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General stability of functional equations of linear type

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Abstract

Making use of a dynamical systems notion called shadowing, we prove a stability result for linear functional equations in metric groups. As a corollary we obtain stability of the quadratic functional equation in the case when the target space is a metric group satisfying some local 2-divisibility condition.

Keywords: Stability; Functional equation; Locally divisible group; Metric group

1. Introduction

The stability theory of functional equations began with the well-known Ulam's Problem [11], concerning the stability of homomorphisms in metric groups:

Problem. *We are given a group* $(G,+)$ *and a metric group* $(X,+)$. *Does for every* $\varepsilon > 0$ *there exist* $\delta > 0$ *such that if* $f : G \to X$ *satisfies*

$$d\big(f(x+y), f(x)+f(y)\big) < \delta \quad \text{for } x, y \in G,$$

then a homomorphism $a : G \to X$ *exists with*

$$d\big(f(x), a(x)\big) < \varepsilon \quad \text{for } x \in G?$$

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The first partial positive answer to this problem in the case where X is a Banach space was given by D.H. Hyers [6]. Since then many authors have studied the question of stability of various functional equations (see [3,7] for the survey of stability results).

There has been a lot of improvement and generalizations of the Hyers Theorem, but little has been done in the case when X is not a vector space. D. Cenzer [1,2] considered the case when X is a unit complex circle $\mathbb{T}$ with multiplication as a group operation, R. Ger and P. Šemrl in [5] showed stability in the case when X is $\mathbb{C} \setminus \{0\}$ and one of the authors obtained in [9] some results in a more general setting.

However, all the above mentioned results concerned the Cauchy functional equation, and cannot be easily modified to other functional equations. *Thus except for the Cauchy functional equation there are no, at least known to the authors, stability results in the case when the target space has no global divisibility*!

Our aim is to cover this shortage and provide a general stability result for functional equations in the case when the target space is a metric group (with some local divisibility condition). To illustrate the usefulness of our theorem we show stability of the quadratic functional equation.

We would like to mention here an important generalization of the Hyers method done by G.L. Forti [4]. Forti's procedure can be applied if the given stability problem can be reduced to functional inequality in which only functions of one variable are involved.

Our method of reasoning follows a similar idea. We first prove a general result in stability of an equation in one variable, Theorem 1, which is applied later as a tool in proving various stability results. In fact our Theorem 1 is a partial generalization of Theorem 1 from [4] to the case of noninvertible mappings. Thus our results can be regarded as a further step in the direction indicated by Forti.

Before proceeding further we would like to stress that we were able to obtain our results by applying a new method of proof based on the ideas from dynamical systems.

Now we would like first to establish some notation and recall some basic definitions concerning shadowing. For more details we refer the reader to [8,10]. By $\mathbb{N}$ we denote the set of all nonnegative integers, X will denote from now on a complete metric space. Let $\phi : X \to X$ be given.

Definition 1. Let $\delta \geqslant 0$. We say that a sequence $(x_k)_{k\in\mathbb{N}}$ is a *δ-pseudoorbit* (for ϕ) if

$$d\big(x_{k+1}, \phi(x_k)\big) \leqslant \delta \quad \text{for } k \in \mathbb{N}.$$

A 0-pseudoorbit is called an *orbit*.

Thus a sequence $(x_k)_{k\in\mathbb{N}}$ is an orbit if $x_{k+1} = \phi(x_k)$ for $k \in \mathbb{N}$.

Now we proceed to the notion of local invertibility [10] and its use for shadowing. By $B(x, r)$ we denote the closed ball centered at x and with radius r.

Definition 2. Let r, $R > 0$ be given. We say that $\phi : X \to X$ is locally (r, R)-invertible at $x_0 \in X$ if

$$\forall y \in B\big(\phi(x_0), R\big) \quad \exists! x \in B(x_0, r){:} \quad \phi(x) = y.$$

If ϕ is locally (r, R)-invertible at each $x \in X$ then we say that ϕ is locally (r, R)-invertible.

For locally (r, R)-invertible ϕ we define a function $\phi_{x_0}^{-1} : B(\phi(x_0), R) \to B(x_0, r)$ in such a way that $\phi_{x_0}^{-1}(y)$ denotes the unique x from the above definition which satisfies $\phi(x) = y$. Moreover, we put

$$\operatorname{lip}_R \phi^{-1} := \sup_{x_0 \in X} \operatorname{lip}\bigl(\phi_{x_0}^{-1}\bigr).$$

We will need the following result [10]:

Theorem A. *Let $l \in (0,1)$, $R \in (0,\infty)$ be fixed and let $\phi : X \to X$ be locally (lR, R)-invertible. We assume additionally that* $\operatorname{lip}_R(\phi^{-1}) \leqslant l$.

Let $\delta \leqslant (1-l)R$ and let $(x_k)_{k\in\mathbb{N}}$ be an arbitrary δ-pseudoorbit. Then there exists a unique $y \in X$ such that

$$d\bigl(x_k, \phi^k(y)\bigr) \leqslant lR \quad \textit{for } k \in \mathbb{N}.$$

Moreover,

$$d\bigl(x_k, \phi^k(y)\bigr) \leqslant \frac{l\delta}{1-l} \quad \textit{for } k \in \mathbb{N}.$$

2. Homogeneous functions

In this section we prove stability of the homogeneity equation. We assume that G, X are complete metric spaces and $\psi : G \to G$, $\phi : X \to X$ are given. We call a function $f : G \to X$ homogeneous if $f(\psi x) = \phi f(x)$ for $x \in G$.

As a consequence of Theorem A we obtain

Theorem 1. *Let $l \in (0,1)$, $R \in (0,\infty)$ be fixed. We assume that $\phi : X \to X$ is locally (lR, R)-invertible and* $\operatorname{lip}_R(\phi^{-1}) \leqslant l$.

Let $f : G \to X$ be such that

$$\delta := \sup_{x\in G} d\bigl(f(\psi x), \phi f(x)\bigr) \leqslant (1-l)R.$$

Then there exists a unique homogeneous function $F : G \to X$ such that

$$d\bigl(f(x), F(x)\bigr) \leqslant lR \quad \textit{for } x \in G.$$

Moreover,

$$d\bigl(f(x), F(x)\bigr) \leqslant \frac{l\delta}{1-l} \quad \textit{for } x \in G.$$

Before proceeding to the proof we would like to compare the above theorem with Forti's Theorem 1 [4]. Although Forti's result deals with the case when ϕ is globally invertible, it allows a much more general RHS then our theorem. Thus our result is only a partial generalization of Forti's theorem to the case of noninvertible ϕ.

Proof. Fix arbitrarily $x \in G$. Then

$$d\bigl(f\bigl(\psi^{k+1}x\bigr), \phi\bigl(f\bigl(\psi^k x\bigr)\bigr)\bigr) \leqslant \delta \quad \text{for } k \in \mathbb{N},$$

which implies that $(f(\psi^k x))_{k\in\mathbb{N}}$ is a δ-pseudoorbit for ϕ. By Theorem A we obtain that there exists a unique $F_x \in X$ such that

$$d\big(f\big(\psi^k x\big), \phi^k F_x\big) = d\big(f\big(\psi^k x\big), \phi^k F_x\big) \leqslant lR \quad \text{for } k \in \mathbb{N}. \tag{1}$$

We define the function $F: G \to X$ by the formula $F(x) = F_x$. Then setting in (1) $k = 0$ we get that $d(f(x), F(x)) \leqslant lR$.

It remains to prove that F is homogeneous. Replacing in (1) x by ψx we obtain that $F(\psi x)$ is a unique element which satisfies

$$d\big(f\big(\psi^{k+1} x\big), \phi^k F(\psi x)\big) \leqslant lR \quad \text{for } k \in \mathbb{N}.$$

On the other hand, by (1) we get

$$d\big(f\big(\psi^{k+1} x\big), \phi^{k+1} F(x)\big) \leqslant lR \quad \text{for } k \in \mathbb{N}.$$

By the uniqueness we obtain that $F(\psi x) = \phi F(x)$. □

The assumption of (r, R)-invertibility can easily be verified when X is a metric group. We say that $(X, +, d)$ is a *metric group* if X is an Abelian group with a translation invariant metric, that is

$$d(a + x, b + x) = d(a, b) \quad \text{for } a, b, x \in X. \tag{2}$$

To keep the notation consistent with that often used terminology we write $\|x\|$ instead of $d(x, 0)$. Thus we have

$$d(x, y) = \|x - y\|.$$

Given an Abelian group X and $n \in \mathbb{Z}$ we define the mapping $[n_X]: X \to X$ by the formula

$$[n_X](x) := nx \quad \text{for } x \in X.$$

We will need the following proposition.

Proposition 1. *Let X be an Abelian metric group, let $R > 0$, $n \in \mathbb{N}$, $n \geqslant 1$ be fixed. Suppose that*

$$\forall S \in [0, 2R]\ \forall y \in B(0, S) \quad \exists! x \in B(0, S/n): \quad y = nx. \tag{3}$$

Then

$$\big\|[1/n]w\big\| \leqslant \|w\|/n \quad \textit{for } w \in B(0, 2R), \tag{4}$$

$$\big\|[1/n]u - [1/n]v\big\| \leqslant \|u - v\|/n \quad \textit{for } u, v \in B(0, R), \tag{5}$$

where for $z \in B(0, 2R)$ by $[1/n]z$ we understand the unique element from $B(0, 2R/n)$ which satisfies (3).

Proof. Putting in (3) $S = \|w\|$ we obtain (4).

To check (5), fix arbitrary $u, v \in B(0, R)$. We first prove that $[1/n]u - [1/n]v = [1/n](u - v)$. To see this, notice that by (4),

$$\big\|[1/n]u - [1/n]v\big\| \leqslant \big\|[1/n]u\big\| + \big\|[1/n]v\big\| \leqslant \|u\|/n + \|v\|/n \leqslant 2R/n.$$

Clearly, $n([1/n]u - [1/n]v) = u - v$. By the uniqueness we get $[1/n]u - [1/n]v = [1/n](u - v)$. Now by (4) we get

$$\big\|[1/n]u - [1/n]v\big\| = \big\|[1/n](u - v)\big\| \leqslant \|u - v\|/n. \qquad \square$$

Let $\mathbb{T} := [0, 1)$ denote the group with addition modulo 1 (it can be also identified with the complex unit circle with multiplication as a group operation). On $\mathbb{T}$ we define the translation invariant metric by the formula

$$d_{\mathbb{T}}(x, y) := \min\bigl\{|x - y|, 1 - |x - y|\bigr\}.$$

Remark 1. Let $n \in \mathbb{Z}$, $|n| \geqslant 1$ and $R < 1/4$ be arbitrarily fixed. Applying Proposition 1 one can easily see that the mapping $[n_{\mathbb{T}}]$ is locally $(R/|n|, R)$-invertible and $\mathrm{lip}_R([n_{\mathbb{T}}]^{-1}) = 1/|n|$.

As a direct consequence of Theorem 1 we obtain

Corollary 1. *Let G be a set, $\psi : G \to G$, $n \in \mathbb{Z}$, $|n| \geqslant 2$, and let $\delta < \frac{1}{4}(1 - 1/|n|)$ be fixed. Let $f : G \to \mathbb{T}$ be such that*

$$d_{\mathbb{T}}\bigl(f(\psi x), nf(x)\bigr) \leqslant \delta \quad \text{for } x \in G.$$

Then there exists a unique function $F : G \to \mathbb{T}$ such that

$$F(\psi x) = nF(x) \quad \text{for } x \in G,$$

and

$$d_{\mathbb{T}}\bigl(f(x), F(x)\bigr) \leqslant \delta/\bigl(|n| - 1\bigr) \quad \text{for } x \in G.$$

Proof. Take $l = 1/|n|$, $\frac{1}{4}(1 - 1/|n|) < R < 1/4$ and apply Theorem 1. □

3. Linear equation

In this section we prove our main result on the stability of the linear functional equations in metric groups. Due to its generality, it can be applied to show stability of various functional equations.

Theorem 2. *Let $l \in (0, 1)$, $R \in (0, \infty)$, $\delta \in (0, (1 - l)R)$, $\varepsilon > 0$, $m \in \mathbb{N}$, $n \in \mathbb{Z}$. Let G be a commutative semigroup, X a complete Abelian metric group. We assume that the mapping $[n_X]$ is locally (lR, R)-invertible and that $\mathrm{lip}_R([n_X]^{-1}) \leqslant l$.*

Let $f : G \to X$ satisfy the following two inequalities

$$\Biggl\| \sum_{i=1}^{N} a_i f(b_i x + c_i y) \Biggr\| \leqslant \varepsilon \quad \text{for } x, y \in G,$$

$$\bigl\| f(mx) - nf(x) \bigr\| \leqslant \delta \quad \text{for } x \in G,$$

where a_i are endomorphisms in X, b_i, c_i are endomorphisms in G. We assume additionally that there exists $K \in \{1, \dots, N\}$ such that

$$\sum_{i=1}^{K} \mathrm{lip}(a_i)\delta \leqslant (1 - l)R, \qquad \varepsilon + \sum_{i=K+1}^{N} \mathrm{lip}(a_i)\frac{l\delta}{1 - l} \leqslant lR. \tag{6}$$

Then there exists a unique function $F : G \to X$ such that

$$F(mx) = nF(x) \quad \text{for } x \in G, \tag{7}$$

and

$$\|f(x)-F(x)\| \leqslant \frac{l\delta}{1-l} \quad \text{for } x \in G. \tag{8}$$

Moreover, then F satisfies

$$\sum_{i=1}^{N} a_i F(b_i x + c_i y) = 0 \quad \text{for } x, y \in G. \tag{9}$$

Proof. By Theorem 1 there exists a unique function F which satisfies (7) and (8). We are going to prove that F satisfies (9).

Consider arbitrary $x, y \in G$. We are going to show that the sequence

$$\left(\sum_{i=1}^{K} a_i f\big(m^k b_i x + m^k c_i y\big)\right)_{k\in\mathbb{N}}$$

is an $(1-l)R$-pseudoorbit for $[n_X]$. We have

$$\begin{aligned}
&\left\|\sum_{i=1}^{K} a_i f\big(m^{k+1} b_i x + m^{k+1} c_i y\big) - n\sum_{i=1}^{K} a_i f\big(m^k b_i x + m^k c_i y\big)\right\| \\
&\quad\leqslant \sum_{i=1}^{K} \operatorname{lip}(a_i)\big\| f\big(m^{k+1} b_i x + m^{k+1} c_i y\big) - nf\big(m^k b_i x + m^k c_i y\big)\big\| \\
&\quad\leqslant \sum_{i=1}^{K} \operatorname{lip}(a_i)\delta \leqslant (1-l)R.
\end{aligned}$$

By Theorem A there exists a unique $w \in X$ such that

$$\left\|\sum_{i=1}^{K} a_i f\big(m^k b_i x + m^k c_i y\big) - n^k w\right\| \leqslant lR \quad \text{for } k \in \mathbb{N}.$$

But

$$\begin{aligned}
&\left\|\sum_{i=1}^{K} a_i f\big(m^k b_i x + m^k c_i y\big) - n^k \sum_{i=1}^{K} a_i F(b_i x + c_i y)\right\| \\
&\quad= \left\|\sum_{i=1}^{K} a_i f\big(m^k b_i x + m^k c_i y\big) - \sum_{i=1}^{K} a_i F\big(m^k b_i x + m^k c_i y\big)\right\| \\
&\quad\leqslant \sum_{i=1}^{K} \operatorname{lip}(a_i)\big\| f\big(m^k b_i x + m^k c_i y\big) - F\big(m^k b_i x + m^k c_i y\big)\big\| \\
&\quad\overset{\text{by (8)}}{\leqslant} \sum_{i=1}^{K} \operatorname{lip}(a_i)\frac{l\delta}{1-l} \leqslant lR
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \sum_{i=1}^{K} a_i f\left(m^k b_i x + m^k c_i y\right) - \left(-n^k\right) \sum_{i=K+1}^{N} a_i F(b_i x + c_i y) \right\| \\
&\quad \leqslant \left\| \sum_{i=1}^{N} a_i f\left(m^k b_i x + m^k c_i y\right) \right\| \\
&\qquad + \left\| \sum_{i=K+1}^{N} a_i F\left(m^k b_i x + m^k c_i y\right) - \sum_{i=K+1}^{N} a_i f\left(m^k b_i x + m^k c_i y\right) \right\| \\
&\quad \leqslant \varepsilon + \sum_{i=K+1}^{N} \operatorname{lip}(a_i) \frac{l\delta}{1-l} \leqslant lR.
\end{aligned}$$

Consequently

$$w = \sum_{i=1}^{K} a_i F(b_i x + c_i y) = - \sum_{i=K+1}^{N} a_i F(b_i x + c_i y). \qquad \square$$

4. Quadratic functional equation

Making use of Theorem 2 we are going to prove stability of the quadratic functional equation in metric groups. We will need the following easy lemma. In the two following results we use the operation $[1/n]$ defined in Proposition 1.

Lemma 1. *Let X be an Abelian metric group and let $R > 0$ be fixed. Suppose that the following condition*

$$\forall S \in [0, 2R]\ \forall y \in B(0, S) \quad \exists! x \in B(0, S/2): \quad y = 2x \tag{10}$$

holds. Then for every $k \in \mathbb{N}$ the mapping $[2^k_X]$ is locally $(R/2^k, R)$-invertible and $\operatorname{lip}_R([2^k_X]^{-1}) \leqslant 2^{-k}$.

Proof. Since X is a metric group it is enough to check that $[2^k_X]$ is locally $(R/2^k, R)$-invertible at 0 and $\operatorname{lip}([2^k_X]_0^{-1}) \leqslant 1/2^k$. We are going to do this by applying Proposition 1.

Let $y \in B(0, R)$ be arbitrary. We put $x = [1/2]^k y$. Clearly by (4) $\|x\| \leqslant \|y\|/2^k$ and $2^k x = y$. Suppose that there exists $\tilde{x} \in B(0, R/2^k)$, $\tilde{x} \neq x$, such that $2^k \tilde{x} = y$. Then there exists $l \in \{1, \ldots, n\}$ such that $2^l x = 2^l \tilde{x}$, $2^{l-1} x \neq 2^{l-1} \tilde{x}$. We obtain a contradiction with uniqueness in (10).

Making use of (5) we get

$$\left\| [1/2]^k u - [1/2]^k v \right\| \leqslant \|u - v\|/2^k. \qquad \square$$

Theorem 3. *Let $R > 0$, let G be an Abelian group and let X be a complete metric Abelian group satisfying the condition*

$$\forall S \in [0, 2R]\ \forall y \in B(0, S) \quad \exists! x \in B(0, S/2): \quad y = 2x.$$

Let $\varepsilon \leqslant R/8$ be arbitrary and let $f : G \to X$ be such that

$$\left\| f(x+y) + f(x-y) - 2f(x) - 2f(y) \right\| \leqslant \varepsilon \quad \textit{for } x, y \in G. \tag{11}$$

Then there exists a unique function $F : G \to X$ *such that*

$$F(x+y) + F(x-y) = 2F(x) + 2F(y) \quad \textit{for } x, y \in G,$$

and

$$\|F(x) - f(x)\| \leqslant \varepsilon/2 \quad \textit{for } x \in G.$$

Proof. By Lemma 1 we obtain that the mapping $[4_X]$ is locally $(R/4, R)$-invertible and $\mathrm{lip}_R([4_X]_2^{-1}) \leqslant 1/4$.

Putting $x = y = 0$ in (11) we obtain that $\|2f(0)\| \leqslant \varepsilon \leqslant R$. Let

$$\tilde{f}(x) := f(x) - \big(f(0) - [1/2]\big(2f(0)\big)\big).$$

Then

$$\big\|\tilde{f}(x+y) + \tilde{f}(x-y) - 2\big(\tilde{f}(x) + \tilde{f}(y)\big)\big\| \leqslant \varepsilon.$$

As $\|\tilde{f}(0)\| = \|[1/2](2f(0))\| \leqslant \varepsilon/2$, the above inequality yields

$$\big\|\tilde{f}(2x) - 4\tilde{f}(x)\big\| \leqslant \frac{3}{2}\varepsilon.$$

We are going to apply Theorem 2 for the function $\tilde{f}$. So let $l = 1/4$, $\delta = \frac{3}{2}\varepsilon$, $\mathrm{a}_1 = a_2 = 2\,\mathrm{id}_X$, $a_3 = a_4 = \mathrm{id}_X$, $K = 2$. Then

$$\delta \leqslant (1-l)R, \qquad \sum_{i=1}^{2} \mathrm{lip}(a_i)\delta \leqslant (1-l)R, \qquad \varepsilon + \sum_{i=3}^{4} \mathrm{lip}(a_i)\frac{l\delta}{1-l} \leqslant lR.$$

Thus all the assumptions of Theorem 2 are satisfied, and therefore we conclude that there exists a unique function $\tilde{F} : G \to X$ such that

$$\begin{aligned} &\tilde{F}(2x) = 4\tilde{F}(x) \quad \text{for } x \in G, \\ &\tilde{F}(x+y) + \tilde{F}(x-y) = 2\tilde{F}(x) + 2\tilde{F}(y) \quad \text{for } x, y \in G, \\ &\big\|\tilde{f}(x) - \tilde{F}(x)\big\| \leqslant \frac{l\delta}{1-l} = \varepsilon/2 \quad \text{for } x \in G. \end{aligned} \tag{12}$$

We put $F(x) := \tilde{F}(x) + (f(0) - [1/2](2f(0)))$. Then F satisfies

$$\begin{aligned} &F(x+y) + F(x-y) = 2F(x) + 2F(y) \quad \text{for } x, y \in G, \\ &\big\|f(x) - F(x)\big\| \leqslant \varepsilon/2 \quad \text{for } x \in G. \end{aligned} \tag{13}$$

We prove the uniqueness part. Suppose that there exists another function F_1 satisfying (13). Let $\tilde{F}_1(x) := F_1(x) - (f(0) - [1/2](2f(0)))$. Then

$$\begin{aligned} &\tilde{F}_1(x+y) + \tilde{F}_1(x-y) = 2\tilde{F}_1(x) + 2\tilde{F}_1(y) \quad \text{for } x, y \in G, \\ &\big\|\tilde{f}(x) - \tilde{F}_1(x)\big\| \leqslant \varepsilon/2 \quad \text{for } x \in G. \end{aligned}$$

Since $\|\tilde{f}(0)\| \leqslant \varepsilon/2$, we obtain that $\|\tilde{F}_1(0)\| \leqslant \varepsilon$. Since $2\tilde{F}_1(0) = 0$, by the uniqueness of the local division by two we get $\tilde{F}_1(0) = 0$, and consequently $\tilde{F}_1(2x) = 4\tilde{F}_1(x)$. Thus $\tilde{F}_1$ satisfies (12), and by the uniqueness we obtain that $\tilde{F}_1 = \tilde{F}$. Thus $F_1 = \tilde{F}_1 + (f(0) - [1/2](2f(0))) = \tilde{F} + (f(0) - [1/2](2f(0))) = F$. $\square$

As a direct consequence of the previous theorem and Remark 1 we obtain stability of the quadratic functional equation in the case when the target space is the group $\mathbb{T}$.

Corollary 2. *Let G be an Abelian group. Let $\varepsilon \in (0, 1/32)$ be arbitrary and let $f : G \to X$ be such that*

$$d_{\mathbb{T}}\big(f(x+y)+f(x-y), 2f(x)+2f(y)\big) \leqslant \varepsilon \quad \text{for } x, y \in G.$$

Then there exists a unique function $F : G \to \mathbb{T}$ such that

$$F(x+y)+F(x-y)=2F(x)+2F(y) \quad \text{for } x, y \in G,$$

and

$$d_{\mathbb{T}}\big(f(x), F(x)\big) \leqslant \varepsilon/2 \quad \text{for } x \in G.$$

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