ELSEVIER Journal of Computational and Applied Mathematics 106 (1999) 299–305

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.nl/locate/cam

A note on extending Euler's connection between continued fractions and power series

John Gill *

Department of Mathematics, University of Southern Colorado, Pueblo, CO 81001-4901, USA

Received 12 January 1999

Abstract

Euler's Connection describes an exact equivalence between certain continued fractions and power series. If the partial numerators and denominators of the continued fractions are perturbed slightly, the continued fractions equal power series plus easily computed error terms. These continued fractions may be integrated by the series with another easily computed error term.

MSC: 40A15; 30B70

Keywords: Continued fraction; Power series; Fixed point

1. Introduction

Euler, in 1748, described an exact equivalence between a series and a continued fraction (CF) (see e.g. [1]). In terms of power series (PS) with non-zero coefficients this connection may be written

$$c_0 + \frac{c_1\zeta}{1} \; \frac{\frac{c_2}{c_1}\zeta}{-\,1+\frac{c_2}{c_1}\zeta\,-} \cdots \frac{\frac{c_n}{c_{n-1}}\zeta}{-\,1+\frac{c_n}{c_{n-1}}\zeta} \equiv c_0 + c_1\zeta + \cdots + c_n\zeta^n \tag{1}$$

for each n. Or (1) may be expressed in an equivalent form:

$$\frac{1}{1\,-}\;\frac{a_1\zeta}{1+b_1\zeta\,-}\;\frac{a_2\zeta}{1+b_2\zeta\,-}\cdots\frac{a_n\zeta}{-\,1+b_n\zeta} = 1 + a_1\zeta + a_1a_2\zeta^2 + \cdots + a_1a_2\cdots a_n\zeta^n, \tag{1a}$$

where $a_n = b_n = c_{n+1}/c_n$.

* *E-mail address:* jgill@uscolo.edu (J. Gill)

PII: S0377-0427(99)00071-0

If the partial numerators and denominators of the CF in (1a) are perturbed slightly, so that $a_n \neq b_n$, the nth approximants of the new CF no longer represent exactly the nth sum of a finite PS. It is possible, however, to express the nth approximants as the nth partial sum of a particular power series (the "connecting" series) plus a convenient error term:

$$\frac{1}{1} - \frac{a_1\zeta}{1+b_1\zeta} - \frac{a_2\zeta}{1+b_2\zeta} - \cdots - \frac{a_n\zeta}{1+b_n\zeta} = 1 + e_1\zeta + e_2\zeta^2 + \cdots + e_n\zeta^n + E_n(\zeta). \tag{1b}$$

The formula derived in this paper generates the relationship in (1b) in such a manner that, when $|a_n - b_n|$ is small and a simple convergence criterion is met by the CF the expression $E = \lim_{n\to\infty} E_n$ is efficiently bounded: $|E(\zeta)| \leqslant C\varepsilon|\zeta|$ for values of the variable ζ in a disk about the origin. Surprisingly, the formula derives from consideration of the *complex dynamic behavior* of linear fractional transformations (LFTs). As $\varepsilon \to 0$, each $|a_n - b_n| \to 0$ and (1b) tends to (1a). Thus, it is possible to actually observe the continuous coalescence of two distinct arithmetic expansions, a CF and a PS, into a single unified entity.

In the inequality above, C is an easily computed constant, and the value of ε depends upon the distribution of the set of *repelling fixed points* of the LFTs that generate the CF.

If the connecting series is integrated, one obtains the approximate integral of the CF, with an error

$$|E_{\text{int}}(\zeta)| \leqslant C\varepsilon|\zeta|^2.$$

2. Development of the theory

Our immediate goal is to express the special CF and PS in (1a) in terms of *attracting* and *repelling fixed points* of LFTs. Then the relationship will be extended in a very natural manner in this dynamical setting.

Divide both sides of (1a) by a complex number $\beta \neq 0$:

$$\frac{1}{\beta} - \frac{a_1\beta\zeta}{a_1\zeta+1} - \frac{a_2\zeta}{a_2\zeta+1} - \cdots - \frac{a_n\zeta}{a_n\zeta+1} = \frac{1}{\beta} + \frac{a_1}{\beta}\zeta + \frac{a_1a_2}{\beta}\zeta^2 + \cdots + \frac{a_1a_2\cdots a_n}{\beta}\zeta^{n-1}. \tag{2}$$

Next, set $\alpha_n := \beta a_n$ for $n = 1, 2, \ldots$, so that (2) becomes

$$\frac{1}{\beta} - \frac{\alpha_1\zeta}{\frac{\alpha_1}{\beta}\zeta+1} - \frac{\frac{\alpha_2}{\beta}\zeta}{\frac{\alpha_2}{\beta}\zeta+1} - \cdots - \frac{\frac{\alpha_n}{\beta}\zeta}{\frac{\alpha_n}{\beta}\zeta+1} = \frac{1}{\beta} + \frac{\alpha_1}{\beta^2}\zeta + \frac{\alpha_1\alpha_2}{\beta^3}\zeta^2 + \cdots + \frac{\alpha_1\alpha_2\cdots\alpha_n}{\beta^{n+1}}\zeta^n$$

which, through an equivalence transformation, may be written as

$$\frac{1}{\beta} - \frac{\alpha_1\beta\zeta}{\alpha_1\zeta+\beta} - \frac{\alpha_2\beta\zeta}{\alpha_2\zeta+\beta} - \cdots - \frac{\alpha_n\beta\zeta}{\alpha_n\zeta+\beta} = \frac{1}{\beta} + \frac{\alpha_1}{\beta^2}\zeta + \frac{\alpha_1\alpha_2}{\beta^3}\zeta^2 + \cdots + \frac{\alpha_1\alpha_2\cdots\alpha_n}{\beta^{n+1}}\zeta^n. \tag{3}$$

The CF on the left contains a special *T-fraction* [1], and $\{\alpha_n\zeta\}$ and β are fixed points of certain LFTs, as we shall see a little further along.

Next, we will generalize the CF in (3) as follows:

$$\frac{1}{\beta_1} - \frac{\alpha_1\beta_1\zeta}{\alpha_1\zeta+\beta_1} - \frac{\alpha_2\beta_2\zeta}{\alpha_2\zeta+\beta_2} - \cdots, \tag{3a}$$

which reduces to (3) when $\beta_n \equiv \beta$. The sequences $\{\alpha_n\zeta\}$ and $\{\beta_n\}$ connect the CF to the dynamic behavior of LFTs, described briefly in the following background exposition:

The *periodic* CF

$$\frac{a}{b} + \frac{a}{b} + \cdots$$

in most instances converges to the *attracting fixed point* of the generating LFT:

$$f(w) = \frac{a}{b+w}. \tag{4}$$

To see this is the case, assume there are two distinct fixed points of (4), α and β, with unequal moduli. One can write (4) as $f(w) = \alpha\beta/(\alpha + \beta - w)$, which can then be expressed in an implicit form appropriate for iteration:

$$\frac{f(w)-\alpha}{f(w)-\beta} = K\frac{w-\alpha}{w-\beta}, \quad K = \frac{\alpha}{\beta} \quad \text{with } |\alpha| < |\beta|. \tag{5}$$

Here α is the *attracting fixed point* of f, and β is its *repelling fixed point.* Upon iteration, (5) becomes

$$\frac{f^n(w)-\alpha}{f^n(w)-\beta} = K^n\frac{w-\alpha}{w-\beta}.$$

Clearly $f^n(w) \to \alpha$ as $n \to \infty$ if $w \neq \beta$. Therefore,

$$\frac{a}{b} + \frac{a}{b} + \frac{a}{b} + \cdots = \frac{\alpha\beta}{\alpha+\beta} - \frac{\alpha\beta}{\alpha+\beta} - \frac{\alpha\beta}{\alpha+\beta} - \cdots = \alpha.$$

Returning now to the more general setting, we first isolate the regular CF structure imbedded in (3a):

$$\frac{\alpha_1\zeta\beta_1}{\alpha_1\zeta+\beta_1} - \frac{\alpha_2\zeta\beta_2}{\alpha_2\zeta+\beta_2} - \cdots \tag{6}$$

The nth approximant of (6) may be expressed as $F_n(0) = f_1 \circ f_2 \circ \cdots \circ f_n(0)$, where $f_n(w) = \alpha_n\beta_n\zeta/(\alpha_n\zeta + \beta_n - w)$. $\alpha_n\zeta$ and β_n are the fixed points of f_n. Thus, we may write

$$\frac{1}{\beta_1 - F_n(w)} = \frac{1}{\beta_1} - \frac{\alpha_1\beta_1\zeta}{\alpha_1\zeta+\beta_1} - \frac{\alpha_2\beta_2\zeta}{\alpha_2\zeta+\beta_2} - \cdots - \frac{\alpha_n\beta_n\zeta}{\alpha_n\zeta+\beta_n-w}. \tag{7}$$

Although (7) with $w=0$, as a generalization of the CF in (3), no longer equals the nth partial sum of a PS, we shall now expand it in a quasi-series form.

Theorem 1. *If* $F_n(0) = f_1 \circ f_2 \circ \cdots \circ f_n(0)$, *with* $f_n(w) = \alpha_n\zeta\beta_n/(\alpha_n\zeta + \beta_n - w)$,

$$\frac{1}{\beta_1 - F_n(w)} = \sum_{m=1}^{n}\left[\frac{1}{\beta_m}\prod_{j=1}^{m-1}\left(\frac{\alpha_j}{\beta_j}\right)\zeta^{m-1}\right] + \frac{1}{\beta_{n+1} - w}\prod_{j=1}^{n}\left(\frac{\alpha_j}{\beta_j}\right)\zeta^n + E_n(\zeta), \tag{8}$$

where

$$E_n = -\sum_{m=1}^{n}\left[\prod_{j=1}^{m}\left(\frac{\alpha_j}{\beta_j}\right)\zeta^m\delta_m(F_{m+1}^n(w))\right], \quad \delta_m(t) = \frac{\beta_m-\beta_{m+1}}{(\beta_m-t)(\beta_{m+1}-t)},$$

and

$$F_{m+1}^n(w) = f_{m+1}\circ f_{m+2}\circ\cdots\circ f_n(w), \quad F_{n+1}^n(w)=w, \quad \prod_{j=1}^{0}\left(\frac{\alpha_j}{\beta_j}\right) := 1.$$

Proof.
From (5), with $K_n := \alpha_n\zeta/\beta_n$,

$$\frac{f_n(w)-\alpha_n\zeta}{f_n(w)-\beta_n} = K_n\frac{w-\alpha_n\zeta}{w-\beta_n}$$

$$\Rightarrow \frac{f_n(w)-\beta_n+\beta_n-\alpha_n\zeta}{f_n(w)-\beta_n} = K_n\frac{w-\beta_n+\beta_n-\alpha_n\zeta}{w-\beta_n}$$

$$\Rightarrow\cdots\Rightarrow \frac{1}{f_n(w)-\beta_n} = -\frac{1}{\beta_n}+\frac{K_n}{w-\beta_n} = -\frac{1}{\beta_n}+K_n\left(\frac{1}{w-\beta_{n+1}}+\delta_n(w)\right)$$

$$\Rightarrow \frac{1}{f_n(w)-\beta_n} = -\frac{1}{\beta_n}+\frac{K_n}{w-\beta_{n+1}}+K_n\delta_n(w). \tag{9}$$

Repeated application of (9) leads by induction to (8). □

Corollary 1.1. *If $\beta_n\equiv\beta$, then*

$$\frac{1}{\beta}\,\frac{\alpha_1\beta\zeta}{-\alpha_1\zeta+\beta-}\cdots\frac{\alpha_n\beta\zeta}{-\alpha_n\zeta+\beta} = \frac{1}{\beta-F_n(0)} = \sum_{m=1}^{n+1}\left[\frac{1}{\beta^m}\left(\prod_{j=1}^{m-1}\alpha_j\right)\zeta^{m-1}\right],$$

which is consistent with Euler's connection as written in (3).

We must establish conditions on $\{\alpha_n\}$ and $\{\beta_n\}$ that imply convergence of the CF in (8) (with $w=0$) and produce a small and easily estimated E_n. If no such conditions are stipulated the CF and its connecting PS may go their own separate ways.

Example 1. Let $\alpha_n\equiv 1$ and $\beta_n=n$, then the CF in (8) is

$$\frac{1}{1-}\,\frac{\zeta}{\zeta+1-}\,\frac{2\zeta}{\zeta+2-}\cdots = \frac{1}{1-\zeta}$$

if $|\zeta|\leqslant 1$, and the PS is

$$1+\frac{\zeta}{2!}+\frac{\zeta^2}{3!}+\cdots = \frac{e^{\zeta}-1}{\zeta}.$$

Theorem 2. *Suppose there exist positive constants A and B such that $|\alpha_n| \leqslant A$ and $|\beta_n| \geqslant B > 1$ for $n \geqslant 1$. Set $M := \frac{1}{A}\frac{B-1}{B+1}$, then*

$$\frac{1}{\beta_1} - \frac{\alpha_1\beta_1\zeta}{\alpha_1\zeta+\beta_1} - \frac{\alpha_2\beta_2\zeta}{\alpha_2\zeta+\beta_2} - \cdots = \lambda(\zeta),$$

a function analytic in $(|\zeta| \leqslant M)$ with $|\lambda(\zeta)| < 1/(B-1)$. In addition,

$$\frac{1}{\beta_1} - \frac{\alpha_1\beta_1\zeta}{\alpha_1\zeta+\beta_1} - \frac{\alpha_2\beta_2\zeta}{\alpha_2\zeta+\beta_2} - \cdots = \left[\frac{1}{\beta_1} + \frac{1}{\beta_2}\left(\frac{\alpha_1}{\beta_1}\right)\zeta + \frac{1}{\beta_3}\left(\frac{\alpha_1\alpha_2}{\beta_1\beta_2}\right)\zeta^2 + \cdots\right] + E(\zeta)$$

and

$$\int_0^z \left(\frac{1}{\beta_1} - \frac{\alpha_1\beta_1\zeta}{\alpha_1\zeta+\beta_1} - \cdots\right) \mathrm{d}\zeta = \left[\frac{1}{\beta_1}z + \frac{1}{2\beta_2}\left(\frac{\alpha_1}{\beta_1}\right)z^2 + \frac{1}{3\beta_3}\left(\frac{\alpha_1\alpha_2}{\beta_1\beta_2}\right)z^3 + \cdots\right] + E_{\mathrm{int}}(z)$$

where both series converge in $(|z| \leqslant M)$ and

$$|E(\zeta)| \leqslant C\varepsilon|\zeta| \quad \text{and} \quad |E_{\mathrm{int}}(z)| \leqslant C\varepsilon|z|^2,$$

with

$$C := \frac{A}{(B-1)^2(B-AM)} \quad \text{and} \quad |\beta_n - \beta_{n-1}| \leqslant \varepsilon \quad \text{for all } n.$$

Proof.

The classical Pringsheim Convergence Criterion [1] will apply if we can show that $|\alpha_n\zeta + \beta_n| \geqslant |\alpha_n\beta_n\zeta| + 1$. Write

$$|\alpha_n\zeta + \beta_n| \geqslant |\beta_n| - |\alpha_n||\zeta| \geqslant |\beta_n| - AM, \quad \text{and} \quad AM|\beta_n| + 1 \geqslant |\alpha_n\beta_n\zeta| + 1.$$

Now $|\beta_n| - AM \geqslant AM|\beta_n| + 1$ if $|\beta_n| \geqslant B = (1+AM)/(1-AM)$, which is equivalent to $M = (1/A)(B-1)/(B+1)$. Consequently the CF

$$\frac{\alpha_1\beta_1\zeta}{\alpha_1\zeta+\beta_1} - \frac{\alpha_2\beta_2\zeta}{\alpha_2\zeta+\beta_2} - \cdots$$

converges to an analytic function $F(\zeta)$ where $|F(\zeta)| < 1$. Since $|\beta_1| \geqslant B > 1, |\lambda(\zeta)| < 1/(B-1)$.

From (8) of Theorem 1, we have

$$\frac{1}{\beta_1} - \frac{\alpha_1\beta_1\zeta}{\alpha_1\zeta+\beta_1} - \cdots - \frac{\alpha_n\beta_n\zeta}{\alpha_n\zeta+\beta_n} = \sum_{m=1}^{n+1} \frac{1}{\beta_m}\left(\prod_{j=1}^{m-1}\frac{\alpha_j}{\beta_j}\right)\zeta^{m-1} + E_n(\zeta)$$

where

$$|E_n(\zeta)| \leqslant \sum_{m=1}^{n}\left(\prod_{j=1}^{m}\left|\frac{\alpha_j}{\beta_j}\right| |\zeta|^m |\delta_m(F_{m+1}^n(w))|\right)$$

and

$$|\delta_m(F^n_{m+1}(0))| \leqslant \frac{\varepsilon}{|\beta_m - F^n_{m+1}(0)||\beta_{m+1} - F^n_{m+1}(0)|} \leqslant \frac{\varepsilon}{(B-1)^2},$$

since the Pringsheim Criterion applies to each $F^n_{m+1}(0)$, giving $|F^n_{m+1}(0)| < 1$. Thus

$$|E_n(\zeta)| \leqslant \frac{\varepsilon}{(B-1)^2}\sum_{m=1}^{n}\left(\prod_{j=1}^{m}\left|\frac{\alpha_j}{\beta_j}\right||\zeta|^m\right) \leqslant \frac{\varepsilon}{(B-1)^2}\sum_{m=1}^{n}\left(\frac{A}{B}\right)^m|\zeta|^m \leqslant \frac{\varepsilon}{(B-1)^2}\frac{\frac{A}{B}|\zeta|}{1-\frac{A}{B}M}$$

$$= \frac{A}{(B-1)^2(B-AM)}\varepsilon|\zeta| = C\varepsilon|\zeta|.$$

A similar argument verifies the inequality $|E_{\text{int}}(z)| \leqslant C\varepsilon|z|^2$. □

Corollary 2.1. *If $\beta_n \equiv \beta$ and the hypotheses of Theorem 2 are satisfied, then*

$$\int_0^z \left(\frac{1}{\beta} - \frac{\alpha_1\zeta\beta}{\alpha_1\zeta+\beta} - \frac{\alpha_2\zeta\beta}{\alpha_2\zeta+\beta} - \cdots\right)\mathrm{d}\zeta = \sum_{m=1}^{\infty}\left[\frac{1}{m}\prod_{j=1}^{m-1}(\alpha_j)\left(\frac{z}{\beta}\right)^m\right].$$

If $\alpha_n \equiv \alpha$ also,

$$\int_0^z \left(\frac{1}{\beta} - \frac{\alpha\zeta\beta}{\alpha\zeta+\beta} - \frac{\alpha\zeta\beta}{\alpha\zeta+\beta} - \cdots\right)\mathrm{d}\zeta = \int_0^z \frac{1}{\beta-\alpha\zeta}\mathrm{d}\zeta = -\frac{1}{\alpha}\mathrm{Ln}\left(1-\frac{\alpha z}{\beta}\right).$$

Example 2. Set $p_n := 1 + 10^{-k-n}$ for some $k \geqslant 1$. Then set $\alpha_n := 1 - 1/(n+1)$ and $\beta_n := 2p_n$. Here $A = 1$ and $B = 2$, so that $M = \frac{1}{3}$, $C = 0.6$, and $\varepsilon = 1.8 \times 10^{-k-1}$:

$$\frac{1}{2p_1} - \frac{\frac{1}{2}2p_1\zeta}{\frac{1}{2}\zeta+2p_1} - \frac{\frac{2}{3}2p_2\zeta}{\frac{2}{3}\zeta+2p_2} - \cdots - \frac{\frac{n}{n+1}2p_n\zeta}{\frac{n}{n+1}\zeta+2p_n} - \cdots$$

$$= \frac{1}{2p_1} + \frac{2}{2^2p_1p_2}\zeta + \frac{3}{2^3p_1p_2p_3}\zeta^2 + \cdots + E(\zeta)$$

and

$$\int_0^z \left(\frac{1}{2p_1} - \frac{\frac{1}{2}2p_1\zeta}{\frac{1}{2}\zeta+2p_1} - \frac{\frac{2}{3}2p_2\zeta}{\frac{2}{3}\zeta+2p_2} - \cdots - \frac{\frac{n}{n+1}2p_n\zeta}{\frac{n}{n+1}\zeta+2p_n} - \cdots\right)\mathrm{d}z,$$

$$= \frac{1}{2p_1}z + \frac{1}{2^2p_1p_2}z^2 + \frac{1}{2^3p_1p_2p_3}z^3 + \cdots + E_{\text{int}}(\zeta),$$

where $|E(\zeta)| \leqslant (1.08 \times 10^{-k-1})|\zeta|$ and $|E_{\text{int}}(z)| \leqslant (1.08 \times 10^{-k-1})|z|^2$ for $|\zeta|, |z| \leqslant \frac{1}{3}$. For instance, if $|\zeta|, |z| = \frac{1}{3}$ and $k = 3$, the predicted errors are approximately 0.007% the value of the series and

0.002% the value of the integrated series. Both series have an actual radius of convergence $R = 2$, and the CF may converge as a meromorphic function well beyond the limiting value of M.

Reference

[1] W. Jones, W. Thron, Continued Fractions: Analytic Theory & Applications, Addison-Wesley, Reading, MA, 1980.