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Extension of Ambarzumyan's Theorem to General Boundary Conditions

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We extend the classical Ambarzumyan's theorem for the Sturm–Liouville equation (which is concerned only with Neumann boundary conditions) to the general boundary conditions, by imposing an additional condition on the potential function. Our result supplements the Pöschel–Trubowitz inverse spectral theory. We also have parallel results for vectorial Sturm–Liouville systems. © 2001 Academic Press

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1. INTRODUCTION

Consider the Sturm–Liouville problem

$$-y'' + qy = \lambda y \tag{1.1}$$

such that

$$\begin{cases} y(0)\cos\alpha + y'(0)\sin\alpha = 0 \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \end{cases} \tag{1.2}$$

where $q \in L^1(0,\pi), \alpha, \beta \in [0,\pi)$. Ambarzumyan's theorem [1] states that for the Neumann boundary condition ($\alpha = \beta = \frac{\pi}{2}$), if the spectrum

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$\sigma = \{n^2 : n \in \mathbf{N} \cup \{0\}\}$, then the potential function $q = 0$ a.e. The theorem may be viewed as the first theorem in the history of inverse spectral theory. Recently, Shen and one of us (H. H. Chern), using ideas from [3], proved a vectorial version of Ambarzumyan's theory [4].

As all these works only deal with Neumann boundary conditions, it would be interesting to study other boundary conditions. However, Pöschel and Trubowitz in their classical monograph [7] showed that for the typical Dirichlet problem, there exists infinitely many L^2 potentials near zero which correspond to the "zero" spectrum $\sigma = \{n^2 : n \in \mathbf{N}\}$. This means that for the Dirichlet problem, when the spectrum is "zero," the potential is not necessarily zero. The associated Ambarzumyan's theorem is not valid.

However, we can impose some additional conditions on the potential function to arrive at a theorem for the general boundary conditions. Our main theorem is the following.

THEOREM 1.1. *For the Sturm–Liouville problem* (1.1) *and* (1.2), *assume that* $\alpha = \beta \neq \frac{\pi}{2}$. *Then* $\sigma = \{n^2 : n \in \mathbf{N}\}$, *and the potential function* q *satisfies*

$$\int_0^{\pi} q(x) \cos 2(x - \alpha)\, dx = 0$$

if and only if $q = 0$ *a.e.*

Theorem 1.1 adds content to the Pöschel–Trubowitz inverse spectral theory. Its significance will be discussed after its proof in Section 2. The proof makes use of explicit eigenvalue asymptotics instead of the Gelfand–Levitan equation as needed by previous proofs. Applying the theorem, we arrive at the following interesting conclusion.

THEOREM 1.2. *For the Sturm–Liouville problem* (1.1) *and* (1.2), *any two of the following three conditions imply the third one*:

(a) $\alpha = \beta \in [0, \pi)$,

(b) $\sigma = \{n^2 : n \in \mathbf{N}\}$ *is the spectral set of* (1.1) *and* (1.2), *and* $\int_0^{\pi} q(x) \cos 2(x - \alpha)\, dx = 0$

(c) $q = 0$ *a.e.*

From the above theorem (or more explicitly, Theorem 2.1), when the potential function vanishes and the spectrum is "zero," then the phases α, β have to be equal. Thus Ambarzumyan's inverse spectrum theorem is false when $\alpha \neq \beta$.

We are also able to prove an analog of Theorem 1.1 for the vectorial case, generalizing the result in [4]. Since eigenvalue asymptotics are not clear in vectorial Sturm–Liouville problems, except for Dirichlet and Neumann boundary conditions [2], we resort to the (vectorial) Gelfand–Levitan equation developed in [4].

In Section 2, we study the scalar case. We first show that (Theorem 2.1) for the zero potential, the spectrum is "zero" if and only if the boundary phases are equal, i.e., $\alpha = \beta$. Then we prove our main theorems. In Section 3, we study the vectorial case. We prove a parallel result to Theorem 2.1, which says that for the zero potential, its spectrum is "zero" if and only if the boundary conditions are the same. Finally we prove the (vectorial) Ambarzumyan theorems for some other boundary conditions (Theorem 3.4), and then for Dirichlet boundary conditions (Theorem 3.7).

2. SCALAR CASE

THEOREM 2.1. *For the Sturm–Liouville problem* (1.1) *and* (1.2), *when* $q = 0$, *the spectrum* $\sigma = \{n^2 : n \in \mathbf{N}\}$ *if and only if* $\alpha = \beta$.

Proof. When $q = 0$, (1.1) becomes $y'' + \lambda y = 0$. Its general solution is

$$y = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x,$$

where $c_1 c_2 \neq 0$. So $y(0) = c_1$, $y'(0) = c_2\sqrt{\lambda}$, and the boundary condition at the left endpoint 0 becomes

$$c_1 \cos\alpha + c_2\sqrt{\lambda}\sin\alpha = 0. \tag{2.1}$$

On the other hand,

$$\begin{cases} y(\pi) = c_1\cos(\sqrt{\lambda}\pi) + c_2\sin(\sqrt{\lambda}\pi) \\ y'(\pi) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\pi). \end{cases}$$

Substituting into the boundary condition at the right endpoint π, we obtain

$$\begin{aligned} &c_1(\cos\beta\cos(\sqrt{\lambda}\pi) - \sqrt{\lambda}\sin\beta\sin(\sqrt{\lambda}\pi)) \\ &\quad + c_2(\cos\beta\sin(\sqrt{\lambda}\pi) + \sqrt{\lambda}\sin\beta\cos(\sqrt{\lambda}\pi)) = 0. \end{aligned} \tag{2.2}$$

Combining (2.1) and (2.2),

$$\begin{aligned} &\sqrt{\lambda}\sin\alpha(\cos\beta\cos(\sqrt{\lambda}\pi) - \sqrt{\lambda}\sin\beta\sin(\sqrt{\lambda}\pi)) \\ &\quad = \cos\alpha(\cos\beta\sin(\sqrt{\lambda}\pi) + \sqrt{\lambda}\sin\beta\cos(\sqrt{\lambda}\pi)). \end{aligned}$$

After grouping,

$$\begin{aligned} &(\lambda\sin\alpha\sin\beta + \cos\alpha\cos\beta)\sin(\sqrt{\lambda}\pi) \\ &\quad + \sqrt{\lambda}\sin(\beta - \alpha)\cos(\sqrt{\lambda}\pi) = 0. \end{aligned} \tag{2.3}$$

If $\sqrt{\lambda} = n$, then $\sin(\beta - \alpha) = 0$. We have $\alpha = \beta$. On the other hand, if $\alpha = \beta$, then by (2.3),

$$(\lambda\sin^2\alpha + \cos^2\alpha)\sin(\sqrt{\lambda}\pi) = 0.$$

Therefore $\sqrt{\lambda}$ has to be an integer. ∎

Proof of Theorem 1.1. The necessary part is obvious. To prove the sufficiency, observe that by Lemma 2.2 in [6] (see also [5, 8]), we obtain with a scaling,

$$\sqrt{\lambda_n} = n + \frac{1}{n\pi}(\operatorname{scot}\beta - \operatorname{scot}\alpha) + \frac{1}{2n\pi^3} \times \int_0^\pi (1 + \alpha_0 \pi^2 \cos(2nx) q(x))\,dx + O\left(\frac{1}{n^2}\right),$$

as $n \longrightarrow \infty$. Here $\operatorname{scot}\gamma = \cot\gamma$, if $\gamma \neq 0; = 0$ otherwise and $\alpha_0 = -1$, if $\alpha = 0; = 1$, otherwise. Then if $\alpha = \beta$,

$$\begin{aligned}\sqrt{\lambda_n} &= n + \frac{1}{2n\pi^3}\int_0^\pi q + \frac{\alpha_0}{2n\pi}\int_0^\pi \cos(2nx) q(x)\,dx + O\left(\frac{1}{n^2}\right),\\ &= n + \frac{1}{2n\pi^3}\int_0^\pi q + o\left(\frac{1}{n}\right),\end{aligned}$$

by the Riemann–Lebesgue lemma. Thus if $\lambda_n = n^2$ for any $n \in N$, then $\int_0^\pi q = 0$.

Next we show that $\sin(x - \alpha)$ is the first eigenfunction. By the variational principle,

$$\lambda_1 = \inf \frac{\langle y, Hy\rangle}{\langle y, y\rangle},$$

where $y \in C^2[0, \pi]$, satisfying the boundary conditions (1.2) and $Hy = -y'' + qy$. Now $y = \sin(x - \alpha)$ satisfies the boundary conditions (1.2), and so

$$1 = \lambda_1 \leq \frac{\int_0^\pi \sin^2(x-\alpha)(1 + q(x))\,dx}{\int_0^\pi \sin^2(x-\alpha)\,dx}.$$

Hence

$$\begin{aligned}\frac{\pi}{2} &\leq \frac{\pi}{2} + \int_0^\pi q(x)\sin^2(x-\alpha)\,dx,\\ &= \frac{\pi}{2} + \frac{1}{2}\int_0^\pi q - \frac{1}{2}\int_0^\pi q(x)\cos 2(x-\alpha)\,dx.\end{aligned}$$

Since $\int_0^\pi q = 0$, and by assumption, the right hand side is exactly $\frac{\pi}{2}$, the test function $y = \sin(x - \alpha)$ achieves the minimum value and is thus the first eigenfunction. Substituting this into (1.1), we obtain

$$\sin(x-\alpha) + q(x)\sin(x-\alpha) = 1 \cdot \sin(x-\alpha).$$

Therefore $q \equiv 0$ a.e. ∎

Proof of Theorem 1.2. It follows directly from Theorem 2.1, Theorem 1.1, and the classical Ambarzumyan's theorem. ∎

According to the Pöschel–Trubowitz inverse spectral theory [7], any L^2 function defined on $[0, \pi)$ can be decomposed into direct sums of its odd part and even part. Let U and E denote the Banach spaces of these odd parts and even parts. Also let f stand for the function that maps an L^2 potential to its Dirichlet spectrum which is equivalent to another Banach space $F = \mathbf{R} \oplus l_2$. Thus f is a map from $U \oplus E$ to F, and $f(0, 0) = 0$. Applying the implicit function theorem, they showed that near $u_0 = 0$, the even part can be uniquely expressed as a function of its odd part ($e = e(u)$) such that $f(u, e(u)) = 0$. Thus $q = u + e(u)$ is an isospectral L^2 potential. In terms of Fourier series, our main theorem says that if

$$a_2 = \int_0^{\pi} q(x) \cos 2x \, dx = \int_0^{\pi} e(u) \cos 2x \, dx = 0,$$

then all the even Fourier coefficients vanish, i.e., $a_n = 0$ for all n. Thus the odd part also vanishes. In other words, if q is an isospectral potential, then its even part does not vanish, in particular, as our theorem indicates, $a_2 \neq 0$. In this way, our theorem supplements the Pöschel–Trubowitz theory.

3. THE VECTORIAL CASE

Consider the vectorial Sturm–Liouville problem

$$\begin{cases} -\phi'' + P(x)\phi = \lambda\phi \\ A\phi(0) + B\phi'(0) = \mathscr{A}\phi(\pi) + \mathscr{B}\phi'(\pi) = 0, \end{cases} \tag{3.1}$$

where $P(x)$ is a $d \times d$ real symmetric continuous matrix-valued function and the coefficient matrices $A, B, \mathscr{A}, \mathscr{B}$ satisfy

$$\begin{cases} BA^* = AB^*, \qquad \mathscr{B}\mathscr{A}^* = \mathscr{A}\mathscr{B}^* \\ \operatorname{rank}[A, B] = \operatorname{rank}[\mathscr{A}, \mathscr{B}] = d. \end{cases} \tag{3.2}$$

Here $[A, B]$ denotes the $d \times 2d$ augmented matrix composed of A and B. Also let $P = [P_1 P_2 \ldots P_n]$, where

$$P_k = \begin{bmatrix} P_{1k} \\ \vdots \\ P_{dk} \end{bmatrix}.$$

THEOREM 3.1. *Let* $P(x) = 0$. *Then the spectral set of* (3.1) *is* $\{n^2 : n \geq 1\}$ *and each eigenvalue has multiplicity d if and only if* $\mathscr{B}A^* = \mathscr{A}B^*$. *If both* B *and* $\mathscr{B}$ *are invertible, this is also equivalent to* $B^{-1}A = \mathscr{B}^{-1}\mathscr{A}$.

Remark. This means that when B and $\mathscr{B}$ are invertible, the spectral set of (3.1) is $\{n^2 : n \in \mathbf{N}\}$ and $P \equiv 0$ occur simultaneously only when the boundary conditions are the same.

To study the vectorial Sturm–Liouville problem, it is useful and worthy to solve the associated matrix equation as below. For any $\lambda \in C$, the solution of the matrix equation

$$\begin{cases} -Y'' + P(x)Y = \lambda Y \\ Y(0, \lambda) = B^*, \qquad Y'(0, \lambda) = -A^*, \end{cases} \tag{3.3}$$

can be characterized by the Gelfand–Levitan equation as

$$Y(x, \lambda) = C(x, \sqrt{\lambda}) + \int_0^x K(x, t)C(t, \sqrt{\lambda})\, dt, \tag{3.4}$$

where

$$C(x, \mu) = \cos(\mu x)B^* - \frac{\sin(\mu x)A^*}{\mu}, \tag{3.5}$$

and $\mathscr{K}(x, t)$ is determined by a particular wave equation. The eigenvalues are characterized by those values λ so that the matrix-valued function

$$W(\lambda) = \mathscr{B}Y'(\pi, \lambda) + \mathscr{A}Y(\pi, \lambda)$$

is singular or not. In particular, an eigenvalue λ_k is of multiplicity d, if and only if $W(\lambda_k) = 0$ (cf. [4]). Furthermore, it was shown in [4] that

$$K(\pi, \pi) = \frac{1}{2}\int_0^\pi P(x)\, dx. \tag{3.6}$$

We introduce two lemmas before we give the proof of Theorem 3.1.

LEMMA 3.2. *If the two $d \times d$ matrices A, B satisfy*

$$\operatorname{rank}[A, B] = d, \qquad BA^* = AB^*,$$

then the matrix $tB + iA$ is nonsingular for $t \in \mathbf{R}\backslash\{0\}$.

Proof. Notice that

$$(tB + iA)(tB^* - iA^*) = t^2 BB^* + AA^*. \tag{3.7}$$

Let $\langle u, v\rangle$ be the ordinary inner product in $\mathbf{C}^d$. For any $v \in \mathbf{C}^d$, if

$$(tB + iA)(tB^* - iA^*)v = 0,$$

we want to show that $v = 0$. By (3.7), we have

$$\langle v, (t^2BB^* + AA^*)v\rangle = t^2\langle B^*v, B^*v\rangle + \langle A^*v, A^*v\rangle = 0,$$

for all $t \in \mathbf{R}\backslash\{0\}$. This holds true if and only if $B^*v = 0$ and $A^*v = 0$. Observe that the hypothesis $\operatorname{rank}[A, B] = d$ implies that the only intersection between the nullspaces of A^* and B^* is $\{0\}$. ∎

LEMMA 3.3. *Let* $A, B, \mathscr{A}, \mathscr{B}$ *be* $d \times d$ *matrices satisfying* (3.2). *If* $\mathscr{B}A^* = \mathscr{A}B^*$, *then for any* $\lambda > 0$, $\lambda\mathscr{B}B^* + \mathscr{A}A^*$ *is invertible*.

Proof. For $\lambda > 0$, let $\mu = \sqrt{\lambda}$. Since

$$(\mu\mathscr{B} + i\mathscr{A})(\mu B^* - iA^*) = \mu^2\mathscr{B}B^* + \mathscr{A}A^*,$$

by Lemma 3.2, the proof is done. ∎

Proof of Theorem 3.1. As $P(x) = 0$, we have

$$W(\lambda) = -\left(\sqrt{\lambda}\mathscr{B}B^* + \mathscr{A}A^*\frac{1}{\sqrt{\lambda}}\right)\sin\sqrt{\lambda}\pi + \cos\sqrt{\lambda}\pi(\mathscr{A}B^* - \mathscr{B}A^*).$$

By the above identity and Lemma 3.3, the first part is valid.

Then, observe that when B and $\mathscr{B}$ are both invertible, $\mathscr{A}B^* = \mathscr{B}A^*$ if and only if

$$\mathscr{B}^{-1}\mathscr{A} = A^*(B^*)^{-1} = (B^{-1}A)^* = B^{-1}A.$$

The last equality above is due to (3.2). ∎

THEOREM 3.4. *For the vectorial Sturm–Liouville system* (3.1) *and* (3.2), *assume that* B *and* $\mathscr{B}$ *are both invertible and the matrix* $B^{-1}A = \mathscr{B}^{-1}\mathscr{A}$ *has nonzero characteristic values. Let* $\alpha_k \in (0, \pi)\backslash\{\pi/2\}$ $(k = 1, 2, \ldots, d)$ *and* S *be the Hermitian matrix such that*

$$S^{-1}(B^{-1}A)S = \operatorname{diag}\{\cot\alpha_1, \cot\alpha_2, \ldots, \cot\alpha_d\}.$$

Then $P \equiv 0$ *if and only if*

(i) *the eigenvalues are* n^2 $(n \in \mathbf{N})$ *with multiplicity* d; *and*

(ii) *for all* $k = 1, \ldots, d$,

$$\int_0^\pi (S^{-1}P(x)S)_{kk}\cos(2(x - \alpha_k))\,dx = 0.$$

Proof. The sufficiency part is obvious. To prove the converse, consider the matrix differential equation (3.3) for $Y = Y(x, \lambda)$, $\lambda = n^2$ is an eigenvalue of multiplicity d if and only if $W(n^2) = \mathscr{A}Y(\pi, n^2) + \mathscr{B}Y'(\pi, n^2) = 0$, the zero matrix. By (3.4) and (3.5),

$$C'(x, \mu) = -\mu\sin(\mu x)B^* - \cos(\mu x)A^*,$$

and

$$Y'(x, \lambda) = C'(x, \sqrt{\lambda}) + K(x, x)C(x, \sqrt{\lambda}) + \int_0^x \partial_x K(x, t)C(t, \sqrt{\lambda})\,dt.$$

That means

$$Y'(\pi, n^2) = C'(\pi, n) + K(\pi, \pi)C(\pi, n) + \int_0^\pi \partial_x K(x, t)|_{x=\pi}C(t, n)\,dt.$$

Now,

$$C(\pi, n) = (-1)^n B^*, \qquad C'(\pi, n) = (-1)^{n+1} A^*.$$

Hence for all n,

$$\begin{aligned} 0 &= \mathscr{A} Y(\pi, n^2) + \mathscr{B} Y'(\pi, n^2), \\ &= (-1)^n \mathscr{A} B^* + \mathscr{A} \int_0^\pi K(\pi, t) C(t, n)\, dt + (-1)^{n+1} \mathscr{B} A^* \\ &\quad + (-1)^n \mathscr{B} K(\pi, \pi) B^* + \mathscr{B} \int_0^\pi \partial_x K(x, t)|_{x=\pi} C(t, n)\, dt. \end{aligned}$$

By the Riemann–Lebesgue lemma, and the fact that $\mathscr{A} B^* = \mathscr{B} A^*$, we obtain, as n tends to infinity, $\mathscr{B} K(\pi, \pi) B^* = o(1)$. Hence $\mathscr{B} K(\pi, \pi) B^* = 0$. So as $\operatorname{rank} B = \operatorname{rank} \mathscr{B} = d$, we have $K(\pi, \pi) = 0$. Therefore $\int_0^\pi P(x)\, dx = 0$ by (3.6).

Next we want to show that $\phi_k(x) = \sin(x - \alpha_k) S e_k$ is the first eigenfunction corresponding to the first eigenvalue 1, where e_k is the unit vector whose kth component is 1.

First ϕ_k is an admissible function, because

$$\begin{aligned} S^{-1}(B^{-1} A \phi_k(0) + \phi_k'(0)) &= S^{-1} B^{-1} A S \sin(-\alpha_k) e_k + \cos(\alpha_k) e_k \\ &= [\cot \alpha_k \sin(-\alpha_k) + \cos(\alpha_k)] e_k \\ &= 0. \end{aligned}$$

Hence

$$B^{-1} A \phi_k(0) + \phi_k'(0) = 0.$$

Similarly,

$$B^{-1} A \phi_k(\pi) + \phi_k'(\pi) = 0.$$

Second,

$$\begin{aligned} 1 = \lambda_1 &\le \frac{\int_0^\pi (-\phi_k^{*t} \phi_k'' + \phi_k^{*t} P \phi_k)\, dx}{\int_0^\pi \phi_k^{*t} \phi_k\, dx} \\ &= \frac{-\phi_k^{*t} \phi_k'|_0^\pi + \int_0^\pi (\phi_k'^{*t} \phi_k' + \phi_k^{*t} P \phi_k)\, dx}{\int_0^\pi \phi_k^{*t} \phi_k\, dx}, \\ &= \left[-(1/2) \int_0^\pi \sin 2(x - \alpha_k)\, dx + \int_0^\pi \sin^2(x - \alpha_k)\, dx \right. \\ &\quad \left. + \int_0^\pi (S^{-1} P(x) S)_{kk} \sin^2(x - \alpha_k)\, dx \right] \\ &\quad \times \left[\int_0^\pi \sin^2(x - \alpha_k)\, dx \right]^{-1} \\ &= \frac{\pi/2 - (1/2) \sin 2\alpha}{\pi/2 - (1/2) \sin 2\alpha} = 1. \end{aligned}$$

Hence equality holds. Therefore ϕ_k is an eigenfunction corresponding to the first eigenvalue 1. Substituting ϕ_k into (3.1), we obtain

$$-\phi_k'' + P(x)\phi_k = \phi_k.$$

Since $\phi_k'' = -\phi_k$, $P(x)\phi_k = 0$. That is, for all $1 \le k \le d$, for all $x \in (0, \pi)$, $\sin(x - \alpha_k)PSe_k = 0$. Thus $PS_k \equiv 0$ for all k. We conclude that PS and so P is identically zero. ∎

THEOREM 3.5. *For the vectorial Sturm–Liouville system* (3.1) *and* (3.2), B *and* $\mathscr{B}$ *are both invertible. Any two of the following three conditions imply the third one*:

(a) $B^{-1}A = \mathscr{B}^{-1}\mathscr{A}$ *has nonzero characteristic values.*

(b) *The eigenvalues are* n^2 $(n \in \mathbf{N})$ *with multiplicity* d, *and there exist constants* $\alpha_k \in (0, \pi)\backslash\{\pi/2\}$ $(k = 1, \ldots, d)$ *and a Hermitian matrix* S *such that*

$$\int_0^{\pi} (S^{-1}P(x)S)_{kk} \cos(2(x - \alpha_k))\, dx = 0.$$

(c) $P = 0$.

Note that for the Dirichlet boundary conditions, $\mathscr{B} = B = \mathbf{0}$ which is not invertible. Hence the above theorem does not apply. Nevertheless, in [2], Carlson gave an estimate for the Dirichlet eigenvalues λ_n by means of the eigenvalues λ_n^0 associated with the mean potential function $P_0 = \frac{1}{\pi}\int_0^{\pi} P(x)\, dx$, assuming P_0 is a diagonal matrix.

LEMMA 3.6 [2, Corollary 4.2]. *Suppose* P *is a* C^2, *real symmetric matrix function and* $P_0 = \frac{1}{\pi}\int_0^{\pi} P$ *is a diagonal matrix. Then for sufficiently large* n, *for each* $k = 1, \ldots, n$,

$$\left|\lambda_{dn+k} - \lambda_{dn+k}^0\right| \le \frac{M}{n+1}.$$

In fact, P_0 need not be diagonal. Since P is real symmetric, P_0 is also real symmetric. Hence P_0 is diagonalizable. Let S be a similarity matrix of P_0. Note that the diagonal matrix $S^{-1}P_0S$ satisfies

$$S^{-1}P_0S = \frac{1}{\pi}\int_0^{\pi} S^{-1}P(x)S\, dx.$$

Furthermore, the two Sturm–Liouville problems with potential functions $P(x)$ and $S^{-1}P(x)S$ have the same set of eigenvalues.

THEOREM 3.7. *Suppose P is a C^2, and real symmetric matrix function. Then for the Dirichlet boundary conditions, $P \equiv 0$ if and only if*

(i) *the eigenvalues are n^2 ($n \in \mathbf{N}$) with multiplicity d; and*
(ii) $\int_0^\pi (S^{-1}P(x)S)_{kk} \cos(2x)\, dx = 0$, *for all k.*

Proof. We may apply Carlson's result (Lemma 3.6) to see that for $k = 1, \ldots, d$

$$\left|\lambda_{dn+k} - \lambda^0_{dn+k}\right| \leq \frac{M}{n+1}.$$

Let $S^{-1}P_0S = \operatorname{diag}\{q_1, q_2, \ldots, q_d\}$. Then

$$\lambda^0_{dn+k} = (n+1)^2 + q_k.$$

Hence

$$\lambda_{dn+k} = (n+1)^2 + q_k + O\left(\frac{1}{n}\right).$$

But by assumption, $\lambda_{dn+k} = (n+1)^2$. Hence $q_k = 0$, for each $k = 1, \ldots, d$. We conclude that $S^{-1}P_0S = 0$. So $P_0 = 0$, or

$$\int_0^\pi P(x)\, dx = 0.$$

Following the steps as in Theorem 3.4, we have $P \equiv 0$. ∎

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