Existence and Nonexistence of Solutions for
Quasilinear Elliptic Equations

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In this paper we study the problem

\[-\Delta_p u = f(x, u, \nabla u) \quad \text{in } \Omega,\]
\[u = 0 \quad \text{on } \partial \Omega,\]

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geq 2$, and $\Delta_p u = \text{div}(\nabla |u|^{p-2} \nabla u)$ defines the $p$-Laplacian. We provide some necessary and sufficient conditions on $f$ under which the problem admits a weak solution. For the case $p = 2$ we obtain more general conditions on $f$. The main ingredients are degree theory and the super-subsolution method.

Key Words: elliptic; quasilinear; degenerate; $p$-Laplacian; critical growth in the gradient; existence and nonexistence of solution.

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1. INTRODUCTION

During the past 30 years many mathematicians have been investigating quasilinear elliptic problems of the type

\[-\mathcal{L} u = f(x, u, \nabla u) \quad \text{in } \Omega \]
\[u = 0 \quad \text{on } \partial \Omega, \quad (1.1)\]

where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N, \ N \geq 2 \), \( \mathcal{L} \) is a uniformly elliptic operator of second order verifying the strong maximum principle, and \( f: \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \) is a Carathéodory function satisfying appropriate growth conditions. These studies have contributed for a better understanding of several questions related to (1.1), for example, regularity, existence, and nonexistence of a solution. In the literature, problem (1.1) is known to be critical in the gradient if \( f \) has quadratic growth in the gradient. This is a reasonable assumption when we are seeking for a solution. In fact, Serrin [14] has proved that if \( f \) has a superquadratic growth in the gradient, then (1.1) may not have a solution. On the other hand, several authors have obtained results on the existence of a solution when \( f \) has critical growth in the gradient. For instance, see [1, 3, 5, 8, 9, 12, 13, 16]. Part of these works are concerned to the question of existence when \( f \) interacts in some sense with the first eigenvalue of \( \mathcal{L} \); see especially [5, 9, 16]. The basic tools that have been used are a priori estimates, degree theory, and the super-subsolution method.

The present paper deals with the problem

\[-\Delta_p u = f(x, u, \nabla u) \quad \text{in } \Omega \]
\[u = 0 \quad \text{on } \partial \Omega, \quad (1.2)\]

where \( \Omega \) is a \( C^2 \) bounded domain in \( \mathbb{R}^N, \ N \geq 2 \) and \( \Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u) \) denotes the \( p \)-Laplacian with \( 1 < p < \infty \).

Problems of form (1.2) arise naturally as stationary states of certain models in fluids mechanics. Therefore, it is important to obtain information about the existence and nonexistence of solutions for this problem. By a weak solution (supersolution, subsolution) of (1.2) we mean a function \( u \in C^1(\bar{\Omega}) \) satisfying \( f(\cdot, u(\cdot), \nabla u(\cdot)) \in L^p(\Omega), \ u = 0 \ \text{on } \partial \Omega \) and

\[\int_\Omega |\nabla u|^{p-2} \nabla \phi \cdot \nabla \psi dx = (\geq, \leq) \int_\Omega f(x, u, \nabla u) \phi dx\]

for every \( \phi \in C^1_0(\Omega) \) with \( \phi \geq 0 \) in \( \Omega \). Similarly, the problem (1.2) is said to be critical in the gradient if \( f \) has a growth of power \( p \) in the gradient. In this case, some authors have presented various results on the existence of a weak solution for (1.2); see [4, 6]. However, it has not been analyzed what occurs when \( f \) interacts with the spectrum of the \( p \)-Laplacian. In the present work, we obtain several results on the existence of weak solutions.
under assumptions that relate $f$ and the first eigenvalue $\lambda_1$ of the $p$-Laplacian. Some of these results have not been considered even when $p = 2$. Getting a complete understanding, we provide necessary and sufficient conditions for existence of positive weak solutions. For example, we show that the problem

$$-\Delta_p u = \frac{a}{1 + ku} |\nabla u|^p + b(1 + ku)^{p-1} \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial \Omega,$$

where $a, b > 0$ and $k \geq 0$ are constants, admits a positive weak solution if and only if $(k(p - 1) + a)^{p-1} b < (p - 1)^{p-1} \lambda_1$. Hence,

$$-\Delta_p u = a|\nabla u|^p + b \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial \Omega,$$

has a positive weak solution if and only if $a^{p-1} b < (p - 1)^{p-1} \lambda_1$. Besides, we give sufficient conditions under which the problem

$$-\Delta_p u = \frac{a(u)}{1 + ku} |\nabla u|^p + bu^{p-1} \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial \Omega,$$

possesses a positive weak solution. When $p = 2$, we discuss the existence of weak solutions under more general hypotheses on $f$. In particular, we investigate equations of the type

$$-\Delta u = a(x, u)|\nabla u|^2 + b(x, u) \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial \Omega.$$

The outline of the paper is as follows. In Section 2 we provide necessary and sufficient non-resonance conditions for the existence of positive weak solutions for (1.2). In Section 3 we show some results on the existence of positive weak solutions for problems subject to other non-resonance conditions. Finally, in Section 4 we analyze the problem (1.2) in the case $p = 2$. In particular, we give more general sufficient conditions than those of Section 2. Our arguments are based on the Díaz and Saa inequality [7], degree theory, and super-subsolution method.

2. EXISTENCE AND NONEXISTENCE OF POSITIVE SOLUTIONS

In this section we search for necessary and sufficient conditions for the existence of positive weak solutions for the problem

$$-\Delta_p u = a(u)|\nabla u|^p + b(x, u) \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial \Omega,$$

(2.1)
where \( a: \mathbb{R}_+ \to \mathbb{R}_+ \) is a continuous function, \( \mathbb{R}_+ = [0, +\infty) \), and \( b: \Omega \times \mathbb{R}_+ \to \mathbb{R}_+ \) is a Carathéodory function; that is, for a.e. \( x \in \Omega \) the function \( b(x, \cdot) \) is continuous and for every \( s \in \mathbb{R} \) the function \( b(\cdot, s) \) is measurable.

Before stating our first result, let us consider the problem

\[
\begin{align*}
-\Delta_x v &= g(x, v) & \text{in } \Omega \\
v &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

where \( g: \Omega \times \mathbb{R}_+ \to \mathbb{R}_+ \) is a Carathéodory function verifying

\[
g(x, s) \leq c(s^{p-1} + 1)
\]

for every \( x \in \Omega \) a.e., \( s \geq 0 \), and some constant \( c > 0 \).

The following result is taken from [2]:

**Lemma 2.1.** Suppose

\[
\limsup_{s \to +\infty} \frac{g(x, s)}{s^{p-1}} < \lambda_1, \quad \text{uniformly for } x \in \Omega \text{ a.e.}
\]

Then \((2.2)\) admits at least one nonnegative weak solution in \( C^1(\overline{\Omega}) \).

Let \( H: \mathbb{R}_+ \to \mathbb{R}_+ \) be defined by

\[
H(t) = \int_0^t \exp\left(\frac{1}{p-1} \int_0^s a(r) \, dr\right) \, ds.
\]

Assume that the pair \((a, b)\) satisfies

\[
\begin{align*}
\text{(H1)} & \quad \limsup_{t \to +\infty} \frac{H'(t)^{p-1} b(x, t)}{H(t)^{p-1}} < \lambda_1, \quad \text{uniformly for } x \in \Omega \text{ a.e.}, \\
\text{(H2)} & \quad \text{there are constants } c, k > 0 \text{ such that } H'(t)^{p-1} b(x, t) \leq c H(t)^{p-1} \text{ for } x \in \Omega \text{ a.e. and } t \geq k, \\
\text{(H3)} & \quad \sup_{t \in [0,k]} b(\cdot, t) \in L^p(\Omega).
\end{align*}
\]

**Proposition 2.1.** If \( \text{(H1), (H2) and (H3) are fulfilled} \), then \((2.1)\) admits at least one nonnegative weak solution.

**Proof.** Since \( H'(t) \geq 1 \) for \( t \geq 0 \), then

\[
H(t) \to +\infty \quad \text{as } t \to +\infty.
\]

(2.3)
Define \( g(x,s) = (H'(H^{-1}(s)))^{p-1}b(x,H^{-1}(s)). \) From (2.3) and (H1), it follows that
\[
\limsup_{s \to +\infty} \frac{g(x,s)}{s^{p-1}} < \lambda_1, \quad \text{uniformly for } x \in \Omega \text{ a.e.}
\]
Furthermore, by (H2) and (H3), there exists \( c > 0 \) such that
\[ g(x,s) \leq c(s^{p-1} + 1) \quad \text{for every } x \in \Omega \text{ a.e. and } s \geq 0. \]
By Lemma 2.1, (2.2) admits a nonnegative weak solution \( v \in C^1(\overline{\Omega}). \) So, defining \( u = H^{-1}(v), \) we conclude that \( u \) is a nonnegative weak solution of (2.1).

Let \( a,c: \mathbb{R}_+ \to \mathbb{R}_+ \) be continuous functions and \( b,d: \Omega \times \mathbb{R}_+ \to \mathbb{R}_+ \) be Carathéodory functions such that \( a(t_1) \leq c(t_2) \) and \( b(x,t_1) \leq d(x,t_2) \) for every \( x \in \Omega \text{ a.e. and } t_1,t_2 \geq 0. \)

Suppose
\[
\text{(H4) } a(t)\|\xi\|^p + b(x,t) \leq f(x,t,\xi) \leq c(t)\|\xi\|^p + d(x,t)
\]
for every \( x \in \Omega \text{ a.e., } t \geq 0, \) and \( \xi \in \mathbb{R}^N. \)

**Theorem 2.1.** Assume (H4) with \( (a,b) \) and \( (c,d) \) satisfying (H1), (H2), and (H3). Then (1.2) admits at least one nonnegative weak solution.

**Proof.** Let \( \overline{u}, u \in C^1(\overline{\Omega}) \) be nonnegative weak solutions of problems
\[
-\Delta_p u = a(u)\|\nabla u\|^p + b(x,u) \quad \text{in } \Omega
\]
\[
u = 0 \quad \text{on } \partial \Omega,
\]
\[
-\Delta_p \overline{u} = c(\overline{u})\|\nabla \overline{u}\|^p + d(x,\overline{u}) \quad \text{in } \Omega
\]
\[
\overline{u} = 0 \quad \text{on } \partial \Omega.
\]
By (H4) and the comparison principle, we conclude that \( u \) is a subsolution of (1.2), \( \overline{u} \) is a supersolution of (1.2), and \( u \leq \overline{u} \) in \( \Omega. \) Therefore, by \( C^{1,1}(\overline{\Omega}) \) estimates in [11] and monotonic iteration, one concludes that there exists \( u \in C^1(\overline{\Omega}), \) which is a weak solution of (1.2) with \( u \leq u \leq \overline{u}. \)

Now let us give necessary conditions for the existence of a positive weak solution for (2.1).

Assume
\[
\text{(H5) } H'(t)^{p-1}b(x,t) \geq \lambda_1 H(t)^{p-1} \text{ for } x \in \Omega \text{ a.e. and } t > 0, \]
\[
\text{(H6) } H'(t)^{p-1}b(x,t) > \lambda_1 H(t)^{p-1} \text{ for } x \text{ in a subset of positive measure of } \Omega \text{ and } t > 0.
\]
PROPOSITION 2.2. If (H5) and (H6) are satisfied, then (2.1) does not admit a positive weak solution.

Proof. Suppose, by contradiction, that (2.1) admits a positive weak solution \( u \in C^1(\Omega) \). Define \( v = H(u) \). Then \( v \) is a positive weak solution of (2.2), where \( g(x,s) \) is as in the proof of Proposition 2.1. In addition, by (H5) and (H6), one gets

\[
\begin{align*}
g(\cdot, u(\cdot)) &\geq \lambda_1 u^{p-1}(\cdot) \quad \text{in } \Omega \text{ a.e.} \\
g(\cdot, u(\cdot)) &> \lambda_1 u^{p-1}(\cdot) \quad \text{in a subset of positive measure of } \Omega.
\end{align*}
\]

Let \( \varphi_1 \) be a positive eigenfunction associated to \( \lambda_1 \) such that \( \varphi_1 > v \) in \( \Omega \). Since \((\varphi_1^p - v^p)/\varphi_1^{p-1}, (v^p - \varphi_1^p)/v^{p-1} \in W_0^{1,p}(\Omega)\), we can write

\[
I = \int_{\Omega} |\nabla \varphi_1|^{p-2} \nabla \varphi_1 \cdot \nabla \left( \frac{\varphi_1^p - v^p}{\varphi_1^{p-1}} \right) dx + \int_{\Omega} |\nabla v|^{p-2} \nabla v \cdot \nabla \left( \frac{v^p - \varphi_1^p}{v^{p-1}} \right) dx.
\]

From the Dáaz and Saa inequality [7], we know that \( I \geq 0 \). On the other hand, from (2.4) and (2.5), we obtain

\[
I = \int_{\Omega} \lambda_1 (\varphi_1^p - v^p) dx + \int_{\Omega} \frac{g(x,u)}{v^{p-1}} (v^p - \varphi_1^p) dx
\]

\[
= \int_{\Omega} \lambda_1 - \frac{g(x,u)}{v^{p-1}} (\varphi_1^p - v^p) dx < 0,
\]

a contradiction. \( \blacksquare \)

EXAMPLE 2.1. Let us consider the problem

\[
-\Delta_p u = a|\nabla u|^p + b \quad \text{in } \Omega
\]

\[
u = 0 \quad \text{on } \partial \Omega,
\]

where \( a, b > 0 \) are constants. Then, by Propositions 2.1 and 2.2, (2.6) admits a positive weak solution if and only if

\[
a^{p-1}b < (p-1)^{p-1}\lambda_1.
\]

EXAMPLE 2.2. More generally, set the problem

\[
-\Delta_p u = \frac{a}{1 + ku}|\nabla u|^p + b(1 + ku)^{p-1} \quad \text{in } \Omega
\]

\[
u = 0 \quad \text{on } \partial \Omega,
\]
where $a, b > 0$ and $k \geq 0$ are constants. Then, by Propositions 2.1 and 2.2, (2.7) possesses a positive weak solution if and only if

$$(k(p-1) + a)^{p-1} b < (p-1)^{p-1} \lambda_1.$$ 

**Example 2.3.** Consider the problem

$$-\Delta_p u = \frac{a}{(1+u)^{\frac{p}{2}}} |\nabla u|^p + b(1 + u)^{p-1} \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial \Omega,$$

where $a, b > 0$ and $\alpha > 1$ are constants. Then, by an extension of L’Hospital’s rule and Propositions 2.1 and 2.2, it is easy to conclude that (2.8) has a positive weak solution if and only if $b < \lambda_1$.

### 3. Existence of Positive Solutions Under Other Non-Resonance Conditions

In this section we need the following result which is a consequence of the degree theory in cones:

**Lemma 3.1.** Suppose

(i) $\liminf_{s \to +\infty} \frac{g(x,s)}{s^{p-1}} > \lambda_1$, uniformly for $x \in \Omega$ a.e.,

(ii) $\limsup_{s \to 0^+} \frac{g(x,s)}{s^{p-1}} < \lambda_1$, uniformly for $x \in \Omega$ a.e.

Then (2.2) admits at least one positive weak solution.

**Proof.** At first, one shows that there exists $c > 0$ such that $\|u\|_{W^{1,\infty}(\Omega)} \leq c$ for every nonnegative weak solution $u$ of (2.2). This is done by contradiction, supposing that there is a sequence of nonnegative weak solutions $u_n$ such that $\|u_n\|_{W^{1,\infty}(\Omega)} \to \infty$. Dividing the equation by its norms to the power of $p - 1$ and taking the limit, one arrives at the relation

$$-\Delta_p v = g(x)v^{p-1} \quad \text{in } \Omega$$

$$v \geq 0 \quad \text{in } \Omega$$

$$v = 0 \quad \text{on } \partial \Omega,$$

where $v$ is, up to a subsequence, the weak limit of $u_n/\|u_n\|_{W^{1,\infty}(\Omega)}$, $g \in L^\infty(\Omega)$ and, by (i), $g(x) > \lambda_1$ for $x \in \Omega$ a.e. Using the Díaz and Saa inequality [7], it is easy to verify that $v = 0$ in $\Omega$; otherwise it must change.
sign. Taking \( u_n \) as a test function in the equation satisfied by \( u_n \), dividing by its norms to the power of \( p \), and letting \( n \to \infty \), one gets a contradiction. The next step is considering the cone \( C = \{ u \in W_0^{1,p}(\Omega) : u \geq 0 \text{ in } \Omega \} \) and defining the homotopy \( H : [0, +\infty) \times C \to C \) by \( v = H(t, u) \), where

\[
-\Delta_p u = g(x, u) + t \quad \text{in } \Omega \\
v = 0 \quad \text{on } \partial \Omega.
\]

Let \( T(u) = H(0, u) \). Using (ii), one concludes that there exists \( r > 0 \) such that \( u \neq tT(u) \) for every \( t \in [0, 1] \) and \( u \in C \) with \( \|u\|_{W_0^{1,p}(\Omega)} = r \). In addition, suppose that there are \( t \in [0, +\infty) \) and \( u \in C \) satisfying \( u = H(t, u) \). Using (i), one obtains

\[
-\Delta_p u \geq \lambda t^{p-1} + t - c \quad \text{in } \Omega
\]

with \( \lambda > \lambda_1 \). Assuming \( t > c \) and again applying the Diaz and Saa inequality [7], one concludes that \( u \equiv 0 \) in \( \Omega \), contradicting \( t > c \). Hence, \( t \leq c \). Therefore, there exists \( M > r \) such that \( \|u\|_{W_0^{1,p}(\Omega)} < M \) for every \( u \in C \) satisfying \( u = H(t, u) \) for some \( t \in [0, +\infty) \). Now applying the result about expansion of cones due to Krasnoselskii [10] we are done.

Now, assume

\begin{align*}
\text{(H7)} & \quad \lim_{t \to +\infty} \frac{H'(t)^{p-1}b(x, t)}{H(t)^{p-1}} > \lambda_1, \text{ uniformly for } x \in \Omega \text{ a.e.,} \\
\text{(H8)} & \quad \limsup_{t \to 0^+} \frac{H'(t)^{p-1}b(x, t)}{H(t)^{p-1}} < \lambda_1, \text{ uniformly for } x \in \Omega \text{ a.e.}
\end{align*}

**Proposition 3.1.** If (H2), (H3), (H7), and (H8) are fulfilled, then (2.1) admits at least one positive weak solution.

**Proof.** Let \( g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+ \) be as in the proof of Proposition 2.1. By (H7) and (H8), we get

\[
\lim_{s \to +\infty} \frac{g(x, s)}{s^{p-1}} > \lambda_1 \quad \text{and} \quad \limsup_{s \to 0^+} \frac{g(x, s)}{s^{p-1}} < \lambda_1,
\]

both uniformly for \( x \in \Omega \) a.e. Furthermore, by (H2) and (H3), there exists a constant \( c > 0 \) such that

\[
g(x, s) \leq c(s^{p-1} + 1) \quad \text{for } x \in \Omega \text{ a.e. and } s \geq 0.
\]
Therefore, by Lemma 3.1, (2.2) admits a positive weak solution $v \in C^1(\Omega)$. So, defining $u = H^{-1}(v)$, we conclude that $u$ is a positive weak solution of (2.1).

Note that (H8) implies $u \equiv 0$ in $\Omega$ is also a solution of (2.1). By virtue of Proposition 3.1 and with the aid of the auxiliary problem (2.1), a similar procedure used in the proof of Theorem 2.1 allow us to state the following result.

**Theorem 3.1.** Assume (H4) with $(a, b)$ and $(c, d)$ satisfying (H2), (H3), (H7), and (H8). Then (1.2) admits at least one positive weak solution.

**Example 3.1.** Let us consider the equation

$$-\Delta_p u = \frac{a(u)}{1 + ku} |\nabla u|^p + bu^{p-1} \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial \Omega,$$

where $a: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a bounded continuous function and $b, k > 0$ are constants. Clearly, $u \equiv 0$ is a trivial solution of (3.1). However, if

$$\liminf_{t \rightarrow +\infty} a(t) = a,$$

$$b < \lambda_1$$

and

$$\left( \frac{a}{k(p-1)} + 1 \right)^{p-1} b > \lambda_1,$$

then, by Proposition 3.1, (3.1) admits a positive weak solution. In particular, if

$$\left( 1 - \frac{1}{p} \right)^{p-1} \lambda_1 < b < \lambda_1,$$

then the problem

$$-\Delta_p u = \frac{1}{1 + u} |\nabla u|^p + bu^{p-1} \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial \Omega,$$

has a positive weak solution.
4. THE CASE $p = 2$

In this section we generalize Proposition 2.1 and Theorem 2.1 for $p = 2$. Consider the problem

$$-\Delta v + \sum_{i=1}^{N} B_i(x,v) D_i v + C(x,v) v = g(x,v) \quad \text{in } \Omega$$

$$v = 0 \quad \text{on } \partial \Omega,$$

where $B_i, C, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions.

The next lemma follows as a particular case of a result in [15].

**LEMMA 4.1.** Assume

(i) There is a constant $c > 0$ such that $|B_i(x,s)| \leq c$ and $|C(x,s)| \leq c$ for $x \in \Omega$ a.e. and $s \in \mathbb{R}$. Define

$$\mu_1 = \lim_{\|v\|_{L^2(\Omega)} \to +\infty} \inf_{v \in H^1(\Omega)} \frac{J(v,v)}{\|v\|_{L^2(\Omega)}^2},$$

where

$$J(v,v) = \int_{\Omega} |\nabla v|^2 \, dx + \sum_{i=1}^{N} \int_{\Omega} B_i(x,v) D_i v \, dx + \int_{\Omega} C(x,v) v^2 \, dx.$$ 

Then $\mu_1 \in (-\infty, +\infty)$. In addition, if

(ii) $\limsup_{|s| \to +\infty} \frac{g(x,s)}{s} < \mu_1$, uniformly for $x \in \Omega$, a.e.,

(iii) there is a constant $c_1 > 0$ such that $|g(x,s)| \leq c_1(|s| + 1)$ for $x \in \Omega$ a.e. and $s \in \mathbb{R}$

then (4.1) admits at least one weak solution.

**Remark 4.1.** There is $c_0 > 0$ such that if $B_i$ and $C$ verify (i) of Lemma 4.1 with $c_0$ in place of $c$, then $J \geq 0$, $\mu_1 > 0$, and the full elliptic operator of the left-hand side of (4.1) satisfies the strong maximum principle. In this case, if $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$, then Lemma 4.1 provides the existence of a nonnegative weak solution.

Now let us discuss the problem

$$-\Delta u = a(x,u)|\nabla u|^2 + b(x,u) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega,$$

(4.2)
where \( a: \overline{\Omega} \times \mathbb{R}_+ \to \mathbb{R}_+, D_ia, D_{ii}a: \overline{\Omega} \times \mathbb{R}_+ \to \mathbb{R} \) are continuous functions and \( b: \Omega \times \mathbb{R}_+ \to \mathbb{R}_+ \) is a Carathéodory function, where \( D_ia(x,t) = D_{i,ii}a(x,t) \) and \( D_{ii}a(x,t) = D_{i,iii}a(x,t) \).

Assume the assumption below

\((H9)\) There is a constant \( k > 0 \) such that \( |\int_0^s D_ia(x,r) \, dr| \leq k \) and \( |\int_0^s D_{ii}a(x,r) \, dr| \leq k \) for every \((x,t) \in \Omega \times \mathbb{R}_+\).

Define \( H: \overline{\Omega} \times \mathbb{R}_+ \to \mathbb{R}_+ \) by

\[
H(x,t) = \int_0^t \exp \left( \int_0^s a(x,r) \, dr \right) ds.
\]

Since \( H \in C^2(\overline{\Omega} \times \mathbb{R}_+) \) and \( D_iH(x,t) \geq 1 \) for \( x \in \Omega \) and \( t \geq 0 \), by the implicit function theorem or, more precisely, the submersion local form theorem, there exists \( K \in C^2(\overline{\Omega} \times \mathbb{R}_+) \) such that \( s = H(x,K(x,s)) \). Let

\[
\begin{align*}
J(v,v) &= \int_{\Omega} |\nabla v|^2 \, dx + \sum_{i=1}^N \int_{\Omega} B_i(x,v) D_iuv \, dx + \int_{\Omega} C(x,v) v^2 \, dx, \\
& \quad \text{for every } v \in W^{1,2}_0(\Omega), \text{ where}
\end{align*}
\]

\[
B_i(x,s) = \frac{2H_{i,s}(x,K(x,s))}{H_i(x,K(x,s))},
\]

\[
C(x,s) = \sum_{i=1}^N \left( \frac{H_{i,s}(x,K(x,s))}{s} \right) - \frac{2H_{i,i}(x,K(x,s))H_{i,s}(x,K(x,s))}{H_i(x,K(x,s))s}.
\]

By \((H9)\), \( B_i \) and \( C \) verify (i) of Lemma 4.1, so \( \mu_1 \) is finite. Suppose \( J \geq 0 \), \( \mu_1 > 0 \), and the full elliptic operator of the left-hand side of (4.1) verifies the strong maximum principle; see Remark 4.1.

Assume that \( b \) satisfies

\((H10)\) \( \lim_{t \to +\infty} \frac{H_i(x,t)b(x,t)}{H(x,t)} < \mu_1 \), uniformly for \( x \in \Omega \) a.e.,

\((H11)\) there are constants \( c, k > 0 \) such that \( H_i(x,t)b(x,t) \leq cH(x,t) \) for \( x \in \Omega \) and \( t \geq k \).

**Proposition 4.1.** If \((H3), (H9), (H10), \) and \((H11)\) are fulfilled, then (4.2) admits at least one nonnegative weak solution.
Proof. Define \( g(x,s) = H_j(x,K(x,s))b(x,K(x,s)) \). From (H3), (H9), (H10), and (H11), one has

\[
\limsup_{s \to +\infty} \frac{g(x,s)}{s} < \mu_1, \quad \text{uniformly for } x \in \Omega \text{ a.e.},
\]

\[
g(x,s) \leq c(s + 1), \quad \text{for all } x \in \Omega \text{ a.e. and } s \geq 0.
\]

By Lemma 4.1 and Remark 4.1, (4.1) admits a nonnegative weak solution \( v \in C^1(\mathbf{\Omega}) \cap W^{2,p}(\Omega) \). Now, denoting \( u = K(x,v) \) we get \( u \in C^1(\mathbf{\Omega}) \cap W^{2,p}(\Omega) \); let us verify that \( u \) satisfies (4.2). Indeed, since \( v(x) = H(x,u(x)) \) we get

\[
v_{x_i} = H_{x_i} + H_{u_{x_i}},
\]

\[
v_{x_i x_i} = H_{x_i x_i} + 2H_{x_i}u_{x_i} + H_{u_{x_i}}u_{x_i}^2 + H_{u_{x_i} x_i}.
\]

Then

\[
\Delta v = \sum_{i=1}^{N} \left( H_{x_i x_i} + 2H_{x_i}u_{x_i} \right) + H_{u_i} |\nabla u|^2 + H_{u} \Delta u.
\]

Inserting \( v \) in the expression of \( B_i \) and \( C \) yields

\[
\sum_{i=1}^{N} B_i(x,v) D_i v + C(x,v) v = \sum_{i=1}^{N} \left( H_{x_i x_i} + 2H_{x_i}u_{x_i} \right).
\]

Consequently

\[
-\Delta v + \sum_{i=1}^{N} B_i(x,v) D_i v + C(x,v) v
\]

\[
= -H_{u_i}v |\nabla u|^2 - H_{u} \Delta u.
\]

Since \( g(x,v) = H_j(x,u)b(x,u) \), it remains to note that \( H_{u_i} = aH_i \). \[ \blacksquare \]

Finally, let \( a,c: \Omega \times \mathbb{R}_+ \to \mathbb{R}_+ \) be continuous functions and \( b,d: \Omega \times \mathbb{R}_+ \to \mathbb{R}_+ \) Carathéodory functions such that \( a(x,t_1) \leq c(x,t_2) \) and \( b(x,t_1) \leq d(x,t_2) \) for every \( x \in \Omega \text{ a.e. and } t_1, t_2 \geq 0 \).

Suppose

(H12) \( a(x,t)|\xi|^2 + b(x,t) \leq f(x,t,\xi) \leq c(x,t)|\xi|^2 + d(x,t) \), for every \( x \in \Omega \text{ a.e. and } t \geq 0 \).
An analogous reasoning used in the proof of Theorem 2.1 produces the following result:

**THEOREM 4.1.** Suppose \( H_{12} \) with \( a, b \), and \( c, d \) verifying \( H_3, H_9, H_{10}, \) and \( H_{11} \). Then (1.2) admits at least one nonnegative weak solution.

When \( a(x, t) = a(t) \), one has \( B_t = 0 = C \). Consequently, \( \mu_1 = \lambda_1 \). Hence, when \( p = 2 \), Proposition 4.1 and Theorem 4.1 generalize Proposition 2.1 and Theorem 2.1, respectively. We finish by giving an example where the hypotheses of Proposition 2.1 are not necessarily satisfied; however, Proposition 4.1 applies.

**EXAMPLE 4.1.** Let us set the problem

\[-\Delta u = \frac{a(\varepsilon x)}{(1 + u)^{\alpha}}|\nabla u|^2 + b(x)(1 + u) \quad \text{in } \Omega \quad \text{(4.3)}\]

\[u = 0 \quad \text{on } \partial\Omega,\]

where \( \alpha > 0 \), \( a: \mathbb{R}^N \rightarrow \mathbb{R}_+ \) is a function of class \( C^2 \) such that \( D_1 a \) and \( D_{ij} a \) are bounded functions in \( \mathbb{R}^N \), and \( b: \Omega \rightarrow \mathbb{R}_+ \) is a bounded measurable function.

If \( a(x) = a \in \mathbb{R} \) for every \( x \in \mathbb{R}^N \) and \( b(x) < \lambda_1 \) uniformly for \( x \in \Omega \) a.e., then, by Proposition 2.1, (4.3) admits a positive weak solution. On the other hand, if \( a \) is not constant, there exists \( \varepsilon_0 > 0 \) such that the conditions given in Remark 4.1 hold for every \( 0 < \varepsilon < \varepsilon_0 \). Hence, if \( 0 < \varepsilon < \varepsilon_0 \) and

\[b(x) < \mu_1, \quad \text{uniformly for } x \in \Omega \text{ a.e.}, \quad (4.4)\]

then problem (4.3) possesses a positive weak solution. In particular, there is \( \varepsilon_0 > 0 \) such that if \( 0 < \varepsilon < \varepsilon_0 \) and (4.4) holds, then the problem

\[-\Delta u = \frac{\cos^2(x \sum_{i=1}^{N} x_i)}{(1 + u)^{\alpha}}|\nabla u|^2 + b(x)(1 + u) \quad \text{in } \Omega \]

\[u = 0 \quad \text{on } \partial\Omega,\]

has a positive weak solution.

**REFERENCES**


