h \neq 0 \right\}.
\]

The square of this norm of the operator represents the maximum ratio between the energy in the system input and the energy in the system output.

Hankel operators, denoted \( \Gamma \), were introduced as operators on \( \ell_2(\mathbb{N}) \) that could be represented by matrices each of whose entries depends only on the sum of its indices. A natural interpretation is that Hankel operators map system inputs for time \( t < 0 \) to outputs for time \( t > 0 \). That is, the input of a system’s Hankel operator can be thought of as setting up a given initial state, and the output of the Hankel operator as being the corresponding initial condition response [6–8].

The Hankel operator and its norm, called the Hankel norm, plays a key role in approximation problems such as model order reduction and model matching. In certain cases, the solution of the \( H_\infty \) control design problem is characterized using the Hankel operator and its norm [9].

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0898-1221/$ – see front matter © 2010 Published by Elsevier Ltd

doi:10.1016/j.camwa.2009.08.016
Let the operator $\Pi_{+}: L_2(\mathbb{R}^+, \mathbb{R}^+) \rightarrow L_2(0, \infty)$ be defined by $\Pi_{+} f(t) = f(t)$ for $t \geq 0$ for any $f \in L_2(\mathbb{R}^+, \mathbb{R}^+)$. Any linear time-invariant operator $K$, can be written as

$$\langle Ku(t) \rangle (t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau,$$

where $h(t)$ is the impulse response of the system. The Hankel operator, $\Gamma_K$, for the operator $K$ is given by

$$\langle \Gamma_K u_- \rangle (t) = \int_{0}^{\infty} h(t+\tau)u_-(\tau)d\tau,$$

where $u_-(t) = \Pi_{-}u(-t)$.

The norm of the Hankel operator determines the relative amount of energy that can be retrieved from the system in the future compared to the amount of energy put into the system in the past. The operator norm of any compact Hermitian operator is the value of the largest-magnitude eigenvalue of the operator. Hence obtaining an estimate of the largest eigenvalue of an operator gives an estimate for the operator norm. For the Hankel operator, this estimate becomes an estimate of the maximum retrievable energy. Next, we consider the Rayleigh–Ritz method for estimating the eigenvalues of a compact Hermitian operator.

2. Rayleigh–Ritz method for estimation of eigenvalues

The Rayleigh–Ritz method is a method of determining the eigenvalues and eigenfunctions for an operator devised by Rayleigh [10] and Ritz [11]. The Rayleigh–Ritz method applies to operators on any Hilbert space $\mathcal{H}$, and is stated in a general form in this section. In the remaining sections of the paper, the result is applied for the special case where $\mathcal{H} = L_2(0, \infty)$.

Let $K \in \mathcal{B}(\mathcal{H})$. Let $\lambda_1^+$ denote the largest positive eigenvalue of $K$ and let $\lambda_1^-$ denote the smallest negative eigenvalue of $K$. If $K$ has no positive eigenvalues, then $\lambda_1^+ = 0$. Similarly, if $K$ has no negative eigenvalues, then $\lambda_1^- = 0$. Define the functionals $J_K(p)$ and $R_K(p)$ by

$$J_K(p) = \langle Kp, p \rangle,$$

and

$$R_K(p) = \frac{\langle Kp, p \rangle}{\langle p, p \rangle}.$$

[12] showed that if $K \in \mathcal{B}(\mathcal{H})$, then

$$\lambda_1^+ = \sup \{J_K(p): p \in \mathcal{H}, \|p\| = 1\}$$

and

$$\lambda_1^- = \inf \{J_K(p): p \in \mathcal{H}, \|p\| = 1\}.$$  \hspace{1cm} (2)

If $K$ has at least one positive eigenvalue, then there is a maximum positive eigenvalue, and the supremum in Eq. (1) is attained. Similarly, if $K$ has at least one negative eigenvalue, the infimum in Eq. (2) is obtained. [12] also give the following lemma.

Lemma 2.1. Let $K \in \mathcal{B}(\mathcal{H})$ be a compact Hermitian operator, and let $p \in \mathcal{H}$ such that $\|p\| = 1$. Then $p$ is a stationary point of $(Kp, p)$ if and only if $p$ is an eigenfunction of $K$. The stationary values are the corresponding eigenvalues of $K$.

In the Rayleigh–Ritz method, an $N$-dimensional subspace, $E_N \subset \mathcal{H}$, spanned by $\{\phi_k\}_{k=1}^{N}$, is selected. $p \in \mathcal{H}$ is approximated by $\hat{p} \in E_N$ defined by $\hat{p} = \sum_{k=1}^{N} a_k \phi_k$ and $\{a_k\}_{k=1}^{N}$ is selected to maximize $J_K(\hat{p})$. This is equivalent to maximizing $J_K(p)$ restricted to $E_N$. Maximizing $J_K(p)$ constrained by $\|p\| = 1$ can be accomplished by way of the Lagrange multiplier method as follows.

Define $J_{mn}$ and $l_{mn}$ to be

$$J_{mn} = \langle K\phi_m, \phi_n \rangle$$

$$= \int_{0}^{\infty} \int_{0}^{\infty} h(t, \tau)\phi_m(t)\phi_n(t)d\tau dt$$

and

$$l_{mn} = \langle \phi_m, \phi_n \rangle.$$  \hspace{1cm} (3)

The matrices $P_N$ and $Q_N$ are given by

$$P_N = \begin{pmatrix} J_{11} & J_{12} & \cdots & J_{1N} \\ J_{21} & J_{22} & \cdots & J_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ J_{N1} & J_{N2} & \cdots & J_{NN} \end{pmatrix},$$

(5)
Therefore, the largest eigenvalue estimate then provides a lower bound on the actual eigenvalue.

Note that

\[
J_{mn} = \int_0^\infty \int_0^\infty h(t, \tau) \phi_m(\tau) \phi_n(t) d\tau dt \\
= \int_0^\infty \int_0^\infty h(\tau, t) \phi_m(\tau) \phi_n(t) dt d\tau = J_{mn}.
\]

Hence, \( P_n \) is a symmetric matrix. Similarly, \( Q_n \) is a symmetric matrix. [12] give the stationary values of \( J_K(p) \) for \( p \in E_N = \text{span}\{\phi_k\}_{k=1}^N \) as the solutions, \( \lambda_{N,k} \), of

\[
\det (P_n - \lambda_{N,k} Q_n) = 0.
\]

Note that \( \lambda_{N,k} \) are the generalized eigenvalues of \( P_n \) with respect to \( Q_n \).

Each of the solutions of Eq. (7) is a stationary value of \( \langle Kp, p \rangle \) restricted to \( E_N \). These stationary values of \( \langle Kp, p \rangle \) restricted to \( E_N \) approximate the stationary values of \( \langle Kp, p \rangle \) on \( \mathcal{H} \), which are the eigenvalues of the operator [12].

If the trial space, \( E_N \) is replaced with a larger trial space, \( E_M \), where \( M > N \), the accuracy of the eigenvalue estimate increases if \( E_N \subset E_M \). In fact, [12] show that if \( P_Nf \rightarrow f \) as \( N \rightarrow \infty \) for all \( f \in \mathcal{H} \), where \( P_N \) is the projection operator from \( \mathcal{H} \) to \( E_N \), then the sequence of estimates converges to the eigenvalue of the operator. It is generally found that the eigenvalue estimates are good approximations for relatively low orders, but the eigenvector estimates are not in general as accurate [12].

3. Application of the Rayleigh–Ritz method to fractional-order systems

The Rayleigh–Ritz method is used to estimate the eigenvalues of the Hankel operator for a fractional-order system as follows. First, the Hankel operator, \( I_K \in \mathcal{B}(L_2[0, \infty)) \) is determined, and it is verified that \( I_K \) is a compact Hermitian operator. Secondly, for \( M \in \mathbb{N} \), \( \{\phi_k\}_{k=1}^M \subset L_2[0, \infty) \), a set of linearly independent functions are selected. \( J_{mn}, I_{mn}, P_N, \) and \( Q_N \) are computed using Eqs. (3)–(6), respectively, for some \( N \in \mathbb{N} \) such that \( N < M \). Eq. (7) is solved to determine the estimates of the eigenvalues of \( K \). Then, more basis functions are added, and \( J_{mn}, I_{mn}, P_N, \) and \( Q_N \) are augmented. Eq. (7) is solved to determine the improved estimates of the eigenvalues of \( I_K \). This process is repeated until sufficient accuracy on the estimates is achieved. Usually this is when the eigenvalue estimates seem to converge to a value.

A sequence of estimates can be obtained by allowing \( N \) to begin at 1 and increase through the integers. The selection of \( \{\phi_k\}_{k=1}^\infty \) has a large impact on the numerical reliability of the computation of the approximates to the eigenvalues, as well as the computation time of \( I_{mn} \) and \( J_{mn} \). For fractional-order systems, because the kernel of the integral operator is complicated, the choice of \( \{\phi_k\}_{k=1}^\infty \) should be made in order to minimize the difficulty of determining \( \langle K\phi_k \rangle(t) \).

In the examples to follow, two sets of trial functions will be used, denoted \( \{\phi_k\}_{k=1}^\infty \) and \( \{\psi_k\}_{k=1}^\infty \). Letting \( \{\phi\}_{k=1}^\infty \) be given by

\[
\phi_k(t) = \sqrt{2k} e^{-kt} \tag{8}
\]

allows \( \langle K\phi_k \rangle(t) \) to be computed relatively easily, and gives \( I_{mn} = 1 \). Letting \( \{\psi\}_{k=1}^\infty \) be given by

\[
\psi_k(t) = \frac{e^t}{n!} \frac{d^n}{dt^n} \left( t^n e^{-t} \right) = L_k(t) e^{-\frac{t}{2}}, \tag{9}
\]

where \( L_k(t) \) is the Laguerre polynomial, improves the numerical condition of \( P_N \) and \( Q_N \) at the cost of increasing the difficulty of calculating \( \langle K\psi \rangle(t) \). \( \{\phi_k\}_{k=1}^\infty \) will be referred to as the normalized exponentials, and \( \{\psi_k\}_{k=1}^\infty \) will be referred to as the Laguerre functions. When a low computation difficulty is desired, it is better to use \( \{\phi_k\}_{k=1}^\infty \). On the other hand, when improved numerical condition of \( P_N \) and \( Q_N \) is needed, it is better to use \( \{\psi_k\}_{k=1}^\infty \).

Note that the Laguerre functions are complete in \( L_2[0, \infty) \) [13]. It is easily shown that the only \( f \in \text{span}\{\psi_k\}_{k=1}^\infty \) such that \( \langle f, \phi_k \rangle = 0 \) for all \( k \in \mathbb{Z}^+ \) is \( f = 0 \). Thus, the normalized exponentials are dense in the Laguerre functions. Thus, both \( \{\phi_k\}_{k=1}^\infty \) and \( \{\psi_k\}_{k=1}^\infty \) are complete in \( L_2[0, \infty) \). Hence for any \( f \in L_2[0, \infty) \), \( P_Nf \rightarrow f \) as \( N \rightarrow \infty \) for the projection from \( L_2[0, \infty) \) to \( \text{span}\{\phi_k\}_{k=1}^N \) and from \( L_2[0, \infty) \) to \( \text{span}\{\psi_k\}_{k=1}^N \). Hence sequences of eigenvalue estimates with these trial functions converge to the eigenvalue of the operator.

The sequence of eigenvalues is a nondecreasing sequence, because the set, \( E_N \), over which the maximum is being taken is becoming larger. Hence, the maximum of \( J_k(p) \) for \( p \in E_{N+1} \) is at least as large as the maximum of \( J_k(p) \) for \( p \in E_N \subset E_{N+1} \). Thus, the largest eigenvalue estimate then provides a lower bound on the actual eigenvalue.

Because this process is applicable only compact operators, we must be sure that our operators are compact.
4. Boundedness and compactness of Hankel operators

In order to apply the Rayleigh–Ritz method to an operator, it must be exhibited that the operator is compact and Hermitian. The following theorem will be utilized to demonstrate the compactness of certain fractional-order operators.

**Theorem 4.1.** Let $h(t, \tau)$ be such that
\[
\int_0^\infty |h(t, \tau)| \, d\tau \leq \alpha < \infty
\]
for all $t \in [0, \infty)$ and
\[
\int_0^\infty |h(t, \tau)| \, dt \leq \beta < \infty
\]
for all $\tau \in [0, \infty)$. Define the operator, $K : L_2[0, \infty) \rightarrow L_2[0, \infty)$, by
\[
(Ku)(t) = \int_0^\infty h(t, \tau)u(\tau) \, d\tau.
\]

Then
1. $K$ is a bounded linear operator with $\|K\| \leq \sqrt{\alpha \beta}$,
2. if $h(t, \tau) = g(t + \tau)$ for some function $g(\xi)$, then $K$ is a compact operator, and
3. if $h(t, \tau)$ is real-valued and $h(t, \tau) = g(t + \tau)$ for some function $g(\xi)$, then $K$ is a Hermitian operator.

**Proof.** 1. This is an application of the Cauchy–Bunyakovsky–Schwarz inequality. The details of this proof are given by [14] in Theorem II.1.6. Note that this upper bound on the operator norm is very coarse.

2. Note that since $h(t, \tau) = g(t + \tau)$, $g(\xi) = h(\xi, 0)$, $\int_0^\infty |h(t, 0)| \, dt \leq \beta < \infty$, $g \in L_1[0, \infty)$. [7] in Corollary 8.11 shows that if $g \in L_1[0, \infty)$, then $\int_0^\infty g(t + \tau)u(\tau) \, d\tau$ is a compact operator on $L_2[0, \infty)$. Hence, $K$ is a compact operator on $L_2[0, \infty)$.

3. Since $h(t, \tau)$ is real-valued and $h(t, \tau) = g(t + \tau)$ for some function $g(\xi)$, then it is clear that $h(t, \tau) = \overline{h(\tau, t)}$. If $(Ku)(t) = \int_0^\infty h(t, \tau)u(\tau) \, d\tau$, then $(K^*u)(t) = \int_0^\infty \overline{h(\tau, t)}u(\tau) \, d\tau$, [14, Example II.2.9]. Thus, $K^* = K$, so $K$ is Hermitian. \qed

5. Examples

5.1. $\frac{1}{s + \sqrt{2}}$ system

First we demonstrate the Rayleigh–Ritz method for a finite-rank operator whose Hankel norm can be easily calculated. Let $\Gamma_{k_4}^2 : L_2[0, \infty) \rightarrow L_2[0, \infty)$ be defined by
\[
(\Gamma_{k_4}^2u)(t) = \int_0^\infty e^{\sqrt{2}(t+\tau)}u(\tau) \, d\tau.
\]

Note that the kernel of the integral operator, $h_4(t, \tau) = g_4(t + \tau)$ where $g(\xi) = e^{-\sqrt{2}\xi}$. Clearly, $g_4(\xi)$ is real-valued on $[0, \infty)$. Also, since
\[
\int_0^\infty |h_4(t, \tau)| \, d\tau = \frac{1}{\sqrt{2}} e^{-\sqrt{2}t} \leq \frac{1}{\sqrt{2}} < \infty
\]
for all $\tau \in [0, \infty)$, and
\[
\int_0^\infty |h_4(t, \tau)| \, dt = \frac{1}{\sqrt{2}} e^{-\sqrt{2}t} \leq \frac{1}{\sqrt{2}} < \infty
\]
for all $t \in [0, \infty)$, by **Theorem 4.1** $\Gamma_{k_4}^2$ is a compact Hermitian operator from $L_2[0, \infty)$ to $L_2[0, \infty)$. Since $h_4(t, \tau) = e^{\sqrt{2}t}e^{\sqrt{2}\tau}$, it is called a degenerate kernel. Hence the operator can be written as
\[
(\Gamma_{k_4}^2u)(t) = e^{-\sqrt{2}t} \int_0^\infty e^{-\sqrt{2}\tau}u(\tau) \, d\tau = \alpha e^{-\sqrt{2}t},
\]
where $\alpha = \int_0^\infty e^{-\sqrt{2}\tau}u(\tau) \, d\tau$. The range of $\Gamma_{k_4}^2$ is [$e^{-\sqrt{2}t}$], and the only possible eigenfunction of $\Gamma_{k_4}^2$ is $e^{-\sqrt{2}t}$. The eigenvalue is simply $\lambda = \int_0^\infty e^{-\sqrt{2}\tau}e^{-\sqrt{2}t} \, d\tau = \frac{\sqrt{2}}{4} \approx 0.353553$. 
Let \( \{ \phi_k \} \subseteq L_2[0, \infty) \) be defined by Eq. (8), let \( f_{mn} \) and \( l_{mn} \) be defined as in Eqs. (3) and (4) respectively. Let \( P_N \) and \( Q_N \) be defined as in Eqs. (5) and (6) respectively. Then the solutions, \( \lambda_{N,k} \), to Eq. (7) are estimates of the eigenvalues of \( K_N \). The sequence of largest generalized eigenvalues of \( P_N \) with respect to \( Q_N \) is given by the sequence,

\[
\{ \lambda_{k1} \} = \{ 0.343146, 0.353247, 0.353544, 0.353551, 0.353552, 0.353553, \ldots \},
\]

which converges to the eigenvalue up to six decimal places within seven terms. Let \( \{ \psi_k \} \subseteq L_2[0, \infty) \) be defined by Eq. (9). Let \( f_{mn} \) and \( l_{mn} \) be defined as in Eqs. (3) and (4) respectively. Let \( P_N \) and \( Q_N \) be defined as in Eqs. (5) and (6) respectively. Then the solutions, \( \lambda_{N,k} \), to Eq. (7) are estimates of the eigenvalues of \( K_N \). The sequence of largest generalized eigenvalues of \( P_N \) with respect to \( Q_N \) is given by the sequence,

\[
\{ \lambda_{k1} \} = \{ 0.343146, 0.353247, 0.353544, 0.353553, 0.353553, \ldots \},
\]

which converges to the eigenvalue up to six decimal places within four terms. The sequences of eigenvalue estimates converge to the eigenvalue of the operator for both sets of trial functions. Now, we consider a fractional-order operator in greater detail.

5.2. \( \frac{1}{\sqrt{\pi + t}} \) system

Let \( I_{K_N} : L_2[0, \infty) \rightarrow L_2[0, \infty) \) be defined by

\[
(I_{K_N} u)(t) = \int_0^\infty \left( \frac{1}{\sqrt{\pi (t + \tau)}} - e^{t+\tau} \text{erfc} \left( \sqrt{t + \tau} \right) \right) u(\tau) d\tau,
\]

where \( \text{erfc}(x) \) is the complementary error function. Note that \( h_1(t, \tau) = g_1(t + \tau) \) for \( g_1(\xi) = \frac{1}{\sqrt{\pi (\xi)}} - e^{\xi} \text{erfc}(\sqrt{\xi}) \). Fig. 1 shows the graph of \( g_1(\xi), g_1(\xi) \) is real-valued on \([0, \infty) \). Note that

\[
\int_0^\infty |h_1(t, \tau)| d\tau = e^\tau \text{erfc} \left( \sqrt{\tau} \right) \leq 1
\]

for all \( \tau \in [0, \infty) \) and

\[
\int_0^\infty |h_1(t, \tau)| d\tau = e^\tau \text{erfc} \left( \sqrt{\tau} \right) \leq 1
\]

for all \( t \in [0, \infty) \). Hence, by Theorem 4.1, \( I_{K_N} \) is a compact Hermitian operator with \( \| I_{K_N} \| \leq \sqrt{1 - 1} = 1 \). Since \( g_1(\xi) \) is the impulse response of the system with transfer function

\[
H_1(s) = \frac{1}{s^2 + 1},
\]

\( I_{K_N} \) is the Hankel operator for the system with that transfer function.

Let the normalized exponentials, \( \{ \phi_k \} \subseteq L_2[0, \infty) \), be defined by Eq. (8), let \( f_{mn} \) and \( l_{mn} \) be defined as in Eqs. (3) and (4) respectively. Let \( P_N \) and \( Q_N \) be defined as in Eqs. (5) and (6) respectively. Then the solutions, \( \lambda_{N,k} \), to Eq. (7) are estimates of the eigenvalues of \( I_{K_N} \).
Estimates on the Three Largest Eigenvalues of $K_1$ Using $\varphi_k$ (Solid) and $v_k$ (Dashed)

Fig. 2. Estimates of the Hankel norm of $\frac{1}{s^2+1}$.

For $N = 1$, $P_1 = \begin{pmatrix} 0.2500 \\ 0.2426 \end{pmatrix}$ and $Q_1 = \begin{pmatrix} 1.0000 \\ 0.9428 \end{pmatrix}$. The generalized eigenvalue of $P_1$ with respect to $Q_1$ is $\lambda_{1,1} = 0.25$.

For $N = 2$, $P_2$ is given by

$$P_2 = \begin{pmatrix} 0.2500 & 0.2426 \\ 0.2426 & 0.2426 \end{pmatrix}$$

and $Q_2$ is given by

$$Q_2 = \begin{pmatrix} 1 \\ 0.9428 \\ 1 \end{pmatrix}.$$ 

The generalized eigenvalues of $P_2$ with respect to $Q_2$ are

$$\{\lambda_{2,k}\}_{k=1}^2 = \{0.2523, 0.0637\}.$$

For $N = 3$, $P_3$ is given by

$$P_3 = \begin{pmatrix} 0.2500 & 0.2426 & 0.2321 \\ 0.2426 & 0.2426 & 0.2361 \\ 0.2321 & 0.2361 & 0.2321 \end{pmatrix}$$

and $Q_3$ is given by

$$Q_3 = \begin{pmatrix} 1.0000 & 0.9428 & 0.8660 \\ 0.9428 & 1.0000 & 0.9798 \\ 0.8660 & 0.9798 & 1.0000 \end{pmatrix}.$$ 

The generalized eigenvalues of $P_3$ with respect to $Q_3$ are

$$\{\lambda_{3,k}\}_{k=1}^3 = \{0.2651, 0.0653, 0.0158\}.$$

Continuing the process gives the generalized eigenvalues for $P_4$, $P_5$, and $P_6$ with respect to $Q_4$, $Q_5$, and $Q_6$ as

$$\{\lambda_{4,k}\}_{k=1}^4 = \{0.2651, 0.0784, 0.0166, 0.0036\},$$

$$\{\lambda_{5,k}\}_{k=1}^5 = \{0.2682, 0.0786, 0.0238, 0.0040, 0.0008\},$$

and

$$\{\lambda_{6,k}\}_{k=1}^6 = \{0.2682, 0.0835, 0.0242, 0.0067, 0.0009, 0.0002\}.$$ 

The sequence of largest generalized eigenvalues of $P_N$ with respect to $Q_N$ is given by the sequence,

$$\{\lambda_{k,1}\} = \{0.2500, 0.252326, 0.265139, 0.26514, 0.268222, 0.268298, 0.269546, 0.269664, 0.270326, 0.270447, 0.270925, \ldots\}.$$ 

Fig. 2 shows the estimates for the largest eigenvalue. After eleven terms, the matrices, $P_N$ and $Q_N$ become numerically ill-conditioned. The Rayleigh–Ritz method was also performed with $\{\varphi_k\}$ defined by the orthonormal exponential functions and
the Laguerre functions, \([\psi_k]_k=1^\infty\). Using the orthonormal exponentials gives the same eigenvalue estimates with numerical instability after eleven bases were used. Using the Laguerre functions, the sequence of estimates on the largest eigenvalue is
\[
\{\lambda_{k,1}\} = \{0.250000, 0.250000, 0.266598, 0.266598, 0.271418, 0.271418, 0.273523, 0.273523, 0.274646, 0.274646, 0.275322, \ldots\}.
\]
Both sets of trial functions give approximations that are close. Because \([\phi_k]_k=1^\infty\) and \([\psi_k]_k=1^\infty\) are complete, the sequences of eigenvalue estimates generated by each of these sets are guaranteed to converge to the same value, the largest.

The estimates seem to be converging to a value greater than, but approximately, 0.273523. Because each of the estimates of the eigenvalue is a lower bound on the eigenvalue of the operator, the largest estimate is used for a lower bound on the eigenvalue.

Because the norm of the operator is equal to the magnitude of the largest eigenvalue of the system, and because \(||I_k|| \leq 1\) by Theorem 4.1, 0.273523 \(\leq ||I_k|| \leq 1\). Recall that the upper bound given by Theorem 4.1 is a coarse bound.

5.3. \(\frac{1}{\sqrt{s+2}}\) system

Let \(I_k^2: L_2[0, \infty) \rightarrow L_2[0, \infty)\) be defined by
\[
(I_k^2u)(t) = \int_0^\infty h_2(t, \tau)u(\tau)d\tau,
\]
where
\[
h_2(t, \tau) = \frac{1}{\sqrt{\pi(t + \tau)}} - 2e^{4(t + \tau)}\text{erfc}(2\sqrt{t + \tau}).
\]
This is the Hankel operator for the system with transfer function
\[
H_2(s) = \frac{1}{\sqrt{s + 2}}.
\]
Note that \(h_2(t, \tau) = g_2(t + \tau)\) where \(g_2(\xi) = \frac{1}{\sqrt{\pi\xi}} - 2e^{4\xi}\text{erfc}(2\sqrt{\xi})\). Fig. 1 shows the graph of \(g_2(\xi)\). Note that \(g_2(\xi)\) is real-valued on \([0, \infty)\). Also note that
\[
\int_0^\infty |h_2(t, \tau)|d\tau = \frac{1}{2}e^{4t}\text{erfc}(2\sqrt{t}) \leq \frac{1}{2},
\]
for all \(t \in [0, \infty)\), and
\[
\int_0^\infty |h_2(t, \tau)|dt = \frac{1}{2}e^{4\tau}\text{erfc}(2\sqrt{\tau}) \leq \frac{1}{2},
\]
for all \(\tau \in [0, \infty)\). Hence, by Theorem 4.1, \(I_k^2\) is a compact Hermitian operator with \(||I_k^2|| \leq \frac{1}{2}\). The Laguerre functions, \([\psi_k]\), are used to determine the estimates on the eigenvalues of this system.

Let \([\psi_k] \subset L_2[0, \infty)\) be defined by Eq. (9). Let \(f_{lm}\) and \(l_{nm}\) be defined as in Eqs. (3) and (4) respectively. Let \(P_N\) and \(Q_N\) be defined as in Eqs. (5) and (6) respectively. Then the solutions, \(\lambda_{N,k}\), to Eq. (7) are estimates of the eigenvalues of \(K_2\). The sequence of largest generalized eigenvalues of \(P_N\) with respect to \(Q_N\) is given by the sequence,
\[
\{\lambda_{k,1}\} = \{0.111111, 0.117175, 0.126458, 0.129394, 0.133069, 0.134734, 0.136711, 0.137771, 0.139001, 0.139729, 0.140564, \ldots\}.
\]
Fig. 3 shows the estimates for the largest eigenvalue. The estimates appear to converge to something near to, but greater than, 0.140564 since the sequence of estimates is nondecreasing. Since \(||I_k^2|| \leq \frac{1}{2}\) by Theorem 4.1, 0.140564 \(\leq ||I_k^2|| \leq 1\).

5.4. \(\frac{1}{\sqrt{s+1}}\) system

Let \(I_k^3: L_2[0, \infty) \rightarrow L_2[0, \infty)\) be defined by
\[
(I_k^3u)(t) = \int_0^\infty h_3(t, \tau)u(\tau)d\tau,
\]
where \(h_3(t, \tau) = g_3(t + \tau)\), where \(g_3(\xi)\) is given by
\[
g_3(\xi) = e^{-\xi}\left(1 - \frac{(\gamma(-\frac{1}{3}, -\xi)}{\Gamma(-\frac{1}{3})} + \frac{(\gamma(-\frac{2}{3}, -\xi)}{\Gamma(-\frac{2}{3})}\right)u_s(\xi),
\]
where $u_4(\xi)$ is the Heaviside unit step function. Fig. 1 shows the graph of $g_3(\xi)$. $g_3(\xi)$ is real-valued on $[0, \infty)$. [15] demonstrated the finite bounds on $\int_0^{\infty} |h_3(t, \tau)|d\tau$ and $\int_0^{\infty} |h_3(t, \tau)|dt$. Thus, by Theorem 4.1, this operator is a compact Hermitian operator from $L_2[0, \infty)$ to $L_2[0, \infty)$. [15] also showed that this operator corresponds to the Hankel operator of the system with transfer function, $H_3(s) = \frac{1}{s^2 + 2}$.

Let $\{\phi_k\} \subset L_2[0, \infty)$ be defined by Eq. (8). Let $J_{mN}$ and $I_{mN}$ be defined as in Eqs. (3) and (4) respectively. Let $P_N$ and $Q_N$ be defined as in Eqs. (5) and (6) respectively. Then the solutions, $\lambda_{N,k}$, to Eq. (7) are estimates of the eigenvalues of $\Gamma_{N,\mathcal{K}}$. The sequence of largest generalized eigenvalues of $P_N$ with respect to $Q_N$ is given by the sequence,

$$\{\lambda_{k1}\} = \{0.166667, 0.169032, 0.182533, 0.182818, 0.187082, 0.187115, 0.189076, 0.189078, \ldots\}.$$  

Fig. 4 shows the estimates for the largest eigenvalue. The estimates seem to be converging to a value near but greater than 0.189078 since the sequence of estimates is nondecreasing. Thus, $\|\Gamma_{N,\mathcal{K}}\| \geq 0.189078$.

6. Conclusions

In this paper the Rayleigh–Ritz method was demonstrated to be usable in the calculation of approximations to the eigenvalues of the Hankel operator of a fractional-order system. These eigenvalues can be used to develop finite-rank and low-order approximations for the fractional-order systems [7]. Determining the eigenvalues of the Hankel operator of the fractional-order system is the first step in determining the $H_{\infty}$-optimal controller. Since the norm of the Hankel operator is the magnitude of the eigenvalue with the largest magnitude, the estimates on the eigenvalue with largest magnitude may be used as an estimate on the Hankel norm of the system.
The Hankel norm is useful because it describes the relationship of the maximum energy that can be taken out of the system in the future based on the amount of energy put into the system in the past. That is
\[
\|I_K\| = \frac{\text{future energy out}}{\text{past energy in}}.
\]
Thus, if the system is excited with the appropriate input, the system can release an amount of energy greater than any of the estimates generated by the Rayleigh–Ritz method times the amount of energy put into the system.

Although there is a lower bound on the maximum possible ratio of energy retrieved from the system to energy put into the system, the Rayleigh–Ritz method does not necessarily yield the input that will allow for the maximum energy to be retrieved from the system. Also, if the conditions of Theorem 4.1 are met, then Theorem 4.1 provides an upper bound for the maximum possible ratio.

Future work involves increasing the accuracy of the estimates of the eigenvalues. This may be accomplished by selecting \(\{\phi_k\}_{k=1}^{\infty}\) so that \(P_N\) and \(Q_N\) are better conditioned. Also, it may be possible to obtain upper bounds on the eigenvalues that are finer than those offered by Theorem 4.1. Other estimation methods may yield more information about the eigenvalues and eigenfunctions for fractional-order systems.

References