FRT-duals as quantum enveloping algebras

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Abstract

The Hopf algebra generated by the l-functionals on the quantum double $\mathbb{C}_q[G] \bowtie \mathbb{C}_q[G]$ is considered, where $\mathbb{C}_q[G]$ is the coordinate algebra of a standard quantum group and q is not a root of unity. It is shown to be isomorphic to $\mathbb{C}_q[G]^{\mathrm{op}} \bowtie U_q(\mathfrak{g})$. This proves a conjecture by T. Hodges. As an algebra it can be embedded into $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$. Here it is proven that there is no bialgebra structure on $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$, for which this embedding becomes a homomorphism of bialgebras. In particular, it is not an isomorphism.

As a preliminary a lemma of Hodges concerning the structure of l-functionals on $\mathbb{C}_q[G]$ is generalized. For the classical groups a certain choice of root vectors is expressed in terms of l-functionals. A formula for their coproduct is derived.
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1. Overview

Let $\mathcal{A}$ be a coquasitriangular Hopf algebra with universal r-form r and let $\mathcal{U}(\mathcal{A})$ be the Hopf subalgebra of the Hopf dual $\mathcal{A}^\circ$ generated by the set of all l-functionals $l^+(a) := r(\cdot \otimes a)$, $l^-(a) := \bar{r}(a \otimes \cdot)$, $a \in \mathcal{A}$. We call it the FRT-dual of $\mathcal{A}$ as it was suggested in [Ho]. There it was shown (the finite-dimensional case is treated already in [Ma]) that there exists an injective algebra homomorphism

$$\iota : \mathcal{U}(\mathcal{A} \bowtie \mathcal{A}) \to \mathcal{U}(\mathcal{A}) \otimes \mathcal{U}(\mathcal{A})$$

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and a surjective Hopf algebra homomorphism

$$\zeta : \mathcal{A}^{\mathrm{op}} \bowtie \mathcal{U}(\mathcal{A}) \to \mathcal{U}(\mathcal{A} \bowtie \mathcal{A}).$$

Here $\mathcal{A} \bowtie \mathcal{B}$ denotes the quantum double of the skew-paired Hopf algebras $\mathcal{A}$ and $\mathcal{B}$. The skew-pairing of $\mathcal{A}$ and $\mathcal{A}$ in $\mathcal{A} \bowtie \mathcal{A}$ is the universal r-form r and the skew-pairing of $\mathcal{U}(\mathcal{A})$ and $\mathcal{A}^{\mathrm{op}}$ in $\mathcal{A}^{\mathrm{op}} \bowtie \mathcal{U}(\mathcal{A})$ is the restriction of the canonical pairing of $\mathcal{A}^{\circ}$ and $\mathcal{A}$. The universal r-form on $\mathcal{A} \bowtie \mathcal{A}$ used to define $\mathcal{U}(\mathcal{A} \bowtie \mathcal{A})$ is $\hat{r} := \bar{r}_{41} \bar{r}_{31} r_{24} r_{23}$ (see Section 2 for details).

In this paper we continue the investigation of these maps under the assumption that $\mathcal{A}$ is the coordinate algebra $\mathbb{C}_q[G]$ of a standard quantum group associated to a connected complex semi-simple Lie group G and q is not a root of unity. The main results are the following facts:

1. As conjectured in [Ho] ζ is an isomorphism in this case (Theorem 1).
2. There exists no bialgebra structure on $\mathcal{U}(\mathbb{C}_q[G]) \otimes \mathcal{U}(\mathbb{C}_q[G])$ such that ι becomes a bialgebra homomorphism (Theorem 2).
   In particular, ι is not an isomorphism (Corollary 1).

We retain the definition of the quantum enveloping algebra $U_q(\mathfrak{g})$ from [Ho]. It differs from the usual one by an extension of the Cartan part. Then $U_q(\mathfrak{g})$ can be identified with $\mathcal{U}(\mathbb{C}_q[G])$, if q is not a root of unity (Proposition 2). Explicitly, one has

$$l^+\big(c^\lambda_{-\mu,\nu}\big) = f^+\big(c^\lambda_{-\mu,\nu}\big) K_\mu, \qquad l^-\big(c^\lambda_{-\mu,\nu}\big) = f^-\big(c^\lambda_{-\mu,\nu}\big) K_{-\nu}$$

for some $f^\pm(c^\lambda_{-\mu,\nu}) \in U_q(\mathfrak{n}_\pm)$ (Proposition 3). Here $c^\lambda_{-\mu,\nu} \in \mathbb{C}_q[G]$ are matrix coefficients of the irreducible representation of $U_q(\mathfrak{g})$ with heighest weight λ. If G is one of the classical Lie groups, then there is a choice of the longest word of the Weyl group of $\mathfrak{g}$, such that the corresponding root vectors of $U_q(\mathfrak{g})$ occur in the above formula as $f^\pm(c^\lambda_{-\mu,\nu})$ for some $c^\lambda_{-\mu,\nu}$ (Proposition 5). As a corollary one obtains a formula for their coproduct (Corollary 2).

There are at least two interpretations of the algebra $\mathbb{C}_q[G] \bowtie \mathbb{C}_q[G]$.

For arbitrary q it is a nonstandard deformation of $\mathbb{C}[G \times G]$. In [Ho] it is therefore denoted by $\mathbb{C}_q[D(G)]$ where $D(G)$ stands for the double group $G \times G$.

If q is real, it becomes a Hopf $*$-algebra which is a deformation of the algebra of polynomial functions in holomorphic and antiholomorphic coordinates on G and describes G as a real Lie group. It appeared first in this role in the q-deformation of the action of $SL(2)$ on Minkowski space, see [PW] or [CSSW].

Many authors proposed definitions of a quantum enveloping algebra corresponding to $\mathbb{C}_q[G] \bowtie \mathbb{C}_q[G]$, in particular, of a q-Lorentz algebra. All are based on the requirement that it should be a Hopf algebra dually paired with $\mathbb{C}_q[G] \bowtie \mathbb{C}_q[G]$. One direct approach to such a Hopf algebra is a dualization of the structure of $\mathbb{C}_q[G] \bowtie \mathbb{C}_q[G]$ in form of a quantum codouble $U_q(\mathfrak{g}) \blacktriangleright\!\!\blacktriangleleft U_q(\mathfrak{g})$. It is shown in [Ma] that ι would be a Hopf algebra homomorphism into such a quantum codouble. Hence it cannot be well-defined by Theorem 2.

In view of the isomorphism $\mathcal{U}(\mathbb{C}_q[G]) \simeq U_q(\mathfrak{g})$ it seems reasonable to consider the FRT-dual $\mathcal{U}(\mathbb{C}_q[G] \bowtie \mathbb{C}_q[G])$ as a rigorously defined alternative. Theorem 1 is then a dual and purely algebraic form of the Iwasawa decomposition introduced in [PW] for the C^*-completion of $\mathbb{C}_q[G] \bowtie \mathbb{C}_q[G]$. Note that the images $\operatorname{Im} \hat{l}^+ \simeq \mathbb{C}_q[G]^{\mathrm{op}}$ and $\operatorname{Im} \hat{l}^- \simeq U_q(\mathfrak{g})$ of the l-functionals on $\mathbb{C}_q[G] \bowtie \mathbb{C}_q[G]$ determine the Iwasawa decomposition.

The rest of this paper is divided into three sections: In order to be self-contained and to fix notations we first recall mainly from [Ho] and [Ma] some facts about $\mathcal{A} \bowtie \mathcal{A}$ and $\mathcal{U}(\mathcal{A} \bowtie \mathcal{A})$ for an arbitrary coquasitriangular Hopf algebra $\mathcal{A}$. In the second section we focus on quantum groups and prove the main results. The last one deals with the relation between l-functionals on $\mathbb{C}_q[G]$ and root vectors of $U_q(\mathfrak{g})$

We essentially retain the notations and conventions from [Ho]. We will freely use material that can be found in standard textbooks such as [Mo].

In the original version of this paper only the classical groups were treated. The author would like to thank T. Hodges and the referee for pointing out that the proof of Theorem 1 works with a minor modification for arbitrary semi-simple groups. They also noticed that the well-definedness of $U_q(\mathfrak{g}) \bowtie U_q(\mathfrak{g})$ was an open problem until now.

2. The Hopf algebras $\mathcal{A} \bowtie \mathcal{A}$ and $\mathcal{U}(\mathcal{A} \bowtie \mathcal{A})$

Let $\mathcal{A}$ be a coquasitriangular Hopf algebra with universal r-form r. Then the quantum double $\mathcal{A} \bowtie \mathcal{A}$ is a Hopf algebra which is the tensor product coalgebra $\mathcal{A} \otimes \mathcal{A}$ endowed with the product

$$(a \otimes b)(c \otimes d) := (ac_{(2)} \otimes b_{(2)}d)\bar{r}(b_{(1)} \otimes c_{(1)})r(b_{(3)} \otimes c_{(3)}).$$

Here $\bar{r}$ denotes the convolution inverse of r and we use Sweedlers notation for the coproduct on the right-hand side. The antipode of $\mathcal{A} \bowtie \mathcal{A}$ is given by $S(a \otimes b) := (1 \otimes S(b))(S(a) \otimes 1)$. See [Ho,Ma] or [KS1] for more information about coquasitriangular Hopf algebras and quantum doubles.

Let $\mathcal{U}(\mathcal{A})$ be the Hopf subalgebra of the Hopf dual $\mathcal{A}^\circ$ generated by the set of all l-functionals

$$l^+(a) := r(\cdot \otimes a), \quad l^-(a) := \bar{r}(a \otimes \cdot), \qquad a \in \mathcal{A}.$$

Following the terminology from [Ho] we call $\mathcal{U}(\mathcal{A})$ the FRT-dual of $\mathcal{A}$.

If r is a universal r-form on $\mathcal{A}$, then $\bar{r}_{21}$ is a universal r-form as well. Note that some formulas in [Ho] differ from those in this paper because there the latter r-form is used.

The Hopf algebra $\mathcal{A} \bowtie \mathcal{A}$ is again coquasitriangular. We define its FRT-dual $\mathcal{U}(\mathcal{A} \bowtie \mathcal{A})$ with respect to the universal r-form $\hat{r} := \bar{r}_{41}\bar{r}_{31}r_{24}r_{23}$, that is,

$$
\begin{aligned}
\hat{r}&\big((a \otimes b) \otimes (c \otimes d)\big) \\
&= \bar{r}(d_{(1)} \otimes a_{(1)})\bar{r}(c_{(1)} \otimes a_{(2)})r(b_{(1)} \otimes d_{(2)})r(b_{(2)} \otimes c_{(2)}) \\
&= \bar{r}(c_{(1)}d_{(1)} \otimes a)r(b \otimes c_{(2)}d_{(2)}).
\end{aligned}
\tag{1}
$$

Consider now the linear maps

$$\theta : \mathcal{A} \bowtie \mathcal{A} \to \mathcal{U}(\mathcal{A}), \quad a \otimes b \mapsto l^+\big(S^{-1}(a)\big)l^-\big(S^{-1}(b)\big)$$

and

$$m : \mathcal{A} \bowtie \mathcal{A} \to \mathcal{A}, \quad a \otimes b \mapsto ab.$$

Recall that the antipode of a coquasitriangular Hopf algebra is always bijective, so θ is well-defined.

Using the formulas $\bar{r}(a \otimes b) = r(S(a) \otimes b)$, $r(a \otimes b) = r(S(a) \otimes S(b))$ (see, e.g., [KS1, Proposition 10.2]) and the fact that the coproduct is an algebra homomorphism one gets

$$
\begin{aligned}
\hat{r}\big((a \otimes b) \otimes (c \otimes d)\big) &= \bar{r}(c_{(1)}d_{(1)} \otimes a)r(b \otimes c_{(2)}d_{(2)}) \\
&= \big\langle l^+\big(S^{-1}(a)\big), (cd)_{(1)}\big\rangle \big\langle l^-\big(S^{-1}(b)\big), (cd)_{(2)}\big\rangle \\
&= \big\langle \theta(a \otimes b), m(c \otimes d)\big\rangle.
\end{aligned}
$$

For the convolution inverse $\bar{\hat{r}} = \bar{r}_{23}\bar{r}_{24}r_{31}r_{41}$ of $\hat{r}$ one obtains similarly

$$\bar{\hat{r}}\big((a \otimes b) \otimes (c \otimes d)\big) = \big\langle S^{-1}\big(\theta(a \otimes b)\big), m(c \otimes d)\big\rangle.$$

We denote the l-functionals of $\mathcal{U}(\mathcal{A} \bowtie \mathcal{A})$ by $\hat{l}^{\pm}$. The preceding equations imply

$$\hat{l}^+ = \theta^\circ \circ m, \qquad \hat{l}^- = m^\circ \circ S^{-1} \circ \theta,$$

where $\theta^\circ : \mathcal{A} \to (\mathcal{A} \bowtie \mathcal{A})^\circ$ and $m^\circ : \mathcal{U}(\mathcal{A}) \to (\mathcal{A} \bowtie \mathcal{A})^\circ$ are linear maps dual to θ and m in the sense

$$\big\langle \theta^\circ(a), b \otimes c\big\rangle := \big\langle \theta(b \otimes c), a\big\rangle, \qquad \big\langle m^\circ(f), a \otimes b\big\rangle := \langle f, ab\rangle.$$

In particular, the images of $\hat{l}^+$ and $\hat{l}^-$ are contained in those of θ° and m°, respectively. The map m is obviously surjective. But $S^{-1} \circ \theta : a \otimes b \mapsto l^-(b)l^+(a)$ is also surjective by the definition of $\mathcal{U}(\mathcal{A})$. Hence one even has

$$\operatorname{Im}\hat{l}^+ = \operatorname{Im}\theta^\circ, \qquad \operatorname{Im}\hat{l}^- = \operatorname{Im}m^\circ.$$

The definition of $\mathcal{U}(\mathcal{A} \bowtie \mathcal{A})$ now implies that the linear map

$$\zeta : \mathcal{A} \otimes \mathcal{U}(\mathcal{A}) \to \mathcal{U}(\mathcal{A} \bowtie \mathcal{A}), \quad \zeta(a \otimes f) := \theta^\circ(a)m^\circ(f)$$

is surjective. It is proven in [Ho] that ζ becomes a Hopf algebra homomorphism, if one considers $\mathcal{A} \otimes \mathcal{U}(\mathcal{A})$ with the Hopf structure $\mathcal{A}^{\mathrm{op}} \bowtie \mathcal{U}(\mathcal{A})$. Here $\mathcal{A}^{\mathrm{op}}$ denotes the opposite algebra of $\mathcal{A}$ and the quantum double is constructed with respect to the canonical pairing of $\mathcal{U}(\mathcal{A})$ and $\mathcal{A}$.

To avoid further notations we will use the product, coproduct and antipode of $\mathcal{A}$ to express those of $\mathcal{A}^{\mathrm{op}}$. So the product of $a, b \in \mathcal{A}^{\mathrm{op}}$ is ba and the coproduct and the antipode of $\mathcal{A}^{\mathrm{op}}$ are Δ and S^{-1}, respectively.

It is also shown in [Ho] that the map

$$\iota : \mathcal{U}(\mathcal{A} \bowtie \mathcal{A}) \to \mathcal{A}^{\circ} \otimes \mathcal{A}^{\circ}, \quad f \mapsto \langle f_{(1)}, (\cdot \otimes 1) \rangle \otimes \langle f_{(2)}, (1 \otimes \cdot) \rangle \tag{2}$$

is an embedding of algebras and that $\iota \circ m^{\circ} = \Delta$ (the coproduct in $\mathcal{U}(\mathcal{A})$) and $\iota \circ \theta^{\circ} = (l^{-} \otimes l^{+}) \circ \Delta$. In particular, $\mathrm{Im}\, \iota \subset \mathcal{U}(\mathcal{A}) \otimes \mathcal{U}(\mathcal{A})$.

If $\mathcal{A}$ is finite-dimensional, then any universal r-form r is simultaneously a universal R-matrix R for the dual Hopf algebra $\mathcal{A}^{\circ}$ which therefore is quasitriangular. This R-matrix can be used to form a quantum codouble $\mathcal{A}^{\circ} \bowtie \mathcal{A}^{\circ}$ of two copies of $\mathcal{A}^{\circ}$, see [Ma]. Its structure is completely dual to that of $\mathcal{A} \bowtie \mathcal{A}$—it is the tensor product algebra $\mathcal{A}^{\circ} \otimes \mathcal{A}^{\circ}$ with a twisted coproduct

$$\Delta(a \otimes b) := a_{(1)} \otimes R(b_{(1)} \otimes a_{(2)}) R^{-1} \otimes b_{(2)}. \tag{3}$$

The map ι becomes a Hopf algebra homomorphism into $\mathcal{A}^{\circ} \bowtie \mathcal{A}^{\circ}$. If $\mathcal{A}$ is in addition factorizable, then both ι and ζ are isomorphisms [Ma, Theorem 7.3.5]. As we will see in the next section, there is no way to define the above coproduct in a rigorous way for arbitrary coquasitriangular Hopf algebras $\mathcal{A}$.

If $\mathcal{A}$ is a Hopf $*$-algebra and r is of real type, i.e., $r(a^* \otimes b^*) = \overline{r(b \otimes a)}$, then $\mathcal{A} \bowtie \mathcal{A}$ is a Hopf $*$-algebra with involution defined by $(a \otimes b)^* := b^* \otimes a^*$ ([Ma, Section 7.3], [KS1, Section 10.2.7]). This applies to the case of the coordinate algebras $\mathbb{C}_q[G]$ treated in the next section if q is real. The involution on $\mathbb{C}_q[G]$ is the unique one, for which the pairing with the compact real form of $U_q(\mathfrak{g})$ [KS1, Section 6.1.7] is a pairing of Hopf $*$-algebras. Then there is a Hopf algebra embedding $a \mapsto 1 \otimes a$ of $\mathcal{A}$ into $\mathcal{A} \bowtie \mathcal{A}$ and any element of $\mathcal{A} \bowtie \mathcal{A}$ can be written uniquely as a^*b with $a, b \in \mathcal{A} \subset \mathcal{A} \bowtie \mathcal{A}$. One says that $\mathcal{A} \bowtie \mathcal{A}$ is a realification of $\mathcal{A}$ (in [Ma] it is called a complexification). There is an involution on $\mathcal{A}^{\mathrm{op}} \bowtie \mathcal{U}(\mathcal{A})$ defined by

$$(a \otimes f)^* := (1 \otimes f^*)\big(S^2(a)^* \otimes 1\big),$$

for which $\mathcal{A}^{\mathrm{op}} \bowtie \mathcal{U}(\mathcal{A})$ becomes a Hopf $*$-algebra and ζ a $*$-homomorphism [Ma, Proposition 7.1.4 and Theorem 7.3.5].

3. Application to quantum groups

We now specialize the preceding considerations to the case where $\mathcal{A}$ is the coordinate algebra of a standard quantum group.

Throughout this section G denotes a connected complex semi-simple Lie group with Lie algebra $\mathfrak{g}$, $\{\alpha_i\}_{i=1,\ldots,N}$ a set of simple roots of $\mathfrak{g}$, $\mathbf{Q} := \sum_{i=1}^{N} \mathbb{Z}\alpha_i$ the root lattice and $\mathbf{L}$ the character group of a maximal torus of G which we identify with a sublattice of the weight lattice of $\mathfrak{g}$ containing $\mathbf{Q}$. For $\lambda, \mu \in \mathbf{L}$ we set $\mu < \lambda$ iff $\lambda - \mu$ is a sum of positive roots.

Furthermore, $\langle \cdot , \cdot \rangle$ denotes the scalar product on $\mathbf{L}$ satisfying $\langle \alpha_i , \alpha_j \rangle = d_i a_{ij}$, where a_{ij} and $d_i a_{ij}$ are the entries of the Cartan matrix and the symmetrized Cartan matrix of $\mathfrak{g}$, respectively.

We retain the convention from [Ho], where the quantum enveloping algebra $U_q(\mathfrak{g})$ has generators K_λ, E_i, F_j, $\lambda \in \mathbf{L}$, $i, j = 1, \dots , N$, fulfilling the relations

$$K_\lambda K_\mu = K_{\lambda + \mu}, \quad [E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}}, \quad K_i := K_{\alpha_i},$$

$$K_\lambda E_i K_\lambda^{-1} = q^{\langle \lambda, \alpha_i \rangle} E_i, \quad K_\lambda F_j K_\lambda^{-1} = q^{-\langle \lambda, \alpha_j \rangle} F_j$$

and the q-Serre relations [KS1, Eqs. (6.8), (6.9)]. The parameter $q \in \mathbb{C} \backslash \{0\}$ is assumed to be not a root of unity. See Eqs. (6.19), (6.20) in [KS1] for the definition of the coproduct, the counit and the antipode of $U_q(\mathfrak{g})$.

Note that $U_q(\mathfrak{g})$ as used here is not the most common one, where K_λ is defined only for $\lambda \in \mathbf{Q}$. It depends on the choice of G. For simply connected G it coincides with $\check{U}$ from [Jo] and for the classical groups with $U_q^{\text{ext}}(\mathfrak{g})$ from [KS1]. There is a $\mathbf{Q}$-grading on $U_q(\mathfrak{g})$ given by

$$U_q(\mathfrak{g}) = \bigoplus_{\lambda \in \mathbf{Q}} U_q^\lambda(\mathfrak{g}), \quad U_q^\lambda(\mathfrak{g}) := \big\{ f \in U_q(\mathfrak{g}) \mid K_\mu f K_\mu^{-1} = q^{\langle \lambda, \mu \rangle} f \ \forall \mu \in \mathbf{L} \big\}.$$

Let $U_q(\mathfrak{h})$, $U_q(\mathfrak{n}_+)$, $U_q(\mathfrak{n}_-)$ be the subalgebras generated by the K_λ, E_i, and F_j, respectively. Setting $U_q^\lambda(\mathfrak{n}_\pm) := U_q^\lambda(\mathfrak{g}) \cap U_q(\mathfrak{n}_\pm)$ we have [Ja, Lemma 4.12]:

Proposition 1. *For* $f \in U_q^\lambda(\mathfrak{n}_+)$ *and* $g \in U_q^\lambda(\mathfrak{n}_-)$ *there are* $f_i' \in U_q^{\mu_i}(\mathfrak{n}_+)$, $f_i'' \in U_q^{\lambda - \mu_i}(\mathfrak{n}_+)$, $0 < \mu_i < \lambda$, *and* $g_j' \in U_q^{\lambda - \nu_j}(\mathfrak{n}_-)$, $g_j'' \in U_q^{\nu_j}(\mathfrak{n}_-)$, $0 > \nu_j > \lambda$, *such that*

$$\Delta(f) = f \otimes K_\lambda + \sum_i f_i' \otimes f_i'' K_{\mu_i} + 1 \otimes f,$$

$$\Delta(g) = g \otimes 1 + \sum_j g_j' K_{\nu_j} \otimes g_j'' + K_\lambda \otimes g.$$

Proof. We can assume without loss of generality that $f = E_{i_1} \cdots E_{i_k}$ and $g = F_{i_1} \cdots F_{i_k}$ with $\pm \lambda = \alpha_{i_1} + \cdots + \alpha_{i_k}$, because $U_q^\lambda(\mathfrak{n}_\pm)$ is spanned by such monomials. The proof is now an easy induction on k. See [Ja] for the details. $\quad \square$

Let W be the Weyl group of $\mathfrak{g}$ generated by the reflections $r_i : \alpha_j \mapsto \alpha_j - a_{ij} \alpha_i$. Let E_{β_k}, F_{β_k}, $k = 1, \dots , n$, be the root vectors of $U_q(\mathfrak{g})$ (see [KS1, Section 6.2.3]) associated to the ordering $\beta_k := r_{i_1} r_{i_2} \cdots r_{i_{k-1}} \alpha_{i_k}$ of the set $\mathbf{R}^+$ of positive roots, where $r_{i_1} r_{i_2} \cdots r_{i_n}$ is a reduced expression of the longest element of W. Then by the Poincaré–Birkhoff–Witt (PBW) theorem the following monomials form a vector space basis of $U_q(\mathfrak{g})$:

$$K_\lambda F_{\mathbf{i}} E_{\mathbf{j}} := K_\lambda F_{\beta_1}^{i_1} \cdots F_{\beta_n}^{i_n} E_{\beta_1}^{j_1} \cdots E_{\beta_n}^{j_n}, \quad \lambda \in \mathbf{L}, \ \mathbf{i}, \mathbf{j} \in \mathbb{N}_0^n.$$

The coordinate algebra $\mathbb{C}_q[G]$ of the standard quantum group associated to G is the Hopf subalgebra of $U_q(\mathfrak{g})^\circ$ spanned by the functionals $c_{u,v}(f) := u(fv)$, $f \in U_q(\mathfrak{g})$, where v is a vector in the irreducible representation of $U_q(\mathfrak{g})$ with heighest weight $\lambda \in \mathbf{L}$ and u is a vector in the dual representation, see [Ho]. If $\{u_n\}$, $\{v_n\}$ is a pair of dual bases in the representation space and its dual, then $\Delta(c_{u,v}) = \sum_n c_{u,v_n} \otimes c_{u_n,v}$. If u, v are weight vectors possessing weights $-\mu$, ν, then $c_{u,v}$ is denoted by $c^\lambda_{-\mu,\nu}$ as well.

If G is simply connected, $\mathbb{C}_q[G]$ equals $R_q[G]$ from [Jo]. The relation with $\mathcal{O}(G_q)$ from [KS1] will be discussed in the next section.

The Hopf algebras $\mathbb{C}_q[G]$ are all coquasitriangular. A universal r-form β is derived in [Ho] from the Rosso form of $U_q(\mathfrak{g})$. To be compatible with [FRT] we use $r := \bar{\beta}_{21}$. This simply exchanges l^+ and l^-. It follows from the construction of r that the l-functionals on $\mathbb{C}_q[G]$ can be identified with elements of $U_q(\mathfrak{g})$. That is, there is a Hopf algebra embedding of $\mathcal{U}(\mathbb{C}_q[G])$ into $U_q(\mathfrak{g})$, By Proposition 4.6 in [HLT] this embedding is in fact surjective. We therefore have:

Proposition 2. *There is an isomorphism* $\mathcal{U}(\mathbb{C}_q[G]) \simeq U_q(\mathfrak{g})$.

This was used tacitly in [Ho]. In what follows, we will not distinguish between $\mathcal{U}(\mathbb{C}_q[G])$ and $U_q(\mathfrak{g})$ any more.

In [Ho] the following description of $\operatorname{Ker} l^\pm$ was given:

$$c_{u,v} \in \operatorname{Ker} l^\pm \quad \Leftrightarrow \quad u\big(U_q(\mathfrak{b}_\mp)v\big) = 0. \tag{4}$$

We will use it to prove the next proposition. It generalizes Lemma 3.3 in [Ho].

Proposition 3. *For* $c^\lambda_{-\mu,\nu} \in \mathbb{C}_q[G]$ *there are* $f^\pm(c^\lambda_{-\mu,\nu}) \in U_q^{\nu-\mu}(\mathfrak{n}_\pm)$ *with*

$$l^+\big(c^\lambda_{-\mu,\nu}\big) = f^+\big(c^\lambda_{-\mu,\nu}\big)K_\mu, \qquad l^-\big(c^\lambda_{-\mu,\nu}\big) = f^-\big(c^\lambda_{-\mu,\nu}\big)K_{-\nu}.$$

Proof. We treat only l^+, the other case is analogous. Let $c^\lambda_{-\mu,\nu} = c_{u,v}$ be given. Fix dual bases $\{u_n\}$, $\{v_n\}$ as above consisting of weight vectors with weights $-\nu_n$, ν_n, such that v is one of the v_n. Let v' be a heighest weight vector and $c^\lambda_{-\mu,\lambda} = c_{u,v'}$. Since l^+ is a coalgebra homomorphism, we have

$$\Delta\big(l^+\big(c^\lambda_{-\mu,\lambda}\big)\big) = \sum_n l^+\big(c^\lambda_{-\mu,\nu_n}\big) \otimes l^+\big(c^\lambda_{-\nu_n,\lambda}\big). \tag{5}$$

It is known that the proposition holds for $\nu = \lambda$ [Ho, Lemma 3.3], so

$$l^+\big(c^\lambda_{-\mu,\lambda}\big) = f^+\big(c^\lambda_{-\mu,\lambda}\big)K_\mu, \qquad l^+\big(c^\lambda_{-\nu_n,\lambda}\big) = f^+\big(c^\lambda_{-\nu_n,\lambda}\big)K_{\nu_n}. \tag{6}$$

By the first equality and Proposition 1 we can express $\Delta(l^+(c^\lambda_{-\mu,\lambda}))$ also as

$$f^+\big(c^\lambda_{-\mu,\lambda}\big)K_\mu \otimes K_\lambda + \sum_i f_i' K_\mu \otimes f_i'' K_{\xi_i+\mu} + K_\mu \otimes f^+\big(c^\lambda_{-\mu,\lambda}\big)K_\mu \tag{7}$$

with $f_i' \in U_q^{\xi_i}(\mathfrak{n}_+)$, $f_i'' \in U_q^{\lambda-\mu-\xi_i}(\mathfrak{n}_+)$, $0 < \xi_i < \lambda - \mu$. If one compares the $U_q(\mathfrak{h})$-parts of the terms in (5) and (7) in the second tensor component, one gets by the second equality in (6) and the PBW theorem

$$\sum_j l^+\big(c^\lambda_{-\mu,\nu_{n_j}}\big) \otimes l^+\big(c^\lambda_{-\nu_{n_j},\lambda}\big) = \sum_k f_{i_k}' K_\mu \otimes f_{i_k}'' K_{\xi_{i_k}+\mu},$$

where the indices n_j and i_k are those with $\nu_{n_i} = \xi_{i_k} + \mu = \nu$.

We claim that the elements $l^+(c^\lambda_{-\nu_n,\lambda})$ are linearly independent. Indeed, assume that there are $x_n \in \mathbb{C}$ with $\sum_n x_n l^+(c_{u_n,\nu'}) = l^+(c_{\sum_n x_n u_n,\nu'}) = 0$. Since ν' is a heighest weight vector, (4) implies $\sum_n x_n u_n = 0$. Hence $x_n = 0$ for all n, because $\{u_n\}$ is a basis. It follows that all $l^+(c^\lambda_{-\mu,\nu_{n_j}})$ are linear combinations of $f_{i_k}' K_\mu$. The considered $l^+(c^\lambda_{-\mu,\nu})$ is one of them, so the proposition follows. $\quad\square$

Now we are ready to prove the main theorem.

Theorem 1. *There is an isomorphism of Hopf algebras*

$$\mathcal{U}\big(\mathbb{C}_q[G] \bowtie \mathbb{C}_q[G]\big) \simeq \mathbb{C}_q[G]^{\mathrm{op}} \bowtie U_q(\mathfrak{g}).$$

Proof. It suffices to prove the injectivity of the epimorphism ζ described in Section 2. We prove that $\zeta' := \iota \circ \zeta : \mathbb{C}_q[G]^{\mathrm{op}} \bowtie U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ with ι from (2) is injective.

Suppose $f \in \operatorname{Ker}\zeta'$, $f = \sum_{\lambda \in \mathbf{L}, \mathbf{i}, \mathbf{j} \in \mathbb{N}_0^n} a_{\lambda\mathbf{ij}} \otimes K_\lambda F_\mathbf{i} E_\mathbf{j}$ with $a_{\lambda\mathbf{ij}} = 0$ for almost all $\lambda\mathbf{ij}$. We have to show that f vanishes.

Order $\mathbb{N}_0^n$ in such a way that the weights $\mu_\mathbf{j}$ of $E_\mathbf{j}$ form a nondecreasing (with respect to $<$) sequence. Let $\mathbf{j}_0$ be the maximal $\mathbf{j}$ for which there exists an $a_{\lambda\mathbf{ij}} \neq 0$. Recall that $\iota \circ m^\circ = \Delta$ and $\iota \circ \theta^\circ = (l^- \otimes l^+) \circ \Delta$. Set $U_q(\mathfrak{b}_\pm) := U_q(\mathfrak{h}) U_q(\mathfrak{n}_\pm)$ and note that by Propositions 3 and 1 we have

$$(l^- \otimes l^+) \circ \Delta(a_{\lambda\mathbf{ij}}) \in U_q(\mathfrak{b}_-) \otimes U_q(\mathfrak{b}_+), \qquad \Delta(K_\lambda F_\mathbf{i}) \in U_q(\mathfrak{b}_-) \otimes U_q(\mathfrak{b}_-).$$

Hence only $\Delta(E_\mathbf{j})$ contribute to the $U_q(\mathfrak{n}_+)$-part in the first tensor component. Expand them according to Proposition 1. Then the PBW theorem implies that

$$\sum_{\lambda\mathbf{i}} (l^- \otimes l^+) \circ \Delta(a_{\lambda\mathbf{ij}_0}) \cdot \Delta(K_\lambda F_\mathbf{i}) \cdot (E_{\mathbf{j}_0} \otimes K_{\mu_{\mathbf{j}_0}})$$

is linearly independent from the other terms occurring in $\zeta'(f)$ and vanishes separately. Since $U_q(\mathfrak{g}) \oplus U_q(\mathfrak{g}) = U_q(\mathfrak{g} \otimes \mathfrak{g})$ is free of zero divisors [DK, Corollary 1.8], we get

$$\sum_{\lambda\mathbf{i}} (l^- \otimes l^+) \circ \Delta(a_{\lambda\mathbf{ij}_0}) \Delta(K_\lambda F_\mathbf{i}) = 0.$$

The same argument applied to the maximal $\mathbf{i}_0$ and the second tensor component shows

$$\sum_\lambda (l^- \otimes l^+) \circ \Delta(a_{\lambda \mathbf{i}_0 \mathbf{j}_0}) \cdot \Delta(K_\lambda) = 0.$$

By Proposition 3 we can write $(l^- \otimes l^+) \circ \Delta(a_{\lambda \mathbf{i}_0 \mathbf{j}_0})$ as $\sum_{\xi \in \mathbf{L}} u_{\lambda \xi} K_\xi \otimes v_{\lambda \xi} K_{-\xi}$ for some $u_{\lambda \xi} \otimes v_{\lambda \xi} \in U_q(\mathfrak{n}_-) \otimes U_q(\mathfrak{n}_+)$. Then the last equation becomes

$$\sum_{\xi \lambda} u_{\lambda \xi} K_{\xi + \lambda} \otimes v_{\lambda \xi} K_{-\xi + \lambda} = 0.$$

This implies $u_{\lambda \xi} \otimes v_{\lambda \xi} = 0$ for all λ, ξ. Finally

$$(l^- \otimes l^+) \circ \Delta(a_{\lambda \mathbf{i}_0 \mathbf{j}_0}) = 0$$

implies $a_{\lambda \mathbf{i}_0 \mathbf{j}_0} = 0$, because $(l^- \otimes l^+) \circ \Delta$ is injective by the definition of $\mathcal{U}(\mathbb{C}_q[G])$. Proceeding by induction with the lower $\mathbf{i}, \mathbf{j}$ the claim follows. □

In contrast to their h-adic counterparts $U_h(\mathfrak{g})$ defined over the ring of formal power series $\mathbb{C}[[h]]$, the Hopf algebras $U_q(\mathfrak{g})$ over $\mathbb{C}$ are not quasitriangular. Nevertheless, parts of the theory of $U_h(\mathfrak{g})$ carry over to $U_q(\mathfrak{g})$, since the l-functionals encode the R-matrix of $U_h(\mathfrak{g})$ to some extent. Hence it is not a priori clear that there is no way to define the twisted coproduct (3) as well on $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$. But we show now that this is in fact impossible.

Theorem 2. *There exists no bialgebra structure on $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ such that ι becomes a homomorphism of bialgebras.*

Proof. Suppose that the opposite holds. Then $\iota \circ \theta^\circ$ is a bialgebra homomorphism as well. Note that $\pm(\nu - \mu) \notin \sum_{i=1}^N \mathbb{N}_0 \alpha_i$ implies $l^\pm(c^\lambda_{-\mu, \nu}) = 0$ by (4). Using this and Proposition 3 one computes

$$\begin{aligned}
\Delta(K_\lambda \otimes K_{-\lambda}) &= \Delta \circ \iota \circ \theta^\circ (c^\lambda_{\lambda, -\lambda}) \\
&= (\iota \circ \theta^\circ \otimes \iota \circ \theta^\circ) \circ \Delta (c^\lambda_{\lambda, -\lambda}) \\
&= \sum_n K_\lambda \otimes f^+ (c^\lambda_{\lambda, \nu_n}) K_{-\lambda} \otimes f^- (c^\lambda_{-\nu_n, -\lambda}) K_\lambda \otimes K_{-\lambda}.
\end{aligned}$$

This must be an invertible element of $U_q(\mathfrak{g})^{\otimes 4}$, because Δ is an algebra homomorphism and $K_\lambda \otimes K_{-\lambda}$ is invertible. Since $K_\lambda \otimes K_{-\lambda} \otimes K_\lambda \otimes K_{-\lambda}$ is invertible, $\sum_n f^+(c^\lambda_{\lambda, \nu_n}) \otimes f^-(c^\lambda_{-\nu_n, -\lambda})$ is an invertible element of $U_q(\mathfrak{g})^{\otimes 2}$.

An invertible element of a graded algebra must be homogeneous—the product of the homogeneous components of heighest degrees n_0, m_0 of the element and its inverse must be of degree zero, so $m_0 = -n_0$, the same must hold for the components of lowest degrees n_1, m_1, so $m_1 = -n_1$ and $n_1 \leqslant n_0$ and $m_1 \leqslant m_0$ implies then $m_0 = m_1 = -n_0 = -n_1$.

By Proposition 3 $\sum_n f^+(c^\lambda_{\lambda,v_n}) \otimes f^-(c^\lambda_{-v_n,-\lambda})$ is not homogeneous with respect to the $\mathbf{Q} \times \mathbf{Q}$-grading of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$, so we obtain a contradiction. $\quad \square$

Corollary 1. *The map ι is not surjective.*

4. L-functionals and root vectors

The root vectors of $U_q(\mathfrak{g})$ are defined in terms of the action of the braid group of $\mathfrak{g}$ on $U_q(\mathfrak{g})$. Since this action is not given by coalgebra homomorphisms, it is not possible to compute their coproduct directly from their definition.

However, it is mentioned in [KS1] on p. 278 that for $G = SL(N + 1)$ there is a choice of $r_{i_1} r_{i_2} \cdots r_{i_n}$, such that the root vectors are certain $f^\pm(c^\lambda_{-\mu,v})$ from Proposition 3. This allows to compute their coproduct explicitly.

In this section we generalize this result to the other classical Lie groups. The main tool will be the following proposition:

Proposition 4. *For $i < j$ there are $x_{ij}(\mathbf{k}), y_{ij}(\mathbf{k}) \in \mathbb{C}$, such that*

$$
E_{\beta_i} E_{\beta_j} - q^{\langle \beta_i, \beta_j \rangle} E_{\beta_j} E_{\beta_i} = \sum_{\mathbf{k} \in \mathbb{N}_0^{j-i-1}} x_{ij}(\mathbf{k}) E_{\beta_{i+1}}^{k_1} \cdots E_{\beta_{j-1}}^{k_{j-i-1}}, \tag{8}
$$

$$
F_{\beta_i} F_{\beta_j} - q^{-\langle \beta_i, \beta_j \rangle} F_{\beta_j} F_{\beta_i} = \sum_{\mathbf{k} \in \mathbb{N}_0^{j-i-1}} y_{ij}(\mathbf{k}) F_{\beta_{i+1}}^{k_1} \cdots F_{\beta_{j-1}}^{k_{j-i-1}}. \tag{9}
$$

If

$$
\beta_i + \beta_j \neq \sum_{l=1}^{j-1} k_l \beta_{i+l},
$$

then $x_{ij}(\mathbf{k}) = y_{ij}(\mathbf{k}) = 0$.

Proof. The two relations (8), (9) are proven in [KS2, Theorem 3.2.3].
Conjugating (8) with K_λ one gets

$$
\sum_{\mathbf{k}} \left(q^{\langle \lambda, \beta_i + \beta_j \rangle} - q^{\langle \lambda, k_1 \beta_{i+1} + \cdots + k_{j-i-1} \beta_{j-1} \rangle} \right) x_{ij}(\mathbf{k}) E_{\beta_{i+1}}^{k_1} \cdots E_{\beta_{j-1}}^{k_{j-i-1}} = 0.
$$

The PBW theorem implies $q^{\langle \lambda, \beta_i + \beta_j \rangle} = q^{\langle \lambda, k_1 \beta_{i+1} + \cdots + k_{j-i-1} \beta_{j-1} \rangle}$ or $x_{ij}(\mathbf{k}) = 0$. Since q is not a root of unity and λ was arbitrary, the additional claim follows for the $x_{ij}(\mathbf{k})$. The same argument applies to the $y_{ij}(\mathbf{k})$. $\quad \square$

We will use a special ordering of the positive roots, in which most if not all terms on the right-hand side of (8), (9) vanish. To define it, we first arrange the positive roots in the following way as parts of matrices:

$$
\beta_{ij} = \begin{cases}
\sum_{k=i}^{j-1} \alpha_k & & \mathfrak{g} = \mathfrak{sl}_{N+1}, \\[2mm]
\sum_{k=i}^{j-1} \alpha_k & j \leqslant N+1 & \\[2mm]
\sum_{k=i}^{N} \alpha_k + \sum_{k=j'}^{N} \alpha_k & j > N+1 & \mathfrak{g} = \mathfrak{so}_{2N+1}, \\[2mm]
\sum_{k=i}^{j-1} \alpha_k & j \leqslant N+1 & \\[2mm]
\sum_{k=i}^{N} \alpha_k + \sum_{k=j'}^{N-1} \alpha_k & j > N+1 & \mathfrak{g} = \mathfrak{sp}_{2N}, \\[2mm]
\sum_{k=i}^{j-1} \alpha_k & j \leqslant N & \\[2mm]
\sum_{k=i}^{N-2} \alpha_k + \alpha_N & j = N+1 & \\[2mm]
\sum_{k=i}^{N} \alpha_k & j = N+2 & \mathfrak{g} = \mathfrak{so}_{2N}, \\[2mm]
\sum_{k=i}^{N} \alpha_k + \sum_{k=j'}^{N-2} \alpha_k & j > N+2 &
\end{cases}
$$

where $j' := 2N + 2 - j$ for $\mathfrak{g} = \mathfrak{so}_{2N+1}$, $j' := 2N + 1 - j$ for $\mathfrak{g} = \mathfrak{sp}_{2N}, \mathfrak{so}_{2N}$ and the indices take the values

$$
\begin{aligned}
i = 1, \dots, N, \ \ j = i+1, \dots, N+1 & \qquad \mathfrak{g} = \mathfrak{sl}_{N+1}, \\
i = 1, \dots, N, \ \ j = i+1, \dots, i'-1 & \qquad \mathfrak{g} = \mathfrak{so}_{2N+1}, \\
i = 1, \dots, N, \ \ j = i+1, \dots, i' & \qquad \mathfrak{g} = \mathfrak{sp}_{2N}, \\
i = 1, \dots, N-1, \ \ j = i+1, \dots, i'-1 & \qquad \mathfrak{g} = \mathfrak{so}_{2N}.
\end{aligned}
$$

Now we fix the expression $\prod_{k=N}^{1} a_k$ for the longest word of W, where

$$
a_k := \begin{cases}
\prod_{i=1}^{k} r_i & & \mathfrak{g} = \mathfrak{sl}_{N+1}, \\[2mm]
\left(\prod_{i=k}^{N} r_i\right)\left(\prod_{j=N-1}^{k} r_j\right) & & \mathfrak{g} = \mathfrak{so}_{2N+1}, \mathfrak{sp}_{2N}, \\[2mm]
1 & k = N & \\[2mm]
\left(\prod_{i=k}^{N-2} r_i\right) r_N \left(\prod_{j=N-1}^{k} r_j\right) & N - k \neq 0 \text{ odd} & \mathfrak{g} = \mathfrak{so}_{2N}. \\[2mm]
\left(\prod_{i=k}^{N-1} r_i\right) r_N \left(\prod_{j=N}^{k} r_j\right) & N - k \neq 0 \text{ even} &
\end{cases}
$$

Then the induced ordering $\prec$ of $\mathbf{R}^+$ is as follows:

$$
\beta_{ij} \prec \beta_{kl} \quad \Leftrightarrow \quad
\begin{cases}
i < k \text{ or } i = k, j < l & \mathfrak{g} = \mathfrak{sl}_{N+1}, \\[2mm]
k < i \text{ or } i = k, l < j & \mathfrak{g} = \mathfrak{so}_{2N+1}, \mathfrak{so}_{2N}, \\[2mm]
k < i \text{ or } i = k, j = N+1 \text{ or} & \\
i = k, l < j, j \neq N+1, l \neq N+1 & \mathfrak{g} = \mathfrak{sp}_{2N}.
\end{cases}
$$

Originally the quantum group coordinate algebras were defined only for the classical groups in terms of generators and relations [FRT]. The generators are the matrix coefficients u^i_j of the vector representation of $U_q(\mathfrak{g})$ (the first fundamental representation which defines $\mathfrak{g}$ as a matrix Lie algebra) with respect to some basis. For the relations we refer to Chapter 9 of [KS1]. There the resulting Hopf algebras are denoted by $\mathcal{O}(G_q)$.

If q is not a root of unity, then $\mathcal{O}(G_q)$ defined in this way is isomorphic to $\mathbb{C}_q[G]$ as used in the last section for all G except $G = SO(2N+1)$. In this case one has $\mathcal{O}(G_{q^2}) \simeq \mathbb{C}_q[G]$.

This is a consequence of the Peter–Weyl theorem [KS1, Theorem 11.22]. The latter is stated in [KS1] under the assumption that q is transcendental. According to Remark 3 on p. 415 of [KS1] and Corollaries 4.15 and 5.22 from [LR] the result holds also for q not a root of unity.

We abbreviate $f \sim g$ iff $f = xg$ with some $x \in \mathbb{C}\backslash\{0\}$ and $(l^{\pm})^i_j := l^{\pm}(u^i_j)$. Then the following statement holds:

Proposition 5. *If $i\,j$ appear as indices of a positive root β_{ij}, then*

$$(l^+)^i_j \sim (l^+)^i_i E_{\beta_{ij}}, \qquad (l^-)^j_i \sim (l^-)^i_i F_{\beta_{ij}},$$

except if $\mathfrak{g} = \mathfrak{sp}_{2N}$ and $j = i'$. In this case, there are $x, y \in \mathbb{C}$, such that

$$(l^+)^i_{i'} \sim (l^+)^i_i (E_{\beta_{ii'}} - x E_{\beta_{ii'-1}} E_i),$$

$$(l^-)^{i'}_i \sim (l^-)^i_i (F_{\beta_{ii'}} - y F_{\beta_{ii'-1}} F_i).$$

Proof. Since this is known for $\mathfrak{g} = \mathfrak{sl}_{N+1}$, we consider only the remaining cases. We also will consider only the $E_{\beta_{ij}}$. The $F_{\beta_{ji}}$ are treated similarly.

The proof is by induction over $j - i$. By the lists of $(l^{\pm})^i_j$ in Section 8.5.2 of [KS1] the claim holds for $j - i = 1$. All occurring $(l^+)^i_j$ except $(l^+)^{N-1}_{N+1}$ for $\mathfrak{g} = \mathfrak{sp}_{2N}, \mathfrak{so}_{2N}$ can be calculated from the recurrence relation

$$(q - q^{-1})(l^+)^i_j = -\big[(l^+)^i_k, (l^+)^k_j\big](l^-)^k_k. \tag{10}$$

Here k with $i < k < j$ is arbitrary with $k \neq i', j'$ [KS1, Proposition 8.29].

We choose $k = j - 1$. This is admissible in all cases except $\mathfrak{g} = \mathfrak{sp}_{2N}, \mathfrak{so}_{2N}$ and $j = N + 1$. These must be treated separately afterwards.

By the explicit lists of the $(l^{\pm})^i_j$ in [KS1] there are $\lambda_k \in \mathbf{L}$ such that

$$(l^{\pm})^k_k = K_{\pm\lambda_k}, \quad (l^+)^{j-1}_j \sim (l^+)^{j-1}_{j-1} E_{f(j-1)}, \quad f(k) := \begin{cases} k & k \leqslant N, \\ k' - 1 & k > N. \end{cases}$$

Inserting this and the induction hypothesis into (10) we get

$$(l^+)^i_j \sim (l^+)^i_i \big(E_{f(j-1)} E_{\beta_{ij-1}} - q^{-g(i,j-1)} E_{\beta_{ij-1}} E_{f(j-1)}\big) \tag{11}$$

with $g(i, j - 1) = \langle \lambda_{j-1}, \beta_{ij-1} \rangle - \langle \lambda_i, \alpha_{f(j-1)} \rangle$.

Inserting the explicit formulas for $\lambda_k, \beta_{ij}, \langle \alpha_i, \alpha_j \rangle$ one gets after some lengthy calculations

$$g(i, j-1) = \begin{cases} 2 & \mathfrak{g} = \mathfrak{sp}_{2N}, \, j = i', \\ -\langle \alpha_{f(j-1)}, \beta_{ij-1} \rangle & \text{otherwise.} \end{cases} \tag{12}$$

In our ordering of $\mathbf{R}^+$ we have $\alpha_{f(j-1)} \prec \beta_{ij-1}$ for $i < j - 1$ and $j \neq i'$ which holds in all cases except $\mathfrak{g} = \mathfrak{sp}_{2N}, \, j = i'$. Since $\mathfrak{g} = \mathfrak{so}_{2N}, \, j = N + 1$ was excluded we furthermore have $\alpha_{f(j-1)} + \beta_{ij-1} = \beta_{ij}$ and there is no other linear combination of roots between $\alpha_{f(j-1)}$ and β_{ij-1} equal to β_{ij}. Hence the exponent in (11) is in all considered cases except $\mathfrak{g} = \mathfrak{sp}_{2N}, \, j = i'$ the same as the one which appears on the left-hand side of (8) and the claim reduces to Proposition 4 (note that for the classical groups $(l^+)^i_j \neq 0$ for all $i \leqslant j$, as follows for example from [KS1, Theorem 8.33].

For $\mathfrak{g} = \mathfrak{sp}_{2N}, \, j = i'$ we obtain

$$(l^+)^i_{i'} \sim (l^+)^i_i \big(E_i E_{\beta_{ii'-1}} - q^{-2} E_{\beta_{ii'-1}} E_i \big) \sim (l^+)^i_i (E_{\beta_{ii'}} - x E_{\beta_{ii'-1}} E_i)$$

for some $x \in \mathbb{C}$, because $\langle \alpha_i, \beta_{ij-1} \rangle = 0$.

It remains to treat the excluded cases $(l^+)^i_{N+1}$ for $\mathfrak{g} = \mathfrak{sp}_{2N}, \mathfrak{so}_{2N}$.

By the explicit lists of $(l^+)^i_j$ in [KS1] we have for $\mathfrak{g} = \mathfrak{sp}_{2N}, \, i = N - 1$

$$\begin{aligned}
(l^+)^{N-1}_{N+1} &\sim (l^+)^{N-1}_{N-1} \big(E_N E_{N-1} - q^{-2} E_{N-1} E_N \big) \\
&= (l^+)^{N-1}_{N-1} \big(E_N E_{N-1} - q^{\langle \alpha_{N-1}, \alpha_N \rangle} E_{N-1} E_N \big) \\
&\sim (l^+)^{N-1}_{N-1} E_{\alpha_{N-1}+\alpha_N}
\end{aligned}$$

by the same argument as above. For $\mathfrak{g} = \mathfrak{so}_{2N}$ the lists directly contain

$$(l^+)^{N-1}_{N+1} \sim (l^+)^{N-1}_{N-1} E_N,$$

so the proposition holds in these cases.

For $i < N - 1$ we need a second induction on i starting with $i = N - 1$. We again use the recurrence relation (10), but now with $k = i + 1$ (which is possible for $i < N - 1$) getting by induction

$$\begin{aligned}
(l^+)^i_{N+1} &\sim \big[(l^+)^i_{i+1}, (l^+)^{i+1}_{N+1} \big] (l^-)^{i+1}_{i+1} \\
&\sim (l^+)^i_i \big(E_i E_{\beta_{i+1N+1}} - q^{\langle \lambda_{i+1}, \alpha_i \rangle - \langle \lambda_i, \beta_{i+1N+1} \rangle} E_{\beta_{i+1N+1}} E_i \big).
\end{aligned}$$

In all cases $\langle \lambda_{i+1}, \alpha_i \rangle = 1$ and the second term in the exponent vanishes, since in β_{i+1N+1} only α_j with $j > i$ occur. Since $\langle \alpha_i, \beta_{i+1N+1} \rangle = -1$ and $\alpha_i \succ \beta_{i+1N+1}$, the same argumentation as above yields

$$\begin{aligned}
(l^+)^i_{N+1} &\sim (l^+)^i_i \big(E_{\beta_{i+1N+1}} E_i - q^{\langle \alpha_i, \beta_{i+1N+1} \rangle} E_i E_{\beta_{i+1N+1}} \big) \\
&\sim (l^+)^i_i E_{\beta_{iN+1}}. \qquad \square
\end{aligned}$$

Corollary 2. *We have*

$$\Delta(E_{\beta_{ij}}) \sim \left((l^-)^i_i \otimes (l^-)^i_i\right) \sum_{k=i}^{j} (l^+)^i_k \otimes (l^+)^k_j,$$

$$\Delta(F_{\beta_{ij}}) \sim \left((l^+)^i_i \otimes (l^+)^i_i\right) \sum_{k=i}^{j} (l^-)^j_k \otimes (l^-)^k_i$$

except when $\mathfrak{g} = \mathfrak{sp}_{2N}$ *and* $j = i'$. *In this case, we have*

$$\Delta(E_{\beta_{ii'}}) \sim x\,\Delta(E_{\beta_{ii'-1}}E_i) + \left((l^-)^i_i \otimes (l^-)^i_i\right) \sum_{k=i}^{i'} (l^+)^i_k \otimes (l^+)^k_{i'},$$

$$\Delta(F_{\beta_{ii'}}) \sim y\,\Delta(F_{i\beta_{i'-1}}F_i) + \left((l^+)^i_i \otimes (l^+)^i_i\right) \sum_{k=i}^{i'} (l^-)^{i'}_k \otimes (l^-)^k_i.$$

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