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FINITE CARDINALS IN GENERAL TOPOI

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Finite cardinals in topoi are introduced without assuming the existence of a natural number object. It is proved that this notion is equivalent to the conjunction of K-finiteness and simple finiteness. In this way, simple finiteness is viewed as a notion which corrects some defects of K-finiteness to obtain cardinal finiteness.

Introduction

In topos theory, the notion of finite object has been approached in different ways. In the presence of a natural numbers object ('NNO'), finite cardinals as investigated by Bénabou [2] is the most adequate. In that case, the full subcategory of finite cardinals is a subtopos satisfying the axiom of choice (every epi may be split). Without natural numbers object, finite objects in the sense of Kuratowski ('K-finite') may be investigated as has been done by Kock et al. [5]. But the full subcategory of K-finite objects is not in general a topos, unless the base topos be boolean. The aim of this paper is to define a notion of finite cardinal which coincides with Bénabou's notion in the presence of a NNO but which in a general topos generates a subtopos satisfying the axiom of choice. This result is obtained via the introduction of 'simple finiteness', a notion which corrects some defeats of K-finiteness.

Our starting point is to be found in the analysis of Brook [3]. In [3], an object is finite if it admits an order with minimal choice, whose opposite order is also an order with minimal choice. Brook shows that in a boolean topos with NNO, finite cardinals are finite objects in his sense. Unfortunately, the boolean character of the topos is not only a sufficient condition, but a necessary one; as an unpleasant consequence, in non-boolean topoi, there are finite cardinals which are not finite objects. We studied in [7] the well-ordering theorem and showed that the notion of order with minimal choice is ill-adapted to the non-boolean case. We substitute it by the concept of simple well-ordering and define an object to be simply finite if it admits a simple well-ordering whose opposite is also a simple well-ordering. We prove the properties obtained by Brook plus the stability by finite sums. This is exactly what is required to prove that the objects which are K-finite and simply finite (these will be our finite cardinals) have the desired properties.

1. Simply finite objects

Let X be an object of the topos $\mathscr{E}$, together with a relation $R \rightarrowtail X \times X$. Recall that R is *simply inductive* if the following holds:

$$\vdash \forall a\,[\forall x\,(\forall x'\,(R(x', x) \Rightarrow x' \in a) \Rightarrow x \in a) \Rightarrow \forall x\,(x \in a)]\,.$$

We define a *simple well-ordering* of X to be a linear ordering of X satisfying trichotomy (i.e. $\vdash x < y \vee x = y \vee y < x$) and such that the strict ordering associated to it is simply inductive. A *double simple well-ordering* of X is a simple well-ordering of X whose opposite is also a simple well-ordering. Remark that trichotomy in the above can be replaced by the condition that the order is discrete (i.e. $\vdash x \leq y \Rightarrow x = y \vee x < y$).

Definition 1.1. An object X of a topos $\mathscr{E}$ is *simply finite* if it admits a double simple well-ordering.

Remark that trichotomy in the notion of simple well-ordering implies that simply finite objects are decidable.

We will denote by $\mathscr{I}_<(x, a)$ the formula $\forall x'\ (x' < x \Rightarrow x' \in a) \Rightarrow x \in a$ and by $\mathscr{I}_<(a)$ the formula $\forall x\ \mathscr{I}_<(x, a)$.

Proposition 1.2. *Every subobject of a simply finite object is simply finite.*

Proof. Let X be a simply finite object and $m : X' \rightarrowtail X$ a subobject of X. Clearly $m(x') \leq m(y')$ defines a linear ordering of X' satisfying trichotomy. Denote by $\theta(y, a')$ the formula $\forall z'\ (m(z') = y \Rightarrow z' \in a')$. It is easy to show that

$$\vdash \mathscr{I}_<(a') \Rightarrow \mathscr{I}_<(x', a')$$

and

$$\vdash \forall y\ (y < m(x') \Rightarrow \theta(y, a')) \Rightarrow \forall y'\ (m(y') < m(x') \Rightarrow y' \in a')\,,$$

from which

$$\vdash \mathscr{I}_<(a') \Rightarrow \forall x'\,[\forall y\,(y < m(x') \Rightarrow \theta(y, a')) \Rightarrow x' \in a']$$

follows. On the other hand,

$$\vdash \forall x'\,[\forall y\,(y < m(x) \rightarrow \theta(y, a')) \rightarrow x' \in a'] \rightarrow \mathscr{I}_<(\{y \mid \theta(y, a')\})$$

and since $<$ is simply inductive,

$$\vdash \mathscr{I}_<(a') \Rightarrow \forall y\ (\theta(y, a'))\,.$$

Using the trivial

$$\vdash m(z') = m(z'),$$

it is easy to conclude that the strict ordering induced on X' is simply inductive and this proves the thesis. $\square$

Let $F: \mathscr{E} \to \mathscr{E}'$ be a logical functor between topoi. With every well-formed expression (term or formula) T of $\mathscr{L}(\mathscr{E})$, we associate a well-formed expression $F(T)$ of $\mathscr{L}(\mathscr{E}')$ by the following inductive definition:
- if x is a variable of type X, then Fx is a variable of type FX;
- if $T = f(t)$, then $F(T) = F(f)(F(t))$;
- if $T = t \in_X a$, then $F(T) = F(t) \in_{FX} F(a)$;
- if $T = R(t)$, then $F(T) = F(R)[F(t)]$;
- if $T = \psi \circ \theta$, then $F(T) = F(\psi) \circ F(\theta)$ (with $\circ \in \{\wedge, \vee, \Rightarrow\}$);
- if $T = \rceil \psi$, then $F(T) = \rceil F(\psi)$;
- if $T = \delta x \psi$, then $F(T) = \delta Fx F(\psi)$ (with $\delta \in \{\forall, \exists\}$).

Clearly if φ is an $\mathscr{E}$-valid formula, then $F(\varphi)$ is $\mathscr{E}'$-valid. Also, if X is simply finite in $\mathscr{E}$, then FX is simply finite in $\mathscr{E}'$.

Theorem 1.3. *Let $\mathscr{E}$ be a topos and $p: X \to I$ an object in $\mathscr{E}/I$. If I is a simply finite object in $\mathscr{E}$, then X is simply finite in $\mathscr{E}$ iff $p: X \to I$ is simply finite in $\mathscr{E}/I$.*

Proof. The condition is necessary, for, if X is simply finite, $I^*(X)$ is simply finite (since I^* is logical) and since p is a subobject of $I^*(X)$, p is simply finite by Proposition 1.2. The condition is also sufficient. To prove this, let I be simply well-ordered by $\leq$ in $\mathscr{E}$ and p simply well-ordered by $\leq_p$ in $\mathscr{E}/I$. The relation defined by

$$x \leq_p x' \vee p(x) < p(x')$$

is a linear discrete ordering. The associated strict ordering is given by

$$x <_p x' \vee p(x) < p(x').$$

But clearly

$$\vdash \mathscr{I}_<(a) \Rightarrow [\forall i' \, (i' < i \Rightarrow \forall x \, (p(x) = i' \Rightarrow x \in a)) \Rightarrow \forall x \, (p(x) = i \Rightarrow \mathscr{I}_{<_p}(x, a))].$$

Since $<_p$ is simply inductive,

$$\vdash \mathscr{I}_<(a) \Rightarrow \forall i \, \mathscr{I}_<(i, \{j \mid \forall x \, (p(x) = j \Rightarrow x \in a)\}).$$

From this, since the ordering on I is simply inductive

$$\mathscr{I}_<(a) \Rightarrow \forall i \, \forall x \, (p(x) = i \Rightarrow x \in a).$$

Using the trivial

$$\vdash \forall x\, (p(x) = p(x))$$

it is easy to conclude that < is simply inductive on X. This proves the sufficiency of the condition. □

Stability properties of simple finiteness are summarized in the following theorem:

Theorem 1.4. *If $\mathscr{E}$ is a topos, then the full sub-category $\mathscr{E}_{\mathrm{sf}}$ of simply finite objects is finitely complete, contains the initial object and is stable under finite internal and external sums.*

Proof. $\mathscr{E}_{\mathrm{sf}}$ is finitely complete by Theorem 1.3 and Proposition 1.2. Stability under finite internal sums is given by the 'only if' part of Theorem 1.3. The initial object 0 is simply finite as a sub-object of the simply finite object 1. It remains to prove the stability under finite external sums. This follows from the observation that if X and Y are simply well-ordered, then

$$\begin{aligned} &\exists x\, \exists x'\, (x \le x' \wedge i_X(x) = \nu \wedge i_X(x') = \nu') \\ &\quad \vee \exists x\, \exists y\, (i_X(x) = \nu \wedge i_Y(y) = \nu') \\ &\quad \vee \exists y\, \exists y'\, (y \le y' \wedge i_Y(y) = \nu \wedge i_Y(y') = \nu') \end{aligned}$$

defines a simple well-ordering on the sum $X \xrightarrow{i_X} X + Y \xleftarrow{i_Y} Y$. □

In contrast with Theorem 1.4, it should be noted that $\mathscr{E}_{\mathrm{sf}}$ is not in general finitely cocomplete. Here is an example of two parallel morphisms between simply-finite objects whose coequaliser is not simply finite.

In Sierpinski's topos, an object is simply finite iff it is an injection between finite sets. Consider in that topos the two parallel morphisms (f, f') and (g, g') defined by

$$\begin{array}{ccc} \{b\} & \overset{f'}{\underset{g'}{\rightrightarrows}} & \{0,1\} \\ \big\downarrow & & \big\| \\ \{a,b\} & \overset{f}{\underset{g}{\rightrightarrows}} & \{0,1\} \end{array}$$

$$f'(b) = g'(b) = 1 \qquad f(a) = 0, \qquad g(a) = g(b) = f(1) = 1\,.$$

The coequaliser of this pair is clearly not an injection.

Let us also mention that for the finite objects studied by Brook [3], stability under finite external sum implies the boolean character of the topos.

2. Natural well-ordering and finite cardinals.

We showed in [8] that the canonical ordering on N, the NNO of a topos $\mathscr{E}$ is always a simple well-ordering. Let $1 \xrightarrow{n} N$ be a global section of N. We denote by $[\![n]\!]$ the subobject of N which is the extension of the formula $x \leq n$. Subobjects $[\![n]\!]$ of N inherit properties of the canonical simple well-ordering of N which are not true in general of a simply well-ordered object with global support: such as the existence of a smallest element.

Definition 2.1. (1) An ordering of X satisfies the *existence property of a partial successor morphism* (EPSM) if:

$$\vdash \forall x\, (\exists y\, (x < y) \Rightarrow \exists y\, \forall t\, (x < t \Leftrightarrow y \leq t))\,.$$

(2) A *natural well-ordering* of X is a simple well-ordering of X having a smallest element and satisfying EPSM. A double natural well-ordering of X is a natural well-ordering of X whose opposite is also a natural well-ordering.

(3) An object X is *naturally finite* iff X has a double natural well ordering.

Let X be a natural finite object. We denote by x_0 its smallest element, by x_M its greatest element, by s_X its endomorphism defined by

$$\vdash (x = x_M \Rightarrow s_X(x) = x) \wedge (x \neq x_M \Rightarrow x_M \Rightarrow \forall t\, (x < t \Leftrightarrow s_X(x) \leq t))$$

and by p_X its endomorphism defined by

$$\vdash (x = x_0 \Rightarrow p_X(x) = x) \wedge (x \neq x_0 \rightarrow \forall t\, (t < x \Leftrightarrow t \leq p_X(x)))\,.$$

The canonical well-ordering of N is clearly a natural well-ordering and its opposite satisfies EPSM. It is also clear that for every $n : 1 \rightarrow N$, $[\![n]\!]$ is naturally finite (for the restriction of the order of N).

Let $f : X \rightarrow X$ and $x : 1 \rightarrow X$. We will say that (X, x, f) satisfies Peano's fifth axiom if the only subobject $i : X' \rightarrowtail X$ such that there exists $x' : 1 \rightarrow X'$ and $f' : X' \rightarrow X'$ with $ix' = x$ and $if' = fi$ is the greatest subobject.

Proposition 2.2. *For every double natural well-ordering on X, (X, x_0, s_X) and (X, x_M, p_X) satisfy Peano's fifth axiom.*

Proof. Let $i : X' \rightarrowtail X$, $x_0' : 1 \rightarrow X'$, $s_X' : X' \rightarrow X'$ be such that $ix_0' = 0$ and $is_X' = s_X i$. Consider the adjoint $\ulcorner X' \urcorner : 1 \rightarrow PX$ of the characteristic function of X'. From the definitions of x_0, s_X, p_X and the discreteness of the ordering, it is easy to derive

$$\vdash (\forall x\, \forall x'\, (x' < x \Rightarrow x' \in \ulcorner X' \urcorner) \Rightarrow x \in \ulcorner X' \urcorner)\,,$$

from which, using the simply inductive character of $<$, one shows that (X, x_0, s_X)

satisfies Peano's fifth axiom. That (X, x_M, p_X) satisfies Peano's fifth axiom is proved similarly. □

Let X be simply well-ordered. We call *support* of this simple well-ordering, the subobject X_p of X which is the extension of the formula $\exists y\,(x < y)$.

Definition 2.3. A finite cardinal in a topos $\mathscr{E}$ is an object X of $\mathscr{E}$ which is the support of a double natural well-ordering.

It is easy to see that if X is a finite cardinal, then $X + 1$ is naturally finite.

Theorem 2.4. *If $\mathscr{E}$ has a NNO N, then Y is a finite cardinal iff Y is isomorphic to $[n]$ for some $n : 1 \to N$.*

Proof. The sufficient condition is clear. To show necessity, let Y be a finite cardinal, i.e. the support of a double natural well-ordering on, say, X. Consider the relation R defined by

$$u(n) = x \wedge \forall n'\,(n' < n \Leftrightarrow u(n') < x)$$

where $u : N \to X$ is such that $u0 = x_0$ and $s_X u = us$. The relation R is clearly functional. By Peano's fifth axiom applied to (X, x_0, s_X), R is also everywhere defined. Hence, there exists a morphism $v : X \to N$ such that

$$\vdash v(x) = n \Leftrightarrow u(n) = x \wedge \forall n'\,(n' < n \Leftrightarrow u(n') < x)\,.$$

v is a section of u; hence u is epic. For x_M, the following holds:

$$\vdash \forall n'(n' < n \Leftrightarrow u(n') < x_M)\,.$$

From all this, it follows that $Y \cong [v(x_M)]$. □

Theorem 2.4 shows in particular that finite cardinals are simply finite. It should be noted that the converse is not true in general: consider e.g. Sierpinski's topos. However in the boolean case simply finite = finite cardinal:

Proposition 2.5. *If $\mathscr{E}$ is a boolean topos, then X is simply finite iff X is a finite cardinal.*

Proof. Let $\mathscr{E}$ be boolean. Simple well-ordering and order with minimal choice coincide in $\mathscr{E}$ (see [8]). Hence every simply well-ordered object X satisfies EPSM. Moreover if X has a global support, then X has a smallest element. Hence every simply well-ordered X with global support is naturally well-ordered. □

Proposition 2.5. generalizes a similar proposition established by Brook for boolean topoi with NNO.

3. K-finiteness, simple finiteness and finite cardinals

Recall that an object X of a topos $\mathscr{E}$ is said to be *K-finite* if for every filtered poset P of $\mathscr{E}$,

$$\vdash \forall \alpha \, \exists p \, \forall x \, (\alpha(x) \leq p) \quad (\alpha \text{ variable of type } P^X) .$$

K-finiteness has been introduced by Kock et al. in [5]. We use its properties without proof. See also [1, 4, 6].

Let P be a poset. We abbreviate by Fil(a) the following formula:

$$\exists p \, (p \in a) \wedge \forall p \, \forall p' \, (p \in a \wedge \ p' \in a$$
$$\Rightarrow \exists z \, (z \in a \ \wedge \ p \leq z \ \wedge \ p' \leq z))$$

which defines the object of filtered subobjects of P. Using this notation, K-finiteness may be expressed by: X is K-finite iff for every poset P

$$\vdash \forall a \, [\mathrm{Fil}(a) \Rightarrow \forall \alpha \, (\forall x \, (\alpha(x) \in a)$$
$$\Rightarrow \exists p \, (p \in a \ \wedge \forall x \, (\alpha(x) \leq p)))] .$$

In general, neither does simple finiteness nor K-finite imply the other. To see this, again consider Sierpinski's topos: a simply finite object is an injection between finite sets; a K-finite object is a surjection between finite sets.

We proceed to show now that the conjunction of K-finiteness and simple finiteness amounts to being a finite cardinal.

Lemma 3.1. *Every simple well-ordering on a K-finite object X with global support is a double natural well-ordering.*

Proof. The opposite order being linear, the existence of the smallest element follows from antisymmetry and K-finiteness applied to $1_X : 1 \to X^X$. The existence of the greatest element follows from the fact that the opposite ordering is linear. The EPSM is proved as follows: since the ordering is linear and X is decidable, there exists $\alpha_{\_} : X \to X^X$ such that

$$\vdash \forall x \, [\forall t \, (t \leq x \Rightarrow \alpha_x(t) = x_M) \wedge (x < t \Rightarrow \alpha_x(t) = t)]$$

but

$$\vdash \forall x \, (\exists y \, (x < y) \Rightarrow \mathrm{Fil}(\{ y \mid x < y \}));$$

hence, by the definition of α_-,

$$\vdash \forall x\, (\exists y\, (x < y) \Rightarrow \forall t\, (x < \alpha_x(t)));$$

by the K-finiteness of X,

$$\vdash \forall x\, \exists y\, (x < y) \Rightarrow \exists y\, (x < y \wedge \forall t\, (y \leq \alpha_x(t)));$$

but

$$\vdash \forall x\, (x < y \;\wedge\; \forall t\, (y \leq \alpha_x(t)) \Rightarrow \forall t\, (x < t \Leftrightarrow y \leq t))\,,$$

from which EPSM follows. EPSM for the opposite ordering is now immediate. □

Remark that the existence of a smallest element may be obtained without the discreteness hypothesis. Hence, every linear ordering on a K-finite object with global support always has a smallest and a greatest element.

Lemma 3.2. *In a topos $\mathscr{E}$, X is naturally finite iff X has global support and is both K-finite and simply finite.*

Proof. If X is naturally finite, then by definition it is simply finite and has global support. It is also K-finite; to see this, let P be a filtered poset in $\mathscr{E}$; if x_0 is the smallest element of X, then

$$\vdash \forall \alpha\, \exists p\, \forall x'\, (x' \leq x_0 \Rightarrow \alpha(x') \leq p);$$

P being filtered,

$$\vdash \forall x\, (\exists p\, \forall x'\, (x' \leq x \Rightarrow \alpha(x') \leq p) \Rightarrow \exists p\, \forall x'\, (x' \leq s_X(x) \Rightarrow \alpha(x') \leq p));$$

by Peano's fifth axiom,

$$\vdash \forall \alpha\, \forall x\, \exists p\, \forall x'\, (x' \leq x \Rightarrow \alpha(x') \leq p);$$

finally, using the existence of the greatest element,

$$\vdash \forall \alpha\, \exists p\, \forall x\, (\alpha(x) \leq p)\,.$$

Conversely, let X have global support, be K-finite and simply finite. Then X is naturally finite by Lemma 3.1. □

Recalling that $X + Y$ is K-finite iff X and Y are K-finite, we obtain

Theorem 3.3. *In a topos $\mathscr{E}$, X is a finite cardinal iff X is K-finite and simply finite.*

Proof. Let X be a finite cardinal. It is simply finite by Proposition 1.2. Recalling that $X+1$ is naturally finite, $X+1$ is K-finite by Lemma 3.2. Hence X is K-finite. Conversely, let X be K-finite; then $X+1$ is K-finite with global support. If moreover X has a double simple well-ordering, then by Lemma 3.2. the double simple ordering of $X+1$ defined as in the proof of Theorem 1.4, is a double natural well-ordering with support X. □

We now proceed to extend a well-known theorem concerning finite cardinals in a topos with NNO.

Let $\mathscr{E}_{\mathrm{dKf}}$ be the full subcategory of decidable K-finite objects of $\mathscr{E}$. In [1] it is proved that $\mathscr{E}_{\mathrm{dKf}}$ is always a (boolean) topos satisfying the axiom of implicit choice (every epi in $\mathscr{E}_{\mathrm{dKf}}$ is locally split); $1+1$ is the classifying object; the inclusion $\mathscr{E}_{\mathrm{dKf}} \hookrightarrow \mathscr{E}$ preserves exponentiation, finite limits and finite sums and is logical iff $\mathscr{E}$ is boolean. In [3] it is proved that the full subcategory of finite objects (in the sense of Brook) forms a boolean topos, satisfying the axiom of choice if the topos is boolean; the inclusion in $\mathscr{E}$ is logical.

Let $\mathscr{E}_{\mathrm{fc}}$ be the full subcategory of finite cardinals of $\mathscr{E}$. By Theorem 3.3, $\mathscr{E}_{\mathrm{fc}}$ is a full subcategory of the boolean topos $\mathscr{E}_{\mathrm{dKf}}$. By [8, Theorem 1], $\mathscr{E}_{\mathrm{fc}}$ coincides with the full subcategory of finite objects (in the sense of Brook) of the boolean topos $\mathscr{E}_{\mathrm{dKf}}$.

We have thus proved

Theorem 3.4. *The subcategory $\mathscr{E}_{\mathrm{cf}}$ of finite cardinals of a topos $\mathscr{E}$ is a boolean topos satisfying AC. The inclusion $\mathscr{E}_{\mathrm{cf}} \hookrightarrow \mathscr{E}$ is logical iff $\mathscr{E}$ is boolean.* □

We finally remark that all properties of finite cardinals in topoi with NNO carry over so smoothly to general topoi. E.g., if $\mathscr{E}$ has NNO, then K-finite objects of $\mathscr{E}$ are locally quotients of finite cardinals. The following example, suggested to us by F.E.J. Linton, shows that this property is not true anymore of general topoi: let S_{f} be the topos of finite sets and consider the category N with its usual order; clearly $S_{\mathrm{f}}^{N^{\mathrm{op}}}$ is a topos $\mathscr{E}$; define F by

$$F(n) = \{0, 1, \ldots, n\},$$

$$F(n+1) \twoheadrightarrow F(n) \text{ maps} \begin{cases} k \le n & \text{on } k, \\ n+1 & \text{on } n. \end{cases}$$

F is a K-finite object of $\mathscr{E}$ but is not locally a quotient of a finite cardinal.

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References

[1] O. Acuña-Ortega and F.E.J. Linton, Finiteness and decidability I, Lecture Notes in Mathematics 753 (Springer, Berlin, 1979) 80–100.
[2] J. Bénabou, Problèmes dans les topos, Université Catholique de Louvain, Inst. Math. Rapport 34, 1973.
[3] T.G. Brook, Finiteness: Another aspect of topoi, Ph.D. Australian National University, 1974.
[4] P.T. Johnstone and F.E.J. Linton, Finiteness and decidability II, Math. Proc. Cambridge Philos. Soc. (1978) 207–219.
[5] A. Kock, P. Lecouturier and C.J. Mikkelsen, Some topos theoretic concepts of finiteness, Lecture Notes in Mathematics 445 (Springer, Berlin, 1975) 209–283.
[6] B. Lesaffre, Finitude dans les topos, Séminaire Bénabou, Université de Paris-Nord, 1978.
[7] M. Mawanda, Bon ordre, choix et finitude dans des topos, Thèse de doctorat, Université Catholique de Louvain, 1984.
[8] M. Mawanda, Well-ordering and choice in toposes, J. Pure Appl. Algebra 50 (1988) 171–184 (This issue).