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Richardson elements for classical Lie algebras

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Abstract

Parabolic subalgebras of semi-simple Lie algebras decompose as $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$ where $\mathfrak{m}$ is a Levi factor and $\mathfrak{n}$ the corresponding nilradical. By Richardson's theorem [R.W. Richardson, Bull. London Math. Soc. 6 (1974) 21–24], there exists an open orbit under the action of the adjoint group P on the nilradical. The elements of this dense orbits are known as Richardson elements. In this paper we describe a normal form for Richardson elements in the classical case. This generalizes a construction for $\mathfrak{gl}_N$ of Brüstle et al. [Algebr. Represent. Theory 2 (1999) 295–312] to the other classical Lie algebra and it extends the authors normal forms of Richardson elements for nice parabolic subalgebras of simple Lie algebras to arbitrary parabolic subalgebras of the classical Lie algebras [K. Baur, Represent. Theory 9 (2005) 30–45]. As applications we obtain a description of the support of Richardson elements and we recover the Bala–Carter label of the orbit of Richardson elements.

Introduction

The goal of this paper is to describe Richardson elements for parabolic subalgebras of the classical Lie algebras.

Let $\mathfrak{p}$ be a parabolic subalgebra of a semi-simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$ a Levi decomposition. By a fundamental theorem of Richardson [Ri] there always exist

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elements x in the nilradical $\mathfrak{n}$ such that $[\mathfrak{p}, x] = \mathfrak{n}$. In other words, if P is the adjoint groups of $\mathfrak{p}$, then the orbit $P \cdot x$ is dense in $\mathfrak{n}$. It is usually called the Richardson orbit. Richardson orbits have been studied for a long time and there are many open questions related to this setting. Our goal is to give explicit representatives for Richardson elements. In the case of $\mathfrak{gl}_n$ there is a beautiful way to construct Richardson elements that has been described by Brüstle et al. in [BHRR]. Furthermore, Richardson elements with support in the first graded part $\mathfrak{g}_1$ (where the grading is induced from the parabolic subalgebra) have been given for all simple Lie algebras in [Ba].

However, these constructions do not work in general for classical Lie algebras. To fill this gap, we have modified the existing approaches to obtain Richardson elements for parabolic subalgebras of the classical Lie algebras. We do this using certain simple line diagrams. They correspond to nilpotent matrices with at most one non-zero entry in each row and in each column. We show that for most parabolic subalgebras, there exists a simple line diagram that defines a Richardson element. But there are cases where this is not possible as we will see. We expect that the representatives we describe will give more insight and hopefully answer some of the open questions. One of the interesting questions in the theory of Richardson elements is the structure of the support of a Richardson element. Recall that any parabolic subalgebra $\mathfrak{p}$ induces a $\mathbb{Z}$-grading of $\mathfrak{g}$,

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i \quad \text{with } \mathfrak{p} = \bigoplus_{i \geqslant 0} \mathfrak{g}_i = \mathfrak{g}_0 \oplus \left(\bigoplus_{i > 0} \mathfrak{g}_i \right)$$

where $\mathfrak{g}_0$ is a Levi factor and $\mathfrak{n} := \bigoplus_{i>0} \mathfrak{g}_i$ the corresponding nilradical. For details, we refer to our joint work with Wallach [BW]. The support of a Richardson element $X = \sum_{\alpha \text{ root of } \mathfrak{n}} k_\alpha X_\alpha$ are the roots of the nilradical $\mathfrak{n}$ with $k_\alpha \neq 0$ (where X_α spans the root subspace $\mathfrak{g}_\alpha$). The support $\operatorname{supp}(X)$ of X lies in the subspace $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ for some $k \geqslant 1$. For the normal form of Richardson elements we can determine the minimal k_0 such that $\operatorname{supp}(X) \subset \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{k_0}$. We also recover the Bala–Carter label of the dense orbit of Richardson elements, also called the *type* of the orbit. The Bala–Carter label is used in the classification of nilpotent orbits of simple Lie algebras, given in [BC]. For a description of these labels see [CM, Chapter 8]. The type of any nilpotent orbit in a classical Lie algebra has been described by Panyushev [Pa] in terms of the partitions of the orbit.

Before we describe our results and explain the structure of this article, we need to fix some notation. If $\mathfrak{p}$ is a parabolic subalgebra of a semi-simple Lie algebra $\mathfrak{g}$ we can assume that $\mathfrak{p}$ contains a fixed Borel subalgebra. In this case we say that $\mathfrak{p}$ is standard. If $\mathfrak{m}$ is a Levi factor of $\mathfrak{p}$ we say that $\mathfrak{m}$ is standard if it contains a fixed Cartan subalgebra $\mathfrak{h}$ that is contained in the fixed Borel subalgebra.

From now on we will assume that $\mathfrak{g}$ is a classical Lie algebra, unless stated otherwise. As usual, the Cartan subalgebra consists of the diagonal matrices and the fixed Borel subalgebra is the set of upper triangular matrices. Then a standard Levi factor has the shape of a sequence of square matrices (blocks) on the diagonal and zeroes outside. In the case of $\mathfrak{so}_{2n}$, we have to be careful: we will only consider parabolic subalgebras where α_n and α_{n-1} are both roots of the Levi factor or both roots of the nilradical or α_{n-1} a root of the Levi factor and α_n a root of the nilradical. In other words the case α_n a root of the Levi factor and α_{n-1} a root of the nilradical will be identified with this last case since the two parabolic subalgebras are isomorphic. So our standard $\mathfrak{p}$ or $\mathfrak{m}$ are uniquely defined by the sequence $d := \underline{d} = (d_1, \ldots, d_r)$ of the sizes of these blocks (and by specifying the type of the Lie algebra).

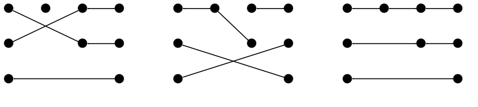
We start by defining line diagrams for dimension vectors in Section 1. It will turn out that each horizontal line diagram corresponds uniquely to an element of the nilradical of the parabolic subalgebra of $\mathfrak{sl}_n$ of the given dimension vector. In Section 2 we gather the necessary properties of Richardson elements. In Section 3 we show that horizontal line diagrams in fact correspond to Richardson elements of the given parabolic subalgebra. The construction of such diagrams for $\mathfrak{gl}_n$ appears first in [BHRR]. We have already mentioned that for the other classical Lie algebras, the horizontal line diagrams do not give Richardson elements. In general, the matrix obtained is not an element of the Lie algebra in question. Thus we will introduce generalized line diagrams in Section 4 to obtain Richardson elements for parabolic subalgebras of the symplectic and orthogonal Lie algebras. As a by-product we obtain the partition of a Richardson element for the so-called simple parabolic subalgebras. The last section discusses the cases where line diagrams do not produce Richardson elements. For these we will allow "branched" diagrams. In Appendix A we add examples illustrating branched diagrams.

1. Line diagrams

Let $d = (d_1, \ldots, d_r)$ be a dimension vector, i.e. a sequence of positive integers. Arrange r columns of d_i dots, top-adjusted. A (*filled*) *line diagram* for d, denoted by $L(d)$, is a collection of lines joining vertices of different columns such that each vertex is connected to at most one vertex of a column left of it and to at most one vertex of a column right of it and such that it cannot be extended by any line.

We say that it is a (*filled*) *horizontal line diagram* if all edges are horizontal lines. Such a diagram will be denoted by $L_h(d)$. We will always assume that the line diagrams are filled and omit the term 'filled.' Line diagrams are not unique. However, for each dimension vector there is a unique horizontal line diagram.

Example 1. As an example, consider the dimension vector $(3, 1, 2, 3)$ and three line diagrams for it, the last one horizontal:



2. Richardson elements

In this section we describe a method to check whether a given nilpotent element of the nilradical of a classical Lie algebra is a Richardson element. The first statement is given in [BW]. Since we will use this result constantly, we repeat its proof.

Theorem 2.1. *Let* $\mathfrak{p} \subset \mathfrak{g}$ *be a parabolic subalgebra of a semi-simple Lie algebra* $\mathfrak{g}$, *let* $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$ *where* $\mathfrak{m}$ *is a Levi factor and* $\mathfrak{n}$ *the corresponding nilradical. Then* $x \in \mathfrak{n}$ *is a Richardson element for* $\mathfrak{p}$ *if and only if* $\dim \mathfrak{g}^x = \dim \mathfrak{m}$.

Proof. Denote the nilradical of the opposite parabolic by $\bar{\mathfrak{n}}$ (the opposite parabolic is defined as the parabolic subalgebra whose intersection with $\mathfrak{p}$ is equal to $\mathfrak{m}$). If $x \in \mathfrak{n}$ then $\mathrm{ad}(x)\mathfrak{g} = \mathrm{ad}(x)\bar{\mathfrak{n}} + \mathrm{ad}(x)\mathfrak{p}$. Now $\mathrm{ad}(x)\mathfrak{p} \subset \mathfrak{n}$ and $\dim \mathrm{ad}(x)\bar{\mathfrak{n}} \leqslant \dim \bar{\mathfrak{n}}$. Thus

$$\dim \mathrm{ad}(x)\mathfrak{g} \leqslant 2 \dim \mathfrak{n}.$$

This implies for $x \in \mathfrak{n}$ that $\dim \mathfrak{m} \leqslant \dim \mathfrak{g}^x$ and equality implies that $\dim \mathrm{ad}(x)\mathfrak{p} = \dim \mathfrak{n}$. Thus equality implies that x is a Richardson element.

For the other direction, let x be a Richardson element for $\mathfrak{p}$. We show that the map $\mathrm{ad}(x)$ is injective on $\bar{\mathfrak{n}}$: Let $y \in \bar{\mathfrak{n}}$ with $\mathrm{ad}(x)y = 0$. Then

$$0 = B\big(\mathrm{ad}(x)y, \mathfrak{p}\big) = B\big(y, \mathrm{ad}(x)\mathfrak{p}\big) = B(y, \mathfrak{n}).$$

In particular, $y = 0$. So $\mathrm{ad}(x)$ is injective on $\bar{\mathfrak{n}}$, giving $\dim \mathrm{ad}(x)\bar{\mathfrak{n}} = \dim \mathfrak{n}$. Thus

$$\dim \overbrace{\mathrm{ad}(x)\mathfrak{p}}^{\mathfrak{n}} + \dim \overbrace{\mathrm{ad}(x)\bar{\mathfrak{n}}}^{\bar{\mathfrak{n}}} = 2 \dim \mathfrak{n} = \dim \mathrm{ad}(x)\mathfrak{g} = \dim \mathfrak{g} - \dim \mathfrak{g}^x.$$

So $\dim \mathfrak{g}^x + \dim \mathfrak{n} = \dim \mathfrak{g} - \dim \mathfrak{n} = \dim \mathfrak{p} = \dim \mathfrak{m} + \dim \mathfrak{n}$, i.e. $\dim \mathfrak{m} = \dim \mathfrak{g}^x$. $\square$

Corollary 2.2. *Let* $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$ *be a parabolic subalgebra of a semi-simple Lie algebra. Let* $X \in \mathfrak{n}$ *be a Richardson element. Then* $\dim \mathfrak{g}^X \leqslant \dim \mathfrak{g}^Y$ *for any* $Y \in \mathfrak{n}$.

Theorem 2.1 gives us a tool to decide whether an element of the nilradical of a parabolic subalgebra is a Richardson element. Namely, we have to calculate its centralizer. Centralizers of nilpotent elements of the classical Lie algebras can be computed using their Jordan canonical form. This well-known result is due to Kraft and Procesi, cf. [KP].

Theorem 2.3. *Let* $(n_1, \ldots, n_r)$ *be the partition of the Jordan canonical form of a nilpotent matrix* x *in the Lie algebra* $\mathfrak{g}$, *let* $(m_1, \ldots, m_s)$ *be the dual partition. Then the dimension of the centralizer of* x *in* $\mathfrak{g}$ *is*

$$\begin{cases} \sum_i m_i^2, & \text{if } \mathfrak{g} = \mathfrak{gl}_n, \\ \sum_i \frac{m_i^2}{2} + \frac{1}{2}\big|\{i \mid n_i \text{ odd}\}\big|, & \text{if } \mathfrak{g} = \mathfrak{sp}_{2n}, \\ \sum_i \frac{m_i^2}{2} - \frac{1}{2}\big|\{i \mid n_i \text{ odd}\}\big|, & \text{if } \mathfrak{g} = \mathfrak{so}_N. \end{cases}$$

So it remains to determine the Jordan canonical form of a given nilpotent element x. It is given by the dimensions of the kernels of the maps x^j, $j \geqslant 1$:

Lemma 2.4. *Let x be a nilpotent $n \times n$ matrix with $x^{m-1} \neq 0$ and $x^m = 0$, set $b_j := \dim\ker x^j$ $(j = 1, \dots, m)$. Define*

$$a_j := \begin{cases} 2b_1 - b_2, & j = 1, \\ 2b_j - b_{j-1} - b_{j+1}, & j = 2, \dots, m-1, \\ b_m - b_{m-1}, & j = m. \end{cases}$$

Then the Jordan canonical form of x has a_s blocks of size s for $s = 1, \dots, m$.

Corollary 2.5. *With the notation of Lemma* 2.4 *above, the Jordan canonical form of x is given by the partition*

$$\bigl(1^{a_1}, 2^{a_2}, \dots, (m-1)^{a_{m-1}}, m^{a_m}\bigr).$$

3. The special linear Lie algebra

We now describe how to obtain a Richardson element from a (horizontal) line diagram. Recall that a standard parabolic subalgebra of $\mathfrak{sl}_n$ is uniquely described by the sequence of lengths of the blocks in $\mathfrak{m}$ (the standard Levi factor). Let $d = (d_1, \dots, d_r)$ be the dimension vector of these block lengths.

We form the horizontal line diagram $L_h(d)$ and label its vertices column wise by the numbers $1, 2, \dots, n$, starting with column 1, labeling top-down. This labeled diagram defines a nilpotent element as the sum of all elementary matrices E_{ij} such that there is a line from i to j, where $i < j$:

$$X(d) = X\bigl(L_h(d)\bigr) = \sum_{i — j} E_{ij}.$$

Example 2. Let $\mathfrak{p} \subset \mathfrak{sl}_9$ be given by the dimension vector $(3, 1, 2, 3)$. We label its horizontal line diagram,

$$\begin{array}{l} 1 — 4 — 5 — 7 \\ 2 ——— 6 — 8 \\ 3 ————— 9, \end{array}$$

and obtain $X(d) = E_{1,4} + E_{4,5} + E_{5,7} + E_{2,6} + E_{6,8} + E_{3,9}$, an element of the nilradical $\mathfrak{n}$ of $\mathfrak{p}$. Using Lemma 2.4 and Corollary 2.5 one checks that the dimension of the centralizer of $X(d)$ is equal to the dimension of the Levi factor. Thus $X(d)$ is a Richardson element for $\mathfrak{p}$ (by Theorem 2.3).

By construction, the matrix $X(d)$ is nilpotent for any dimension vector d. It is in fact an element of the nilradical $\mathfrak{n}$ of the parabolic subalgebra $\mathfrak{p} = \mathfrak{p}(d)$: If $d = (n)$, this is obvious, the constructed nilpotent element is the zero matrix. If $d = (d_1, d_2)$ then the nonzero coefficients of the matrix of $X(d)$ are in the rows $1, \dots, d_1$ and columns $d_1 + 1, \dots, d_2$. In other words, they lie in the $d_1 \times d_2$-block in the upper right corner. The standard Levi factor consists of the blocks $d_1 \times d_1$, $d_2 \times d_2$ on the diagonal. In particular, $X(d_1, d_2)$ is a matrix that lies above the Levi factor. This generalizes to dimension vectors with more entries. So we get part (1) of the following lemma. For part (2) we introduce a new notion.

Definition 1. If there exists a sequence of k connected lines in a line diagram $L(d)$ that is not contained in a longer sequence we say that $L(d)$ has a *k-chain* or a *chain of length k*. A *subchain of length k* (or k-subchain) is a sequence of k connected lines in $L(d)$ that maybe contained in a longer chain. A chain of length 0 is a single vertex that is not connected to any other vertex.

Lemma 3.1. (1) *The element $X(d)$ is an element of the nilradical of $\mathfrak{p}(d)$.*

(2) *For $k \geqslant 1$, the rank of $X(d)^k$ is equal to the number of k-subchains of lines in $L_h(d)$.*

Proof of (2). It is clear that the rank of $X = X(d)$ is the number of lines in the diagram: to construct X, we sum over all lines of the diagram. Since these lines are disjoint (each vertex i is joint to at most one neighbor j with $i < j$) the rows and columns of X are linearly independent. Therefore the rank of X is equal to the number of vertices i such that there is a line from i to some j with $i < j$.

For any $k > 0$, the matrix X^k consists of linearly independent rows and columns. It is clear that an entry (ij) of $X \cdot X$ is non-zero if and only if there is a line i—k—j in $L_h(d)$: $X \cdot X = \sum_{i-k} E_{ik} \sum_{l-j} E_{lj}$ where $E_{ik}E_{lj} = \delta_{kl}E_{ij}$. Similarly, the rank of X^k is the number of vertices i such that there exist vertices $j_1 < j_2 < \cdots < j_k$ and lines i—j_1—$\cdots$—j_k joining them, i.e. the number of k-subchain. □

It turns out that $X(d)$ is a Richardson element for $\mathfrak{p}(d)$, as we will show below. This fact follows also from the description of Brüstle et al. in [BHRR] of Δ-filtered modules without self-extension of the Auslander–Reiten quiver of type A_r (the number r is the number of blocks in the standard Levi factor of the parabolic subalgebra).

Theorem 3.2. *The mapping $d \mapsto X(d)$ associates to each dimension vector with $\sum d_i = n$ a Richardson element for the corresponding parabolic subalgebra $\mathfrak{p} = \mathfrak{p}(d)$ of $\mathfrak{sl}_n$.*

We give here an elementary proof of Theorem 3.2 above. We will use the ideas of this proof to deal with the other classical groups (where we will have to use line diagrams that are not horizontal in general). The main idea is to use the dimension of the centralizer of a Richardson element and the partition of the Jordan canonical form of a nilpotent element.

Proof. Let d be the dimension vector corresponding to the parabolic subalgebra $\mathfrak{p} = \mathfrak{p}(d)$. Let $X = X(d)$ be the nilpotent element associated to it (through the horizontal line diagram). By Theorem 2.1 we have to calculate the dimension of the centralizer of X

and of the Levi factor $\mathfrak{m}$ of $\mathfrak{p}$. By Theorem 2.3, $\dim \mathfrak{g}^X$ is equal to $\sum_i m_i^2 - 1$ where $(m_1, \ldots, m_s)$ is the dual partition to the partition of X. The parts of the dual partition are the entries d_i of the dimension vector as is shown in Lemma 3.3 below. In particular, $\dim \mathfrak{l} = \sum_i d_i^2 - 1 = \dim \mathfrak{g}^X$. □

The following result shows how to obtain the partition and the dual partition of the Jordan canonical form of the nilpotent element associated to the dimension vector d.

Lemma 3.3. *Let d be the dimension vector for $\mathfrak{p} \subset \mathfrak{sl}_n$, $X = X(d)$ the associated nilpotent element of $\mathfrak{sl}_n$. Order the entries $d_1, \ldots, d_r$ of the dimension vector in decreasing order as $D_1, D_2, \ldots, D_r$ (i.e. such that $D_i \geqslant D_{i+1}$ for all i). Then the Jordan canonical form of X is*

$$1^{D_1 - D_2}, 2^{D_2 - D_3}, \ldots, (r-1)^{D_{r-1} - D_r}, r^{D_r}$$

and the dual partition is

$$D_r, D_{r-1}, \ldots, D_1.$$

In other words, the dual partition for $X(d)$ is given by the entries of the dimension vector. Furthermore, for every i-chain in $L_h(d)$ (i.e. for every sequences of length i, $i \geqslant 0$, that is not contained in a longer sequence) the partition has an entry $i + 1$.

Proof. Let $d = (d_1, \ldots, d_r)$ be the dimension vector of $\mathfrak{p}$ and $D_1, \ldots, D_r$ its permutation in decreasing order, $D_i \geqslant D_{i+1}$. To determine the Jordan canonical form of $X = X(d)$ we have to compute the rank of the powers X^s, $s \geqslant 1$, cf. Lemma 2.4.

Since the nilpotent matrix X is given by the horizontal line diagram $L_h(d)$, the rank of X^s is easy to compute: by Lemma 3.1(2), the rank of X^s is the number of s-subchains. In particular, $\operatorname{rk} X = n - D_1$ and $\operatorname{rk} X^2 = n - D_1 - D_2$, $\operatorname{rk} X^3 = n - D_1 - D_2 - D_3$, etc. This gives

$$b_s := \dim \ker X^s = D_1 + \cdots + D_s \quad \text{for } s = 1, \ldots, r.$$

And so, by Lemma 2.4, we obtain $a_1 = D_1 - D_2$, $a_2 = D_2 - D_3$, $\ldots$, $a_r = D_r$ proving the first statement. The statement about the dual partition (i.e. the partition given by the lengths of the columns of the partition) follows then immediately. □

4. Richardson elements for the other classical Lie algebras

In this section we will introduce generalized line diagrams to deal with the symplectic and orthogonal Lie algebras. Having introduced them, we show that they correspond to Richardson elements for the parabolic subalgebra in question. Then we discuss some properties and describe the dual of the partition of a nilpotent element given by such a generalized line diagram. Furthermore, we describe the support of the constructed $X(d)$

and relate it to the Bala–Carter label of the G-orbit through $X(d)$ where G is the adjoint group of $\mathfrak{g}$.

To define the orthogonal Lie algebras, we use the skew diagonal matrix J_n with ones on the skew diagonal and zeroes else. The symplectic Lie algebras $\mathfrak{sp}_{2n}$ are defined using $\left[\begin{smallmatrix} 0 & J_n \\ -J_n & 0 \end{smallmatrix}\right]$. (For details we refer the reader to [GW].) So $\mathfrak{so}_n$ consists of the $n \times n$-matrices that are skew-symmetric around the skew-diagonal and $\mathfrak{sp}_{2n}$ is the set of $2n \times 2n$-matrices of the form

$$\begin{bmatrix} A & B \\ C & A^* \end{bmatrix}$$

where A^* is the negative of the skew transpose of A. Thus in the case of the symplectic and orthogonal Lie algebras, the block sizes of the standard Levi factor form a palindromic sequence.

If there is an even number of blocks in the Levi factor, the dimension vector is of the form $(d_1, \ldots, d_r, d_r, \ldots, d_1)$. We will refer to this situation as type (a). If there is an odd number of blocks in the Levi factor, type (b), the dimension vector is $(d_1, \ldots, d_r, d_{r+1}, d_r, \ldots, d_1)$.

By the (skew) symmetry around the skew diagonal, the entries below the skew diagonal of the matrices $X(d)$ are determined by the entries above the skew diagonal. In terms of line diagrams: For $\mathfrak{sp}_N$ and $\mathfrak{so}_N$ there is a line $(N-j+1)$—$(N-i+1)$ whenever there is a line i—j. We will call the line $(N-j+1)$—$(N-i+1)$ the *counterpart* of i—j and will sometimes denote counterparts by dotted lines. In particular, it suffices to describe the lines attached to the left to vertices of the first r columns for both types (a) and (b).

The (skew)-symmetry will give constraints on the diagram—there will also appear negative entries. For the moment, let us assume that $L(d)$ is a diagram defining an element of the nilradical of the parabolic subalgebra in question. Then part (2) of Lemma 3.1 still holds.

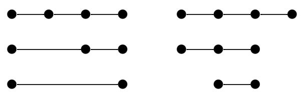
Lemma 4.1. *If $X(d)$ is defined by $L(d)$ then the rank of the map $X(d)^k$ is the number of k-subchains of lines in the diagram.*

This uses the same argument as Lemma 3.1 since by construction, $X(d)$ only has linearly independent rows and columns and the product $X(d)^2$ only has nonzero entries E_{il} if $X(d)$ has an entry E_{ij} and an entry E_{jl} for some j.

The following remark allows us to simplify the shapes of the diagrams we are considering. If $d = (d_1, \ldots, d_r)$ is an r-tuple in $\mathbb{N}^r$, and $\sigma \in S_r$ (where S_r is the permutation group on r letters) we define d_σ as $(d_{\sigma 1}, d_{\sigma 2}, \ldots, d_{\sigma r})$. By abuse of notation, for $d = (d_1, \ldots, d_r, d_r, \ldots, d_1)$ in $\mathbb{N}^{2r}$, we write $d_\sigma = (d_{\sigma 1}, \ldots, d_{\sigma r}, d_{\sigma r}, \ldots, d_{\sigma 1})$ and for $d = (d_1, \ldots, d_r, d_{r+1}, d_r, \ldots, d_1)$ in $\mathbb{N}^{2r+1}$, we define d_σ to be the $2r+1$-tuple $(d_{\sigma 1}, \ldots, d_{\sigma r}, d_{r+1}, d_{\sigma r}, \ldots, d_{\sigma 1})$. It will be clear from the context which tuple we are referring to.

Remark 4.2. For $d = (d_1, \ldots, d_r)$ the diagrams $L_h(d)$ and $L_h(d_\sigma)$ have the same chains of lines for any $\sigma \in S_r$. In other words: for any $k \geqslant 1$, the number of chains of lines of length

k in $L_h(d)$ is the same as the number of lines of length k in $L_h(d_\sigma)$. As an illustration, consider the permutation 1243 of $d = (3, 1, 2, 3)$:



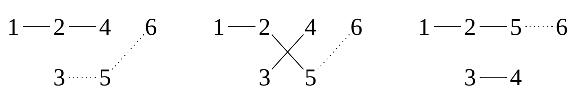
Similarly, for $f = (f_1, \dots, f_r, f_r, \dots, f_1)$, respectively for $g = (g_1, \dots, g_r, g_{r+1}, g_r, \dots, g_1)$, if $L(f)$ and $L(g)$ are line diagrams for $\mathfrak{sp}_{2n}$ or $\mathfrak{so}_N$ then for any $\sigma \in S_r$, the diagrams $L(f_\sigma)$, respectively $L(g_\sigma)$, are also diagrams for the corresponding Lie algebras and have the same exactly the same chains as $L(f)$, respectively as $L(g)$.

We have an immediate consequence of Remark 4.2 and Lemma 4.1:

Corollary 4.3. *Let $d = (d_1, \dots, d_r, d_r, \dots, d_1)$ or $d = (d_1, \dots, d_r, d_{r+1}, d_r, \dots, d_1)$ be the dimension vector of a parabolic subalgebra of a symplectic or orthogonal Lie algebra and $X(d)$ be given by the appropriate line diagram. In calculating the rank of $X(d)^k$ we can assume that $d_1 \leqslant \cdots \leqslant d_r$.*

We will make frequent use of this property. Now we will finally be able to construct diagrams for the other classical cases. We have already mentioned that the horizontal line diagrams do not produce Richardson elements. One reason is that the counterpart of a line i—j is not always horizontal. The other reason is that we have to introduce negative signs for the symplectic and orthogonal cases when we associate a nilpotent matrix to a diagram: If $\mathfrak{g} = \mathfrak{sp}_{2n}$, in the definition of $X(d)$ we subtract E_{ij} whenever there is a line i—j with $n < i < j$. If $\mathfrak{g} = \mathfrak{so}_N$ we subtract E_{ij} whenever there is a line i—j with $i + j > N$.

Example 3. Let $(1, 2, 2, 1)$ be the dimension vector of a parabolic subalgebra of $\mathfrak{sp}_6$. Then the following three line diagrams determine elements of the nilradical of $\mathfrak{p}$:



The last diagram is just a reordering of the second. The nilpotent elements are $X_1 = E_{12} + E_{24} + E_{35} - E_{56}$, respectively $X_2 = E_{12} + E_{25} + E_{34} - E_{56}$. By calculating the Jordan canonical forms for these elements one checks that only the nilpotent element X_2 is a Richardson element.

This example and the discussion above illustrate that for the symplectic and orthogonal Lie algebras, we will use:

(i) non-horizontal lines,
(ii) labeling top–bottom and bottom–top,
(iii) negative signs, too.

Before we start defining these line diagrams we introduce a new notion.

Definition 2. Let $\mathfrak{p}$ be the standard parabolic subalgebra of a symplectic or orthogonal Lie algebra $\mathfrak{g}$. We say that $\mathfrak{p}$ is *simple* if $\mathfrak{p} \subset \mathfrak{g}$ is of one of the following forms:

(1) A parabolic subalgebra of $\mathfrak{sp}_{2n}$ with an even number of blocks in the standard Levi factor.
(2) A parabolic subalgebra of $\mathfrak{so}_{2n}$ with an even number of blocks in the standard Levi factor such that odd block lengths appear exactly twice.
(3) A parabolic subalgebra of $\mathfrak{sp}_{2n}$ with an odd number of blocks in the Levi factor and such that each odd d_i that is smaller than d_{r+1} appears exactly twice.
(4) A parabolic subalgebra of $\mathfrak{so}_N$ with an odd number of blocks in the Levi factor such that either all d_i are odd or there is an index $k \leqslant r$ such that all d_i with $i \leqslant k$ are even, d_j odd for $j > k$ and the even d_i are smaller than $d_{k+1}, \ldots, d_r$. Furthermore, the even block lengths that are larger than d_{r+1} appear only once among $d_1, \ldots, d_k$.

Definition 3. (*Type* (a)) Let $\mathfrak{p}$ be a simple parabolic subalgebra of $\mathfrak{sp}_{2n}$ or $\mathfrak{so}_{2n}$, given by the dimension vector $d = (d_1, \ldots, d_r, d_r, \ldots, d_1)$. Then we define the *line diagram* $L_{\text{even}}(d)$ *associated to* d (and $\mathfrak{g}$) as follows:

(1) Draw $2n$ vertices in $2r$ columns of length $d_1, \ldots,$ top-adjusted. Label the first r columns with the numbers $1, \ldots, n$, top–bottom. Label the second r columns with the numbers $n+1, \ldots, 2n$, bottom–top.
(2) Join the first r columns with horizontal lines as for $\mathfrak{sl}_n$. Draw the counterparts of these lines in the second r columns.
(3) (i) If $\mathfrak{g} = \mathfrak{sp}_{2n}$, add the lines k—$(2n-k+1)$.
    (ii) If $\mathfrak{g} = \mathfrak{so}_{2n}$, one adds the lines $(2l-1)$—$(2n-2l+1)$ and their counterparts $2l$—$(2n-2l+2)$ if n is even. If n is odd, the lines $2l$—$(2n-2l)$ and their counterparts $(2l+1)$—$(2n-2l+1)$.

Definition 4. (*Type* (b)) Let $\mathfrak{p}$ be a simple parabolic subalgebra of $\mathfrak{sp}_{2n}$ or of $\mathfrak{so}_N$, given by the dimension vector $d = (d_1, \ldots, d_r, d_{r+1}, d_r, \ldots, d_1)$. Then we define the *line diagram* $L_{\text{odd}}(d)$ *associated to* d (and $\mathfrak{g}$) as follows:

(1) Draw $2r+1$ columns of length $d_1, \ldots,$ top-adjusted. Label them with the numbers $1, \ldots$ in increasing order, top–bottom in each column.
(2) (i) For $\mathfrak{sp}_{2n}$: If $\min_i\{d_i\} \geqslant 2$, draw a horizontal of lines in the first row and all their counterparts, forming a sequence joining the lowest vertices of each column. Repeat this procedure as long as the columns of the remaining vertices are all at least of length two.
    (ii) For $\mathfrak{so}_N$: If d_1 is odd, go to step (3)(ii). If d_1 is even, do as in (2)(i), drawing lines in the first row and their counterparts joining the lowest vertices. Repeat until either the first of the remaining columns has odd length or there are no vertices left to be joined. Continue as in (3)(ii).

(3) (i) For $\mathfrak{sp}_{2n}$: For the remaining vertices: draw horizontal lines following the top-most remaining vertices and simultaneously their counterparts (the lowest remaining vertices).
(ii) For $\mathfrak{so}_N$: All columns have odd length. Connect the central entries of each column. The remaining column lengths are all even; they are joined as in (2)(ii).

Theorem 4.4. *Let d be the dimension vector for a simple parabolic subalgebra of $\mathfrak{sp}_{2n}$ or $\mathfrak{so}_N$. Then the associated diagram $L_{\text{even}}(d)$, respectively $L_{\text{odd}}(d)$, determines a Richardson element for $\mathfrak{p}(d)$ by setting*

$$X(d) = \sum_{i—j,\ i\leqslant n} E_{ij} - \sum_{i—j,\ i>n} E_{ij} \qquad \text{for } \mathfrak{sp}_{2n},$$

$$X(d) = \sum_{i—j,\ i+j<N} E_{ij} - \sum_{i—j,\ i+j>N} E_{ij} \qquad \text{for } \mathfrak{so}_N,$$

where the sums are over all lines in the diagram.

We first include some immediate consequences of this result. After that we add an observation about the (dual of the) partition corresponding to $X(d)$ and then we are ready to prove Theorem 4.4.

Theorem 4.4 enables us to determine the minimal k such that the Richardson element $X(d)$ lies in the graded parts $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$. To do so we introduce $s(d)$ as the maximal number of entries $d_i, \ldots, d_{i+s}$ of d that are surrounded by larger entries d_{i-1} and d_{i+s+1}. More precisely, if $d = (d_1, \ldots, d_r, d_r, \ldots, d_1)$ or $d = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_1)$ is the dimension vector, we rewrite d as a vector with increasing indices, $(c_1 \ldots, c_r, c_{r+1}, c_{r+2}, \ldots, c_{2r})$, respectively $(c_1 \ldots, c_r, c_{r+1}, c_{r+2}, \ldots, c_{2r+1})$, and define

$$s(d) := 1 + \max_i \{\text{there are } c_{j+1}, \ldots, c_{j+i} \mid c_j > c_{j+l} < c_{j+i+1} \text{ for all } 1 \leqslant l \leqslant i\}.$$

Corollary 4.5. *Let $\mathfrak{p}(d)$ be a simple parabolic subalgebra of the orthogonal or symplectic Lie algebras. Then the element $X(d)$ belongs to $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{s(d)}$. The same holds for parabolic subalgebras of $\mathfrak{sl}_n$.*

This follows from the fact that E_{ij} with i from column k of the line diagram and j from column $k+s$ is an entry of the graded part $\mathfrak{g}_s$. If, e.g., we have $c_1 > c_j < c_{s+1}$ for $j = 2, \ldots, s$ then there is a line joining columns one and $s+1$. So $X(d)$ has an entry in $\mathfrak{g}_s$.

Corollary 4.6. *For $\mathfrak{sl}_n$, $s(d)$ is equal to one if and only if the dimension vector satisfies $d_1 \leqslant \cdots \leqslant d_t \geqslant \cdots \geqslant d_r$ for some $1 \leqslant t \leqslant r$.*

This well-known result has been observed by Lynch [Ly], Elashvili and Kac [EK], Goodwin and Röhrle [GR], and in our joint work with Wallach [BW].

The next lemma shows how to obtain the dual of the partition of $X(d)$ if $X(d)$ is given by the appropriate line diagram for d.

Lemma 4.7. *If $\mathfrak{p}(d)$ is a simple parabolic subalgebra of a symplectic or orthogonal Lie algebra let $X = X(d)$ be given by the appropriate line diagram $L_{\text{even}}(d)$ or $L_{\text{odd}}(d)$. The dual of the partition of X has the form as follows*:

	Dual of the partition of X	$\mathfrak{g}$	*Type of* $\mathfrak{p}$
(i)	$d_1, d_1, \ldots, d_r, d_r$	$\mathfrak{sp}_{2n}$	(a)
(ii)	$d_{r+1} \cup \left(\bigcup_{d_i \notin D_o} d_i, d_i \right) \cup \left(\bigcup_{d_i \in D_o} d_i - 1, d_i + 1 \right)$	$\mathfrak{sp}_{2n}$	(b)
(iii)	$\left(\bigcup_{d_i \text{ even}} d_i, d_i \right) \cup \left(\bigcup_{d_i \text{ odd}} d_i - 1, d_i + 1 \right)$	$\mathfrak{so}_{2n}$	(a)
(iv)	$d_{r+1} \cup \left(\bigcup_{d_i \notin D^e} d_i, d_i \right) \cup \left(\bigcup_{d_i \in D^e} d_i - 1, d_i + 1 \right)$	$\mathfrak{so}_{2n+1}$	(b)
(v)	$d_{r+1} \cup \left(\bigcup_{d_i \notin D^o} d_i, d_i \right) \cup \left(\bigcup_{d_i \in D^o} d_i - 1, d_i + 1 \right)$	$\mathfrak{so}_{2n}$	(b)

where $D_o := \{d_i \text{ odd} \mid d_i < d_{r+1}\}$, $D^o := \{d_i \text{ odd} \mid d_i > d_{r+1}\}$ and $D^e := \{d_i \text{ even} \mid d_i > d_{r+1}\}$ are subsets of $\{d_1, \ldots, d_r\}$.

In particular, if D_o, D^e or D^o are empty, the partition in the corresponding case (ii), (iv) *or* (v) *has the same parts as the dimension vector. The same is true for* (iii), *if there are no odd d_i.*

The proof consists mainly in counting lines and (sub)chains of lines of the corresponding diagrams. Therefore we postpone it and include it in Appendix A. We are now ready to prove Theorem 4.4 with the use of Theorem 2.3 and of Lemma 4.7.

Proof of Theorem 4.4. We consider the case $\mathfrak{g} = \mathfrak{sp}_{2n}$. For the parabolic subalgebras of an orthogonal Lie algebra, the claim follows using the same methods. The idea is to use the dimension of the centralizer of $X(d)$ and compare it to the dimension of the Levi factor. To calculate the dimension of the centralizer, we use the formulae of Theorem 2.3, i.e. we use the dual of the partition of $X = X(d)$ as described in Lemma 4.7 and the number of odd parts in the partition of X.

• $\mathfrak{sp}_{2n}$, type (a): By Lemma 4.7 the dual partition of the nilpotent element $X = X(d)$ has as parts the entries of d. Since they all appear in pairs, the partition of the orbit has no odd entries. So by the formula of Theorem 2.3 we obtain $\dim \mathfrak{g}^X = \frac{1}{2}(2d_1^2 + \cdots + 2d_r^2)$, the same as the dimension of the Levi factor. In particular, X is a Richardson element for the parabolic subalgebra $\mathfrak{p}(d)$ of $\mathfrak{sp}_{2n}$.

• $\mathfrak{sp}_{2n}$, type (b): As in Lemma 4.7 let $D_o \subset \{d_1, \ldots, d_r\}$ be the possibly empty set of the odd d_i that are smaller than d_{r+1}. Then the dual partition has the parts

$$\{d_i, d_i \mid i < r,\ d_i \notin D_o\} \cup \{d_{r+1}\} \cup \{d_{i+1}, d_{i-1} \mid d_i \in D_o\}.$$

The d_i that are not in D_o come in pairs and do not contribute to odd parts in the partition of $X = X(d)$. In particular, the number of odd parts only depends on d_{r+1} and on the entries of D_o. We write the elements of D_o in decreasing order as $\tilde{d}_1, \ldots, \tilde{d}_s$ (where $s = |D_o|$). By assumption (the parabolic subalgebra is simple) these odd entries are all different, $\tilde{d}_1 > \tilde{d}_2 > \cdots > \tilde{d}_s$. Then the number of odd parts of the partition of X is the same as the number of odd parts of the dual of the partition

$$\widetilde{P}: \quad d_{r+1}, \tilde{d}_1 + 1, \tilde{d}_1 - 1, \ldots, \tilde{d}_s + 1, \tilde{d}_s - 1.$$

This has $d_{r+1} - (\tilde{d}_1 + 1)$ ones, $(\tilde{d}_1 + 1) - (\tilde{d}_1 - 1)$ twos, $(\tilde{d}_1 - 1) - (\tilde{d}_2 + 1)$ threes, and so on. So the number of odd parts in the dual of $\widetilde{P}$ is

$$\begin{aligned}&\big[d_{r+1} - (\tilde{d}_1 + 1)\big] + \big[(\tilde{d}_1 - 1) - (\tilde{d}_2 + 1)\big] + \cdots + \big[(\tilde{d}_{s-1} - 1) - (\tilde{d}_s + 1)\big] + \tilde{d}_s - 1\\ &\quad = d_{r+1} - 2s.\end{aligned}$$

Thus the dimension of the centralizer of X is

$$\begin{aligned}&\frac{1}{2}\left[\left(\sum_{\substack{i<r+1\\ d_i \notin D_o}} 2d_i^2\right) + d_{r+1}^2 + \left(\sum_{d_i \in D_o} (d_i - 1)^2 + (d_i + 1)^2\right) + d_{r+1} - 2s\right]\\ &\quad = \sum_{i \leqslant r} d_i^2 + \binom{d_{r+1} + 1}{2} = \dim \mathfrak{m}. \qquad \square\end{aligned}$$

4.1. Bala–Carter labels for Richardson orbits

The support of the nilpotent element of a simple line diagram is by construction a simple system of root. Namely, for any d, the corresponding $X(d)$ has at most one non-zero element in each row and each column. One can check that none of the corresponding positive roots subtract from each other.

In other words, the support $\mathrm{supp}(X)$ forms a simple system of roots.

Remark 4.8. The converse statement is not true. There are Richardson elements whose support form a simple system of roots but where there is no simple line diagram defining a Richardson element. A family of examples are the Borel subalgebras of $\mathfrak{so}_{2n}$ or more general, parabolic subalgebras of $\mathfrak{so}_{2n}$ where α_n and α_{n-1} are both not roots of the Levi factor

If X is a nilpotent element of $\mathfrak{g}$ we denote the G-orbit through X by $\mathcal{O}_X$ (where G is the adjoint group of $\mathfrak{g}$).

Corollary 4.9. *Let $\mathfrak{p}(d)$ be a parabolic subalgebra of $\mathfrak{sl}_n$. Define $X(d)$ by the line diagram $L_h(d)$ or a simple parabolic subalgebra of* (b)*-type for $\mathfrak{sp}_{2n}$, $\mathfrak{so}_N$ Then the group spanned by* supp $X(d)$ *is equal to the Bala–Carter label of the G-orbit $\mathcal{O}_{X(d)}$.*

Proof. This follows from the characterization of the type (i.e. the Bala–Carter label) of $\mathcal{O}_X$ given by Panyushev in [Pa, Section 3].

For simplicity we assume $d_1 \leqslant \cdots \leqslant d_r$. Note that in any case, the partition of $X(d)$ is given by the chains in the line diagram. The partition of $X(d)$ has entry $i+1$ for every chain of length i.

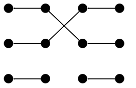
If α given by E_{ij} and β given by E_{kl} are roots of $\operatorname{supp} X(d)$ then they add to a root of $\mathfrak{sl}_n$ if and only if there is a line connecting them. Thus in the case of the special linear Lie algebra a chain of length $i+1$ corresponds to a factor A_i in $\operatorname{supp} X(d)$. Similarly, for $\mathfrak{sp}_{2n}$ and $\mathfrak{so}_N$, a chain of length $i+1$ together with its counterpart give a factor A_i. Finally, the possibly remaining single chain of length $2j+1$ (passing through the central vertex of column $r+1$) in the case of $\mathfrak{so}_{2n+1}$ gives a factor B_j. Then the claim follows with [Pa] where Panyushev describes the type of a nilpotent orbit in terms of its partition. □

5. Branched diagrams

The diagrams we have introduced had at most one line to the left and at most one line to the right of a vertex. We call such a diagram a *simple line diagram*. In the case of simple parabolic subalgebras, we can always choose a simple line diagram to define a Richardson element. However, there are parabolic subalgebras where no simple diagram gives rise to a Richardson elements. After giving an example we characterize the parabolic subalgebras for which there exists a simple line diagram giving a Richardson element. Then we discuss the case of the symplectic Lie algebras. We introduce a branched diagram and obtain a Richardson elements for the parabolic subalgebra in question.

Example 4. (1) Consider the parabolic subalgebra of $\mathfrak{so}_{2n}$ given by the dimension vector (n,n) where n is odd. The element $X = X(n,n)$ given by the diagram $L_{\mathrm{even}}(n,n)$ has rank $n-1$ and so the kernel of the map X^k has dimension $n+1$ or $2n$ for $k=1,2$, respectively. The partition of X is then $1^2, 2^{n-1}$, its dual is $n-1, n+1$. The centralizer of X has dimension $2n^2+1-1$ and the Levi factor of this parabolic subalgebra has dimension n^2. So X is a Richardson element.

(2) Let $\mathfrak{p} \subset \mathfrak{so}_{4d}$ be given by (d,d,d,d) where d is odd. Note that the skew-symmetry of the orthogonal Lie algebra allows at most $d-1$ lines between the two central columns.



The line diagram $L_{\mathrm{even}}(d,d,d,d)$ has $2d+d-1$ lines, $2(d-1)$ two-subchains and $d-1$ three-chains. Calculating the dimensions of the kernel of the map X^k (where $X = X(d,d,d,d)$) yields the partition $2^2, 4^{d-1}$. Its dual is $(d-1)^2, (d+1)^2$, hence the centralizer of X has dimension $2d^2+2$ while the Levi factor has dimension $2d^2$.

Theorem 5.1. *Let $\mathfrak{g}$ be a simple Lie algebra. The parabolic subalgebras $\mathfrak{p}$ of $\mathfrak{g}$ for which there exists a simple line diagram that defines a Richardson element for $\mathfrak{p}$ are*:

The parabolic subalgebras of $\mathfrak{sl}_n$ and the simple parabolic subalgebras of the symplectic and orthogonal Lie algebras.

Proof. By Theorems 3.2 and 4.4 there is always a simple line diagram giving a Richardson element in these cases. It remains to show that these are the only ones. By Corollary 4.3 we can assume w.l.o.g. that $d_1 \leqslant \cdots \leqslant d_r$. Then it turns out that if there is an even number of blocks for $\mathfrak{so}_{2n}$ or if $d_r \leqslant d_{r+1}$ for $\mathfrak{sp}_{2n}$ the problem is translated to the problem of finding a Richardson element in the first graded part $\mathfrak{g}_1$ of $\mathfrak{g}$ because of the following observation: Since $d_1 \leqslant \cdots \leqslant d_r = d_r \geqslant \cdots \geqslant d_1$, or $d_1 \leqslant \cdots \leqslant d_r \leqslant d_{r+1} \geqslant d_r \geqslant \cdots \geqslant d_1$ all lines are connecting neighbored columns. But lines connecting neighbored columns correspond to entries $E_{i,j}$ of the first super diagonal of the parabolic subalgebra, i.e. to entries of $\mathfrak{g}_1$. Then the claim follows from the classification of parabolic subalgebras with a Richardson element in $\mathfrak{g}_1$ for type (a) of $\mathfrak{so}_{2n}$ and if $d_r \leqslant d_{r+1}$ for type (b) parabolic subalgebras of the symplectic Lie algebra. In both cases there exists a Richardson element in $\mathfrak{g}_1$ if and only if each odd block length d_i only appears once among $d_1, \ldots, d_r$, cf. [BW]. If there is no Richardson element in $\mathfrak{g}_1$ then in particular no simple line diagram can give a Richardson element. It remains to deal with (b)-types for $\mathfrak{so}_N$ and (b)-types for $\mathfrak{sp}_{2n}$ where d_{r+1} is not maximal. Both are straightforward but rather lengthy calculation that we omit here. □

By way of illustration we include examples of branched diagrams for non-simple parabolic subalgebras of $\mathfrak{sp}_{2n}$ and of $\mathfrak{so}_N$ in Appendix A. In general, it is not clear how branched diagrams should be defined uniformly for the symplectic and orthogonal Lie algebras. It is clear from the description of simple parabolic subalgebras of $\mathfrak{so}_N$ that this case is more intricate. We assume that Richardson elements can be obtained by adding lines to the corresponding simple line diagrams:

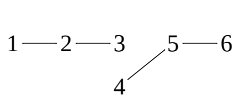
Conjecture 1. *For the* (b)*-type of $\mathfrak{sp}_{2n}$ the appropriate diagram defining a Richardson element is obtained from $L_{\text{odd}}(d)$ by adding a branching for every repetition $d_i = d_{i+1} = \cdots = d_{i+s}$ of odd entries smaller than d_{r+1}.*

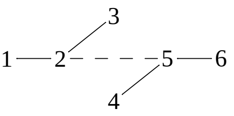
We conclude this section with a remark on the bound $s(d)$ introduced in Section 4. If there is no simple line diagram defining a Richardson element, we can still define $s(d)$ to be the maximal number of a sequence of entries of d that are surrounded by two larger entries. But this will now only be a lower bound, the Richardson element defined by a branched diagram does not necessarily lie in $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{s(d)}$, cf. Examples 5–7.

Appendix A

We discuss some examples of branched line diagrams for $\mathfrak{sp}_{2n}$ and for $\mathfrak{so}_N$ to illustrate Section 5. Recall that the parabolic subalgebras of type (b) of $\mathfrak{sp}_{2n}$ are simple if and only if every odd $d_i < d_{r+1}$ only appears once among $d_1, \ldots, d_r$. In particular, the smallest example of $\mathfrak{sp}_{2n}$ where there is no simple line diagram exists for $n = 3$.

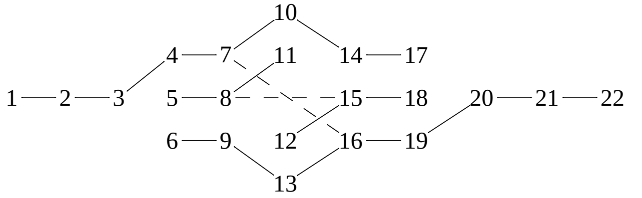
Example 5. Let $\mathfrak{p}$ be the parabolic subalgebra of $\mathfrak{sp}_6$ with dimension vector $(1, 1, 2, 1, 1)$. Consider the diagrams





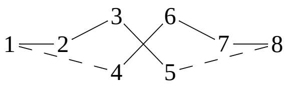
The diagram to the left is a line diagram as in Section 4. The corresponding nilpotent element has a centralizer of dimension 7. However, the Levi factor is five-dimensional. In the second diagram, there is one extra line, connecting the vertices 2 and 5. The defined matrix $X = E_{12} + E_{23} + E_{25} - E_{45} - E_{56}$ has a five-dimensional centralizer as needed.

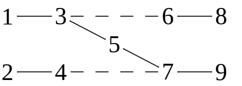
Example 6. The following branched line diagram for the parabolic subalgebra of $\mathfrak{sp}_{22}$ with dimension vector $d = (1, 1, 1, 3, 3, 4, 3, 3, 1, 1, 1)$ gives a Richardson element for $\mathfrak{p}(d)$:



The Levi factor and the centralizer of the constructed X have dimension 31.

Example 7. For the orthogonal Lie algebras, the smallest example are given by $d = (1, 1, 2, 2, 1, 1)$, i.e. (a)-type of $\mathfrak{g} = \mathfrak{so}_8$ and by $d = (2, 2, 1, 2, 2)$ for an odd number of blocks in $\mathfrak{so}_9$. The following branched diagrams give Richardson elements for the corresponding parabolic subalgebras:





Proof of Lemma 4.7. We prove the statement for the symplectic Lie algebras. The corresponding statements for $\mathfrak{so}_N$ are proven similarly.

(i) *Type* (a) *of* $\mathfrak{sp}_{2n}$: Note that the bottom–top ordering of the second half of $L_{\text{even}}(d)$ ensures that the counterpart of a line i—j (for $j \leqslant n$) is again horizontal and that all lines connecting any entry of column r to an entry to its right are horizontal. Therefore the line diagram L_{even} has the same shape as the horizontal line diagram defined for $\mathfrak{sl}_n$. In particular, the orbit of the nilpotent element defined by $L_{\text{even}}(d)$ has the same partition as the one defined by $L_h(d)$. Then the assertion follows with Lemma 3.3.

(ii) *Type* (b) *of* $\mathfrak{sp}_{2n}$: The proof is done by induction on r. Let $d = (d_1, d_2, d_1)$ be the dimension vector. If $d_1 \notin D_o$ (i.e. d_1 is not an odd entry smaller than d_2) then the line diagram $L_{\text{even}}(d_1, d_2, d_1)$ has the same chains of lines as the horizontal diagram for $\mathfrak{sl}_{2n}$.

For $d_1 \in D_o$ the diagram $L_{\text{even}}(d_1, d_2, d_1)$ has $d_1 - 1$ two-chains (chains of length two) and 2 one-chains (i.e. lines). So the kernel of the map X^k has dimension d_2, $d_1 + d_2 + 1$, $2d_1 + d_2$ for $k = 1, 2, 3$, giving the partition $1^{d_2 - d_1 - 1}, 2^2, 3^{d_1 - 1}$ and the dual of it is $d_2, d_1 + 1, d_1 - 1$, as claimed.

Let now $d = (d_1, \ldots, d_r, d_{r+1}, d_r, \ldots, d_1)$ with $d_1 \leqslant \cdots \leqslant d_{r+1}$. For $d' = (d_2, \ldots, d_r, d_{r+1}, d_r, \ldots, d_2)$ is OK. Let d_1 be even. If $d_1 = d_{r+1}$ then the diagram $L_{\text{odd}}(d)$ is the same as $L_h(d)$, the claim follows immediately. If $d_1 < d_{r+1}$, the diagram $L_{\text{odd}}(d)$ is obtained from $L_{\text{odd}}(d')$ by extending d_1 $(2r-2)$-chains to $2r$-chains. The kernels of the map X^k satisfy $\dim\ker X^k = \dim\ker Y^k$ for $k \leqslant 2r-1$, $\dim\ker X^{2r} = 2n - d_1 = \dim\ker Y^{2r} + d_1$ and $\dim\ker X^{2r+1} = 2n = \dim\ker Y^{2r+1} + 2d_1$ where $Y \in \mathfrak{sp}_{2n-2d_1}$ is defined by the line diagram $L_{\text{even}}(d')$. If the partition of Y is $1^{b_1}, 2^{b_2}, \ldots, (2r-1)^{b_{2r-1}}$ then the partition of X is

$$1^{b_1}, \ldots, (2r-2)^{b_{2r-2}}, (2r-1)^{b_{2r-1}-d_1}, (2r)^0, (2r+1)^{d_1}.$$

Thus the dual of this partition is the dual of the partition of Y together with the parts d_1, d_1.

If d_1 is even and $d_1 > d_{r+1}$, the diagram $L_{\text{odd}}(d)$ is obtained from $L_{\text{odd}}(d')$ by extending d_{r+1} $(2r-2)$-chains to $2r$-chains and by extending $d_1 - d_{r+1}$ $(2r-3)$-chains to $(2r-1)$-chains. Here we get $\dim\ker X^k = \dim\ker Y^k$ for $k \leqslant 2r-2$, $\dim\ker X^{2r-1} = \dim\ker Y^{2r-1} + d_1 - d_{r+1}$, $\dim\ker X^{2r} = 2n - dr + 1 = \dim\ker Y^{2r} + 2d_1 - d_{r+1}$ and $\dim\ker X^{2r+1} = 2n = \dim\ker Y^{2r+1} + 2d_1$. So the partition of X can be calculated to be

$$1^{b_1}, \ldots, (2r-3)^{b_{2r-3}}, (2r-2)^{b_{2r-2}-d_1+d_{r+1}}, (2r-1)^{b_{2r-1}-d_{r+1}}, (2r)^{d_1-d_{r+1}}, (2r+1)^{d_{r+1}}$$

with $b_{2r-1} = d_{r+1}$. Again, the dual of the partition of X is obtained from the dual of the partition of Y by adding d_1, d_1.

Let d_1 be odd and $d_1 > d_{r+1}$. In particular, there are no odd d_i that are smaller than d_{r+1}. The shape of $L_{\text{odd}}(d)$ is the same as the diagram for $\mathfrak{sl}_{2n}$ (i.e. they have the same chain lengths). So the dual of the partition is just the dimension vector and we are done. If $d_1 < d_{r+1}$, the diagram $L_{\text{odd}}(d)$ is obtained from $L_{\text{odd}}(d')$ by extending $d_1 - 1$ $(2r-2)$-chains to $2r$-chains and by extending two $(2r-2)$-chains to $(2r-1)$-chains. The calculations of the dimensions of the kernels for X (compared to those for Y) give as partition of X:

$$1^{b_1}, \ldots, (2r-2)^{b_{2r-2}}, (2r-1)^{b_{2r-1}-d_1-1}, (2r)^2, (2r+1)^{d_1-1}.$$

Hence the dual of the partition of X is obtained from the dual of the partition of Y by adjoining $d_1 + 1, d_1 - 1$. □

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