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Local Properties of Ternary Rings of Operators and Their Linking C^*-Algebras

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We show that some local properties (such as nuclearity, exactness, and local reflexivity) of ternary rings of operators (TROs) are closely related to the local properties of their linking C^*-algebras. We also show some equivalent conditions for nuclear TROs, and show that Haagerup's decomposition property for completely bounded maps and Pisier's δ-norm can be naturally generalized to TROs. © 2002 Elsevier Science (USA)

Key Words: operator spaces; TROs; C*-algebras; local theory.

1. INTRODUCTION

A *ternary ring of operators* (or simply, *TRO*) between Hilbert spaces K and H is a norm closed subspace V of $B(K, H)$, which is closed under the triple product

$$(x, y, z) \in V \times V \times V \to xy^*z \in V.$$

A TRO $V \subseteq B(K, H)$ is called a W^*-*TRO* if it is weak* closed (equivalently, weak operator closed, or strong operator closed) in $B(K, H)$. TROs were first introduced by Hestenes [19], and have been intensively studied by Harris [18], Zettl [43], Hamana [16, 17], Exel [14], Kirchberg [25], and Effros–Ozawa–Ruan [12].

It is known (see [12, 39]) that every finite-dimensional TRO can be identified with an ℓ_∞-direct sum of rectangular matrix algebras, i.e. it has the form

$$V = M_{m(1),n(1)} \oplus_\infty \cdots \oplus_\infty M_{m(k),n(k)}.$$

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In general, a TRO V can be identified with the off-diagonal corner (at the (1,2) position) of its *linking C^*-algebra*

$$A(V) = \begin{bmatrix} C & V \\ V^\sharp & D \end{bmatrix}, \tag{1.1}$$

where C and D are C^*-algebras generated by $VV^\sharp$ and $V^\sharp V$ (see details in Section 2). Actually, V is a non-degenerate and faithful Hilbert left-C and right-D bimodule, and is a linking *C–D imprimitive ideal* such that the C^*-algebras C and D are *strongly Morita equivalent* in the sense of Rieffel [36].

If we let $M(C)$ and $M(D)$ denote the multiplier C^*-algebras of C and D, respectively, then V is a Hilbert left-$M(C)$ and right-$M(D)$ bimodule and we may identify V with the off-diagonal corner of the unital C^*-algebra

$$R(V) = \begin{bmatrix} M(C) & V \\ V^\sharp & M(D) \end{bmatrix}. \tag{1.2}$$

If V is a W^*-TRO, then it is known from [12] (or see Proposition 2.3) that $R(V)$ is a von Neumann algebra. In this case, we call $R(V)$ the *linking von Neumann algebra* of V.

TROs and W^*-TROs have been abstractly characterized by Zettl [43]. It is known that TROs share many similar properties of C^*-algebras and von Neumann algebras. For example, it was shown by Harris [43] that every TRO-homomorphism must be a contraction and must be a quotient map onto the range space, which is again a TRO. Every injective TRO-homomorphism must be an isometry. For W^*-TROs, we have the corresponding Kaplansky's density theorem, Tomiyama's conditional expectation theorem, and Sakai's theorem for unique preduals (see [12, 43]).

We also note that every TRO has a very important operator space structure. To see this, let us assume that V is a TRO contained in $B(K, H)$. Then for each $n \in \mathbb{N}$, the matrix space $M_n(V)$ can be identified with a TRO contained in $M_n(B(K, H)) \cong B(K^n, H^n)$. This provides a canonical operator space matrix norm on V such that each $M_n(V)$ is again a TRO. We call this the *TRO-matrix norm* on V (obtained from $B(K, H)$). We will see in Proposition 2.1 that the TRO-matrix norm is uniquely determined on each TRO and does not depend on the choice of representing Hilbert spaces.

TROs form a very interesting class of operator spaces. In many cases, TROs come out more naturally than C^*-algebras in the theory of operator spaces. For instance, it is known by Youngson [42] that TROs are closed under completely contractive projections (comparing Choi and Effros's result [6] that C^*-algebras are closed under completely positive and contractive projections). Some operator space properties for TROs have been studied by Hamana [17], Kirchberg 25, and Effros–Ozawa–Ruan [12].

The aim of this paper is to study the local operator space properties (such as nuclearity, exactness and local reflexivity) for TROs. Quite surprisingly, these local properties on TROs (or on the related imprimitive ideals) have very close connections with (actually, totally determine) the corresponding local properties of their linking C^*-algebras (or the whole Hilbert bimodule systems). For instance, we can show that a TRO V is nuclear (respectively, exact or locally reflexive) if and only if its linking C^*-algebra $A(V)$ is nuclear (respectively, exact or locally reflexive).

The paper is organized as follows. We recall some necessary notions and useful properties for TROs in Section 2, and study the operator space *injective* tensor product $\check{\otimes}$ and *augmented injective* tensor products $:\check{\otimes}$ and $\check{\otimes}:$ for TROs in Section 3. We show in Proposition 3.1 that if V is a TRO and B is a (unital) C^*-algebra, then $V \check{\otimes} B$ is again a TRO with linking C^*-algebra

$$A(V \check{\otimes} B) = A(V) \check{\otimes} B,$$

and show in Proposition 3.2 that $V^{**} : \check{\otimes} B$ and $V \check{\otimes} : B^{**}$ are again TROs. In Section 4, we first recall the generalized Archbold–Batty's conditions C'_λ and C''_λ, which are equivalent to λ-exactness and λ-local reflexivity, for operator spaces. We show that a TRO satisfies condition C'_λ (respectively, satisfies the condition C''_λ) for some $\lambda \geqslant 1$ if and only if it satisfies condition C'_1 (respectively, satisfies condition C''_1). Moreover, a TRO satisfies condition C'_1 (respectively, satisfies condition C''_1) if and only if its linking C^*-algebra $A(V)$ satisfies condition C'_1 (respectively, satisfies condition C''_1).

We note that for general operator spaces, λ-exactness and λ-local reflexivity need not imply 1-exactness and 1-local reflexivity, respectively. For instance, Pisier [33] proved that for $n > 2$, $\ell_1(n)$ with the MAX operator space matrix norm is λ-exact for some $\lambda \geqslant \frac{n}{2\sqrt{n-1}}$, but it is not 1-exact. There are examples of C^*-algebras (such as the full group C^*-algebras $C^*(\mathbb{F})$ on free groups $\mathbb{F}$), which are not 1-locally reflexive, and thus are not λ-locally reflexive for any finite λ. In the Appendix we show that for each $n > 2$, there exists an operator space which is $(n+1)$-locally reflexive, and is only λ-locally reflexive for $\lambda \geqslant \frac{n}{2\sqrt{n-1}}$. Another such kind of example can be found in [21, Proposition 3.12].

Motivated by the C^*-algebra theory, we introduce the *maximal* tensor product $\otimes^{t\max}$ for TROs in Section 5. We show that if V is a TRO and B is a C^*-algebra, then $V \otimes^{t\max} B$ can be identified with the off-diagonal corner of $A(V) \otimes^{\max} B$, where $\otimes^{\max}$ is the maximal C^*-algebra tensor product, and we can obtain the C^*-isomorphism

$$A(V \overset{t\max}{\otimes} B) = A(V) \overset{\max}{\otimes} B.$$

We show, in Section 6, that some corresponding equivalent nuclearity conditions for C^*-algebras hold for TROs (see Theorem 6.5). In Sections 7 and 8, we discuss Haagerups's decomposition property for completely bounded maps on TROs, and show that Pisier's δ-norm and some corresponding C^*-algebra results can be naturally generalized to TROs. At the end of Section 8, we make some remarks on the connection of $R(V)$ with the injectivity of V and the connection of our decomposable maps with the weakly decomposable maps discussed by Kirchberg in his talk [28].

We assume that readers are familiar with the theory of operator spaces, which will play a very important role in this paper. The basic properties of operator spaces and completely bounded maps can be found in [13, 31, 35].

2. PRELIMINARIES

Let V and W be two TROs. A linear map $\theta : V \to W$ is called a *TRO-homomorphism* if it preserves the ternary product

$$\theta(xy^*z) = \theta(x)\theta(y)^*\theta(z)$$

for all $x, y, z \in V$. If, in addition, θ is an injection from V onto W, we call θ a *TRO-isomorphism* from V onto W. If $\theta : V \to W$ is a TRO-homomorphism, then $\theta_n : M_n(V) \to M_n(W)$ is again a TRO-homomorphism and thus is a contraction (by Harris [18]) for every $n \in \mathbb{N}$. This shows that every TRO-homomorphism is actually a complete contraction. Similarly, it is easy to see that every TRO-homomorphism is a complete quotient map onto the range space, and every injective TRO-homomorphism is a complete isometry.

The following result of Hamana and Ruan (cf. [17, Proposition 2.1]), shows that the TRO-matrix norm is uniquely determined on every TRO and does not depend on the choice of representing Hilbert spaces.

PROPOSITION 2.1. *Let $V \subseteq B(K, H)$ and $W \subseteq B(K', H')$ be TROs with the canonical TRO-matrix norms and let $\theta : V \to W$ be a linear isomorphism. Then θ is a TRO-isomorphism if and only if θ is a complete isometry.*

If $V \subseteq B(K, H)$ is a TRO, we let $V^\sharp = \{x^* \in B(H, K) : x \in V\}$ denote the *conjugate space* of V. Then

$$VV^\sharp = \text{span}\left\{ \sum_i v_i w_i^* : v_i, w_i \in V \right\}$$

and

$$V^{\sharp}V = \operatorname{span}\left\{\sum_i v_i^* w_i : v_i, w_i \in V\right\}$$

are $*$-subalgebras of $B(H)$ and $B(K)$, and we let $C(V)$ and $D(V)$ (or simply, C and D if there is no confusion) to denote the C^*-algebras obtained by taking norm closures of $VV^{\sharp}$ and $V^{\sharp}V$, respectively. Then V is a non-degenerate and faithful Hilbert left-C and right-D bimodule such that

$$CV = V \qquad \text{and} \qquad VD = V,$$

and we have the C^*-isomorphisms

$$C = K(V_D) \qquad \text{and} \qquad D^{\text{op}} = K({}_C V),$$

where we let $K(V_D)$ denote the space of all compact right-D module homomorphisms on V and let $K({}_C V)$ denote the space of all compact left-C module homomorphisms on V. It follows that we have

$$\|c\|_C = \sup\{\|cv\|_V : \|v\|_V < 1, v \in V\} \tag{2.1}$$

and

$$\|d\|_D = \sup\{\|vd\|_V : \|v\|_V < 1, v \in V\}. \tag{2.2}$$

It is easy to see that

$$A(V) = \begin{bmatrix} C & V \\ V^{\sharp} & D \end{bmatrix} \tag{2.3}$$

is a C^*-algebra on $H \oplus K$, and we may identify V with the off-diagonal corner of $A(V)$ by the injective TRO-homomorphism

$$\imath_V : v \in V \to \imath_V(v) = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \in A(V).$$

Let $V \subseteq B(K, H)$ and $W \subseteq B(K', H')$ be two TROs and let $\theta : V \to W$ be a TRO-homomorphism from V into W. It is known from Hamana [17] that we may obtain a C^*-homomorphism $\varphi : C(V) \to C(W)$ defined by letting

$$\varphi\left(\sum_i v_i w_i^*\right) = \sum_i \theta(v_i)\theta(w_i)^*, \tag{2.4}$$

and a C^*-homomorphism $\psi : D(V) \to D(W)$ defined by letting

$$\psi\left(\sum_i v_i^* w_i\right) = \sum_i \theta(v_i)^* \theta(w_i) \tag{2.5}$$

for $v_i \in V$ and $w_i \in W$.

PROPOSITION 2.2 (Hamana [17]). *Let V and W be two TROs and let $\theta : V \to W$ be a TRO-homomorphism. Then*

$$\pi_\theta = \begin{bmatrix} \varphi & \theta \\ \theta^* & \psi \end{bmatrix} : A(V) \to A(W)$$

is a well-defined C^*-homomorphism, where $\varphi : C(V) \to C(W)$ and $\psi : D(V) \to D(W)$ are the C^*-homomorphisms given in (2.4) *and* (2.5).

It follows that V is TRO-isomorphic to W if and only if $A(V)$ is C^*-isomorphic to $A(W)$.

This shows that if V is a TRO, then the C^*-algebra $A(V)$ is uniquely determined by V and does not depend on the choice of Hilbert spaces K and H. We call $A(V)$ the *linking C^*-algebra* of V. Without loss of generality, we may always assume that a TRO V is *non-degenerately* represented on Hilbert spaces K and H, i.e. VK is norm dense in H and $V^\sharp H$ is norm dense in K. In this case, it is easy to see that the induced C^*-algebras C and D are non-degenerately represented on H and K, and the identity operators 1_H and 1_K are contained in the multiplier C^*-algebras $M(C)$ and $M(D)$ of C and D, respectively. If we let

$$e = \begin{bmatrix} 1_H & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad e^\perp = \begin{bmatrix} 0 & 0 \\ 0 & 1_K \end{bmatrix}, \tag{2.6}$$

then we may write

$$\iota_V(V) = eA(V)e^\perp. \tag{2.7}$$

We can also identify the C^*-algebras C and D with the diagonal C^*-subalgebras $eA(V)e$ and $e^\perp A(V)e^\perp$ and identify $V^\sharp$ with $e^\perp A(V)e$.

It was indicated by Zettl [43] that if V is a TRO non-degenerately contained in $B(K,H)$, then its weak operator closure coincides with its strong operator closure. Actually, these closures also coincide with the weak* closure of V, i.e. we have

$$\bar{V}^{\text{weak}^*} = \bar{V}^{\text{w.o.t}} = \bar{V}^{\text{s.o.t}} \subseteq B(K,H). \tag{2.8}$$

To see this, we note that for the C^*-algebra $A(V)$, we have

$$\overline{A(V)}^{\text{weak}^*} = \overline{A(V)}^{\text{w.o.t}} = \overline{A(V)}^{\text{s.o.t}} \subseteq B(H \oplus K).$$

Then we may obtain (2.8) from the fact that

$$\bar{V}^{\text{weak}^*} = e\overline{A(V)}^{\text{weak}^*} e^{\perp}, \quad \bar{V}^{\text{w.o.t}} = e\overline{A(V)}^{\text{w.o.t}} e^{\perp} \quad \text{and } \bar{V}^{\text{s.o.t}} = e\overline{A(V)}^{\text{s.o.t}} e^{\perp}.$$

It is known that if V is a TRO, then we have the C^*-isomorphisms

$$M(C) = B(V_D) \qquad \text{and} \qquad M(D)^{\text{op}} = B({}_C V),$$

where we let $B(V_D)$ denote the space of all (bounded) adjointable right-D module homomorphisms and let $B({}_C V)$ denote the space of all (bounded) adjointable left-C module homomorphisms. If V is a non-degenerate W^*-TRO contained in $B(K, H)$, then it is known from Zettl [43, Proposition 4.9] that $M(C)$ and $M(D)$ are von Neumann algebras, and V is a faithful *self-dual* left-$M(C)$ and right-$M(D)$ bimodule. Moreover, we may obtain the following result, which has been discussed in [12]. We include a proof for the convenience of readers.

Proposition 2.3. *Let V be a non-degenerate W^*-TRO contained in $B(K, H)$. Then we have*

$$M(C) = \bar{C}^{\text{weak}^*} \qquad \text{and} \qquad M(D) = \bar{D}^{\text{weak}^*}.$$

Moreover,

$$R(V) = \begin{bmatrix} \bar{C}^{\text{weak}^*} & V \\ V^\sharp & \bar{D}^{\text{weak}^*} \end{bmatrix} = \overline{A(V)}^{\text{weak}^*} = A(V)'' \tag{2.9}$$

is a (non-degenerate) von Neumann subalgebra of $B(H \oplus K)$ and we may identify V with the off-diagonal corner of $R(V)$.

Proof. We will prove $M(C) = \bar{C}^{\text{weak}^*}$. The argument for $M(D) = \bar{D}^{\text{weak}^*}$ is similar. Let us assume that V is a W^*-TRO (which is non-degenerately represented) on Hilbert spaces K and H. Then $C = \overline{VV^\sharp}^{\|\cdot\|}$ is a non-degenerate C^*-algebra on H, and thus

$$M(C) = \{x \in B(H) : xy \in C \text{ and } yx \in C \text{ for all } y \in C\}$$

is a C^*-subalgebra of $\bar{C}^{\text{weak}^*} = C''$ (see [32]). On the other hand, given any $\tilde{c} \in \bar{C}^{\text{weak}^*}$, there exists a net of $c_\alpha \in C$ converging to $\tilde{c}$ in the weak* topology.

Then for every $v \in V$, $c_\alpha v \in V$ converges to $\tilde{c}v$ in the weak* topology on $B(K, H)$. Since V is weak* closed in $B(K, H)$, we must have $\tilde{c}v \in V$. This implies that

$$\tilde{c}(vw^*) = (\tilde{c}v)w^* \in VV^\sharp \subseteq C$$

for all $v, w \in V$. Taking norm limit, we get $\tilde{c}x \in C$ for all $x \in C$. Since we also have $c_\alpha^* \to \tilde{c}^*$ in weak* topology, we obtain

$$x\tilde{c} = (\tilde{c}^* x^*)^* \in C$$

for all $x \in C$. This shows that $\tilde{c} \in M(C)$ and thus $\bar{C}^{\text{weak}^*} \subseteq M(C)$.

Since V is a non-degenerate W^*-TRO contained in $B(K, H)$, then

$$R(V) = \begin{bmatrix} M(C) & V \\ V^\sharp & M(D) \end{bmatrix} = \begin{bmatrix} \bar{C}^{\text{weak}^*} & V \\ V^\sharp & \bar{D}^{\text{weak}^*} \end{bmatrix} = \overline{A(V)}^{\text{weak}^*} = A(V)''$$

is a non-degenerate von Neumann subalgebra of $B(H \oplus K)$ and we may identify V with the off-diagonal corner of $R(V)$. ∎

Let V be a TRO and we let $\pi_u : A(V) \to B(H_u)$ be the non-degenerate *universal representation* of $A(V)$. Then we obtain a non-degenerate faithful representation for the enveloping von Neumann algebra $A(V)^{**}$ such that

$$A(V) \subseteq A(V)^{**} = A(V)'' \subseteq B(H_u).$$

If we let $\{c_\alpha\}$ and $\{d_\alpha\}$ be positive contractive approximate identities of C and D, respectively, they converge in weak* topology to mutually orthogonal projections e and $e^\perp = 1 - e$. These two mutually orthogonal projections split the Hilbert space H_u into $H = eH_u$ and $K = e^\perp H_u$. Then it is easy to see that V is a non-degenerate TRO contained in $B(K, H)$. The weak* topologies on C^{**}, V^{**} and D^{**} coincide with the corresponding weak operator topologies on $B(H)$, $B(K, H)$ and $B(K)$, and V^{**} is a W^*-TRO contained in $B(K, H)$. In this case, we have the identifications

$$C = eA(V)e,\ V = eA(V)e^\perp,\ V^\sharp = e^\perp A(V)e \text{ and } D = e^\perp A(V)e^\perp$$

and

$$C^{**} = eA(V)^{**}e,\ V^{**} = eA(V)^{**}e^\perp,\ V^{\sharp **} = e^\perp A(V)^{**}e \text{ and } D^{**} = e^\perp A(V)^{**}e^\perp.$$

The following proposition is an immediate consequence of Proposition 2.3.

PROPOSITION 2.4. *With the above notation, we have*

$$M(C(V^{**})) = C^{**} \qquad \textit{and} \qquad M(D(V^{**})) = D^{**},$$

and thus

$$R(V^{**}) = \begin{bmatrix} M(C(V^{**})) & V^{**} \\ V^{\sharp **} & M(D(V^{**})) \end{bmatrix} = A(V)^{**}. \tag{2.10}$$

3. INJECTIVE TENSOR PRODUCTS FOR TROS

Let us first recall the *injective* (or *spatial*) tensor product for operator spaces. Given operator spaces $V \subseteq B(H)$ and $W \subseteq B(K)$, we let $V \check{\otimes} W$ denote the norm closure of $V \otimes W$ in $B(H \otimes K)$. It is known from the operator space theory that this tensor product is actually independent of the choice of Hilbert spaces, and is *injective* in the sense that if $\imath_1 : V_1 \to V_2$ and $\imath_2 : W_1 \to W_2$ are completely isometric injections, then the induced tensor map

$$\imath_1 \otimes \imath_2 : V_1 \check{\otimes} W_1 \to V_2 \check{\otimes} W_2$$

is a completely isometric injection. If A and B are C^*-algebras, then $A \check{\otimes} B$ is equal to the *minimal* C^*-tensor product $A \otimes^{\min} B$.

PROPOSITION 3.1. *Let V be a TRO and B a C^*-algebra. Then the canonical TRO-inclusion $\imath_V : V \hookrightarrow A(V)$ induces an injective TRO-homomorphism*

$$\imath_V \otimes id_B : V \check{\otimes} B \to A(V) \check{\otimes} B$$

from which we obtain the TRO-isomorphism

$$V \check{\otimes} B = \imath_V(V) \check{\otimes} B,$$

and the C^*-isomorphism

$$A(V \check{\otimes} B) = A(V) \check{\otimes} B.$$

We also have the C^*-isomorphisms

$$C(V \check{\otimes} B) = C(V) \check{\otimes} B \textit{ and } D(V \check{\otimes} B) = D(V) \check{\otimes} B.$$

Proof. If we let V be a non-degenerate TRO contained in $B(K, H)$, then $A(V)$ is a non-degenerate C^*-subalgebra on $H \oplus K$. Given any C^*-algebra

B, which is non-degenerately represented on a Hilbert space L, it is easy to see that

$$\iota_V \otimes id_B : V\check{\otimes}B \hookrightarrow A(V)\check{\otimes}B \subseteq B((H \oplus K) \otimes L)$$

is an injective TRO-homomorphism such that we can identify $V\check{\otimes}B$ with the off-diagonal corner $\iota_V(V)\check{\otimes}B$ of $A(V)\check{\otimes}B$, and we may obtain the C^*-isomorphism

$$A(V\check{\otimes}B) = A(V)\check{\otimes}B.$$

Moreover, we can obtain the C^*-isomorphisms

$$C(V\check{\otimes}B) = C(V)\check{\otimes}B \qquad \text{and} \qquad D(V\check{\otimes}B) = D(V)\check{\otimes}B. \quad \blacksquare$$

In this paper, we will be mainly interested in the injective tensor products of TROs together with C^*-algebras (rather than with TROs). But it is worth noting that in general, we may consider the injective tensor product $V\check{\otimes}W$ for two TROs V and W, and it is not difficult to verify that this is again a TRO. We leave the details to the readers.

Archbold and Batty introduced condition C and condition C' for C^*-algebras in [1]. These notions, together with condition C'', were generalized to operator spaces by Effros and Haagerup [10]. To study these conditions, we need to recall the augmented injective tensor products for operator spaces (see [13]). Given operator spaces V and W, there is a canonical inclusion

$$\tau_l : V^{**} \otimes W \to (V\check{\otimes}W)^{**}$$

given by

$$\langle \tilde{v} \otimes w, F \rangle = \langle \tilde{v}, F(\cdot \otimes w) \rangle$$

for all $\tilde{v} \in V^{**}, w \in W$ and $F \in (V\check{\otimes}W)^*$. This inclusion induces an injective operator space tensor product, which is called the *left augmented injective tensor product* and is denoted by $:\check{\otimes}$, on $V^{**} \otimes W$. We let $V^{**} :\check{\otimes} W$ denote its completion. Let $\hat{\otimes}$ denote the *operator space projective tensor product*. Then the canonical bilinear map

$$V^* \times W^* \to (V\check{\otimes}W)^* : (f, g) \mapsto f \otimes g$$

extends to a complete contraction

$$\phi : V^* \hat{\otimes} W^* \to (V\check{\otimes}W)^*,$$

and its adjoint map

$$\phi^* : (V \check{\otimes} W)^{**} \to (V^* \hat{\otimes} W^*)^*$$

is a complete contraction from $(V \check{\otimes} W)^{**}$ into $(V^* \hat{\otimes} W^*)^*$. Since

$$V^{**} \check{\otimes} W \hookrightarrow (V^* \hat{\otimes} W^*)^*$$

is a completely isometric inclusion, the identity map on $V^{**} \otimes W$ extends to a complete contraction

$$\phi_l^* : V^{**} : \check{\otimes} W \to V^{**} \check{\otimes} W,$$

which can be identified with the restriction of ϕ^* to $V^{**} : \check{\otimes} W$. Similarly, using the canonical inclusion

$$\tau_r : V \otimes W^{**} \to (V \check{\otimes} W)^{**}$$

given by

$$\langle v \otimes \tilde{w}, F \rangle = \langle \tilde{w}, F(v \otimes \cdot) \rangle$$

for all $v \in V, \tilde{w} \in W^{**}$ and $F \in (V \check{\otimes} W)^*$, we may obtain the *right augmented injective tensor product* $\check{\otimes}$:on $V \otimes W^{**}$. If we let $V \check{\otimes} : W^{**}$ denote its completion, then the identity map on $V \otimes W^{**}$ extends to a complete contraction

$$\phi_r^* : V \check{\otimes} : W^{**} \to V \check{\otimes} W^{**},$$

which can be identified with the restriction of ϕ^* to $V \check{\otimes} : W^{**}$.

PROPOSITION 3.2. *Let V be a TRO and B a C^*-algebra.*

(1) *$V^{**} : \check{\otimes} B$ and $V \check{\otimes} : B^{**}$ are TROs and can be identified with the off-diagonal corners of $A(V)^{**} : \check{\otimes} B$ and $A(V) \check{\otimes} : B^{**}$, respectively.*

(2) *The induced complete contractions ϕ_l^* and ϕ_r^* are TRO-homomorphisms from $V^{**} : \check{\otimes} B$ and $V \check{\otimes} : B^{**}$ onto $V^{**} \check{\otimes} B$ and $V \check{\otimes} B^{**}$, respectively.*

Proof. Let $\iota_V : V \hookrightarrow A(V)$ denote the canonical inclusion from V into $A(V)$. We may identify V with the off-diagonal corner $\iota_V(V) = eA(V)e^{\perp}$ by (2.7), and identify V^{**} with the off-diagonal corner $eA(V)^{**}e^{\perp}$ of $A(V)^{**}$ (see Proposition 2.4). Then it is easy to see that $V^{**} \otimes B$ can be identified with the off-diagonal corner

$$eA(V)^{**}e^{\perp} \otimes B = (e \otimes 1)(A(V)^{**} \otimes B)(e \otimes 1)^{\perp} \qquad (3.1)$$

in the algebraic tensor product $A(V)^{**} \otimes B$, where we let 1 denote the unital element of $M(B)$ and let $(e \otimes 1)^{\perp} = e^{\perp} \otimes 1$.

It is known from Proposition 3.1 that $V \check{\otimes} B$ is a TRO with the linking C^*-algebra $A(V \check{\otimes} B) = A(V) \check{\otimes} B$. Then $(V \check{\otimes} B)^{**}$ is a W^*-TRO and can be identified with the off-diagonal corner of the von Neumann algebra

$$(A(V) \check{\otimes} B)^{**} = \begin{bmatrix} (C(V) \check{\otimes} B)^{**} & (V \check{\otimes} B)^{**} \\ (V^{\sharp} \check{\otimes} B)^{**} & (D(V) \check{\otimes} B)^{**} \end{bmatrix}$$

by Proposition 2.4. Since the canonical inclusion

$$\tau_l : A(V)^{**} : \check{\otimes} B \hookrightarrow (A(V) \check{\otimes} B)^{**} \tag{3.2}$$

is a C^*-inclusion (also see [1] or [13]), we can deduce from (3.1) and the following completely isometric diagram

$$\begin{array}{ccc} A(V)^{**} : \check{\otimes} B & \xrightarrow{\tau_l} & (A(V) \check{\otimes} B)^{**} \\ \uparrow & & \uparrow \\ V^{**} : \check{\otimes} B & \xrightarrow{\tau_l} & (V \check{\otimes} B)^{**} \end{array} \tag{3.3}$$

that $V^{**} : \check{\otimes} B$ is a TRO, which is (completely isometrically) TRO-isomorphic to the off-diagonal corner of $A(V)^{**} : \check{\otimes} B$, i.e. we have the TRO-isomorphism

$$V^{**} : \check{\otimes} B \cong (e \otimes 1)(A(V)^{**} : \check{\otimes} B)(e \otimes 1)^{\perp}.$$

Using the same argument, we can prove that $V \check{\otimes} : B^{**}$ is a TRO and we have the TRO-isomorphism

$$V \check{\otimes} : B^{**} \cong (e \otimes 1)(A(V) \check{\otimes} : B^{**})(e \otimes 1)^{\perp}.$$

To prove (2), we note that since $\phi_l^* : V^{**} : \check{\otimes} B \to V^{**} \check{\otimes} B$ is the completely contractive extension of the identity map on $V^{**} \otimes B$, it is a TRO-homomorphism from $V^{**} : \check{\otimes} B$ into $V^{**} \check{\otimes} B$. Since the range of ϕ_l^* is norm closed and contains the dense subspace $V^{**} \otimes B$, it must be onto. Using similar arguments, we can show that $\phi_r^* : V \check{\otimes} : B^{**} \to V \check{\otimes} B^{**}$ is TRO-homomorphism from $V \check{\otimes} : B^{**}$ onto $V \check{\otimes} B^{**}$. ∎

Remark 3.3. We note that if V is a TRO and B is a C^*-algebra, then we can obtain the TRO-isomorphism

$$V \check{\otimes} : B^{**} = (e \otimes 1)(A(V) \check{\otimes} : B^{**})(e \otimes 1)^{\perp},$$

and thus obtain the C^*-isomorphisms

$$C(V)\check{\otimes} : B^{**} = (e \otimes 1)(A(V)\check{\otimes} : B^{**})(e \otimes 1)$$

and

$$D(V)\check{\otimes} : B^{**} = (e^{\perp} \otimes 1)(A(V)\check{\otimes} : B^{**})(e^{\perp} \otimes 1).$$

Therefore, we can conclude that

$$A(V\check{\otimes} : B^{**}) = A(V)\check{\otimes} : B^{**}.$$

However, the situation is more subtle for the connection between $A(V)^{**} : \check{\otimes} B$ and $V^{**} : \check{\otimes} B$. For the details, the readers are referred to the proof for Theorem 4.3.

4. EXACTNESS AND LOCAL REFLEXIVITY FOR TROS

Let us recall that an operator space V satisfies *condition* C'_λ for some $\lambda \geqslant 1$ if for every operator space W,

$$\phi_r^* : V\check{\otimes} : W^{**} \to V\check{\otimes} W^{**}$$

is a (completely) contractive linear isomorphism with $\|(\phi_r^*)^{-1}\| \leqslant \lambda$, and V satisfies *condition* C''_λ if for every operator space W,

$$\phi_l^* : V^{**} : \check{\otimes} W \to V^{**}\check{\otimes} W$$

is a (completely) contractive linear isomorphism with $\|(\phi_l^*)^{-1}\| \leqslant \lambda$.

Since every operator space is contained in a unital C^*-algebra and the augmented injective tensor products are injective, it suffices to replace W by a unital C^*-algebra B (or simply by $B = B(H)$) in the above definitions. More precisely, we may easily obtain the following lemma.

LEMMA 4.1. *An operator space V satisfies condition C'_λ (respectively, condition C''_λ) for some $\lambda \geqslant 1$ if and only if for every unital C^*-algebra B (or simply $B = B(H)$)*

$$\phi_r^* : V\check{\otimes} : B^{**} \to V\check{\otimes} B^{**}$$

(*respectively*,

$$\phi_l^* : V^{**} : \check{\otimes} B \to V^{**}\check{\otimes} B)$$

is a (completely) contractive linear isomorphism with $\|(\phi_r^*)^{-1}\| \leqslant \lambda$ *(respectively,* $\|(\phi_l^*)^{-1}\| \leqslant \lambda$*).*

Proof. To see this, let us assume that for every $B = B(H)$

$$\phi_r^* : V\check{\otimes} : B^{**} \to V\check{\otimes}B^{**}$$

is a contractive linear isomorphism with $\|(\phi_r^*)^{-1}\| \leqslant \lambda$. Then for any operator space W we may identify W as an operator subspace of some $B = B(H)$ and thus obtain the completely isometric inclusions

$$V\check{\otimes} : W^{**} \hookrightarrow V\check{\otimes} : B^{**} \qquad \text{and} \qquad V\check{\otimes}W^{**} \hookrightarrow V\check{\otimes}B^{**}.$$

Then we can deduce from the diagram

$$\begin{array}{ccc} V\check{\otimes} : B^{**} & \stackrel{\phi_r^*}{\to} & V\check{\otimes}B^{**} \\ \uparrow & & \uparrow \\ V\check{\otimes} : W^{**} & \stackrel{\phi_r^*}{\to} & V\check{\otimes}W^{**} \end{array}$$

that

$$\phi_r^* : V\check{\otimes} : W^{**} \to V\check{\otimes}W^{**}$$

is a completely contractive linear isomorphism with $\|(\phi_r^*)^{-1}\| \leqslant \lambda$. This shows that V satisfies condition C'_λ.

The equivalence for condition C''_λ can be proved by a similar argument. ∎

It is obvious that condition C'_1 (respectively, condition C''_1) implies condition C'_λ (respectively, condition C''_λ). The converse does not hold for general operator spaces (for example, see the Appendix). However, we may obtain the following result for TROs (respectively, for C^*-algebras).

PROPOSITION 4.2. *Let V be a TRO. Then V satisfies condition C'_λ (respectively, condition C''_λ) if and only if V satisfies condition C'_1 (respectively, condition C''_1).*

Proof. Let us assume that V is a TRO satisfying condition C'_λ for some $\lambda \geqslant 1$. Then for any C^*-algebra B,

$$\phi_r^* : V\check{\otimes} : B^{**} \to V\check{\otimes}B^{**}$$

induces a contractive linear isomorphism from $V\check{\otimes} : B^{**}$ onto $V\check{\otimes}B^{**}$. Since V is a TRO, it is known from Proposition 3.2 that $V\check{\otimes} : B^{**}$ is a TRO and ϕ_r^* is a TRO-isomorphism from $V\check{\otimes} : B^{**}$ onto $V\check{\otimes}B^{**}$. It follows from Proposition 2.1 that ϕ_r^* must be a (completely) isometric isomorphism, and thus we must have $\|(\phi_r^*)^{-1}\| = 1$. This shows that a TRO V satisfies condition C'_λ for some $\lambda \geqslant 1$ if and only if it satisfies condition C'_1.

We can similarly prove the equivalence of condition C''_λ and C''_1. ∎

THEOREM 4.3. *Let V be a TRO. Then V satisfies condition C'_1 (respectively, condition C''_1) if and only if its linking C^*-algebra $A(V)$ satisfies condition C'_1 (respectively, condition C''_1).*

Proof. Let us first assume that V satisfies condition C'_1. Then for every unital C^*-algebra B, we have the TRO-isomorphism $V\check{\otimes} : B^{**} = V\check{\otimes}B^{**}$ and thus obtain the C^*-isomorphisms

$$A(V)\check{\otimes} : B^{**} = A(V\check{\otimes} : B^{**}) = A(V\check{\otimes}B^{**})$$

$$= A(V)\check{\otimes}B^{**}$$

from Lemma 4.1, Remark 3.3 and Proposition 3.1. This shows that $A(V)$ satisfies condition C'_1. The other direction is obvious since condition C'_1 passes to subspaces (see [10]).

The proof for condition C''_1 is more complicated since $A(V)^{**}\check{\otimes}B$ and $A(V)^{**} : \check{\otimes}B$ are not equal to the linking C^*-algebras of $V^{**}\check{\otimes}B$ and $V^{**} : \check{\otimes}B$, respectively. Let us assume that B is a unital C^*-algebra. It is known from Proposition 3.2 that

$$V^{**}\check{\otimes}B \cong (e \otimes 1)(A(V)^{**}\check{\otimes}B)(e \otimes 1)^\perp$$

is the off-diagonal corner of the C^*-algebra $A(V)^{**}\check{\otimes}B$. Then we have

$$C(V^{**}\check{\otimes}B) \subseteq (e \otimes 1)(A(V)^{**}\check{\otimes}B)(e \otimes 1) = C(V)^{**}\check{\otimes}B.$$

Since we have $C(V)^{**} = M(C(V^{**}))$ by Proposition 2.4, we can conclude that

$$C(V)^{**}\check{\otimes}B \subseteq M(C(V^{**}\check{\otimes}B)).$$

It follows that for every $u \in C(V)^{**}\check{\otimes}B$, we have

$$\|u\|_{C(V)^{**}\check{\otimes}B} = \sup\{\|ux\|_{V^{**}\check{\otimes}B} : x \in V^{**}\check{\otimes}B, \|x\|_{V^{**}\check{\otimes}B} < 1\}. \tag{4.1}$$

We may also obtain

$$\|u\|_{C(V)^{**}:\check{\otimes}B} = \sup\{\|ux\|_{V^{**}:\check{\otimes}B} : x \in V^{**} : \check{\otimes}B, \|x\|_{V^{**}:\check{\otimes}B} < 1\} \tag{4.2}$$

by applying a similar calculation.

If V satisfies condition C_1'', then for any unital C^*-algebra B we have the TRO-isomorphism $V^{**} : \check{\otimes} B = V^{**} \check{\otimes} B$, and thus for every $u \in C(V)^{**} \otimes B$ we can obtain

$$\begin{aligned}\|u\|_{C(V)^{**}:\check{\otimes}B} &= \sup\{\|ux\|_{V^{**}:\check{\otimes}B} : x \in V^{**} : \check{\otimes}B, \|x\|_{V^{**}:\check{\otimes}B} < 1\} \\ &= \sup\{\|ux\|_{V^{**}\check{\otimes}B} : x \in V^{**}\check{\otimes}B, \|x\|_{V^{**}\check{\otimes}B} < 1\} = \|u\|_{C(V)^{**}\check{\otimes}B}\end{aligned}$$

from (4.1) and (4.2). This shows that

$$C(V)^{**} : \check{\otimes} B = C(V)^{**} \check{\otimes} B.$$

Similarly, we can prove

$$D(V)^{**} : \check{\otimes} B = D(V)^{**} \check{\otimes} B.$$

Therefore, the canonical C^*-homomorphism ϕ_l^* from $A(V)^{**} : \check{\otimes} B$ onto $A(V)^{**} \check{\otimes} B$ must be an injection. This shows that

$$A(V)^{**} : \check{\otimes} B = A(V)^{**} \check{\otimes} B,$$

and thus $A(V)$ satisfies condition C_1''.

The converse is also obvious since condition C_1'' passes to subspaces (see [10]). ∎

We note that conditions C_λ' and C_λ'' are closely related to λ-exactness and λ-local reflexivity, respectively. We recall that an operator space V is said to be *λ-exact* (for some $\lambda \geqslant 1$) if for every finite-dimensional subspace $E \subseteq V$ and $\varepsilon > 0$, there exists a linear isomorphism $\varphi : E \to S$ from E onto a subspace S of some M_n such that $\|\varphi\|_{\mathrm{cb}} \|\varphi^{-1}\|_{\mathrm{cb}} < \lambda + \varepsilon$. An operator space is said to be *λ-locally reflexive* if for every finite-dimensional subspace $E \subseteq V^{**}$, there exists a net of complete bounded maps $\varphi_\alpha : E \to V$ such that $\|\varphi_\alpha\|_{\mathrm{cb}} \leqslant \lambda$ and $\varphi_\alpha \to \iota_E$ in the point-weak* topology. An operator space is usually called *locally reflexive* if it is 1-locally reflexive. The exactness for C^*-algebras was first introduced by Kirchberg [23], and this was extended to operator spaces by Pisier [33]. The local reflexivity for operator spaces was first introduced by Effros–Haagerup [10].

It is known (see [13, Chap. 14]) that an operator space satisfies condition C_1' (respectively, condition C_1'') if and only if it is 1-exact (respectively, locally reflexive). Using a similar argument, we can easily show that an operator space V satisfies condition C_λ' (respectively, condition C_λ'') if and only if V is λ-exact (respectively, λ-locally reflexive). Then we may summarize our results in the following theorems.

THEOREM 4.4. *Let V be a TRO. Then the following are equivalent*:

(1) V *is* 1-*exact* (*or equivalently*, λ-*exact*),
(2) V satisfies condition C_1' (*or equivalently*, *condition* C_λ'),
(3) $A(V)$ *satisfies condition* C_1' (*or equivalently*, *condition* C_λ'),
(4) $A(V)$ *is* 1-*exact* (*or equivalently*, λ-*exact*).

It was shown in [12, Sect. 4] that every 1-exact operator space is locally reflexive. Then we can conclude from Theorem 4.4 that every λ-exact TRO must be locally reflexive. However, this is still an open question for general operator spaces.

THEOREM 4.5. *Let V be a TRO. Then the following are equivalent*:

(1) V *is locally reflexive* (*or equivalently*, λ-*locally reflexive*),
(2) V *satisfies condition* C_1'' (*or equivalently*, *condition* C_λ''),
(3) $A(V)$ *satisfies condition* C_1'' (*or equivalently*, *condition* C_λ''),
(4) $A(V)$ *is locally reflexive* (*or equivalently*, λ-*locally reflexive*).

5. MAXIMAL TRO TENSOR PRODUCT

Given C^*-algebras A and B, the operator space injective tensor product $A\check{\otimes}B$ is just the *minimal* C^*-algebra tensor product on $A \otimes B$. On the other hand, there is a *maximal* C^*-algebra tensor product on $A \otimes B$ given by

$$\|x\|_{\max} = \sup\{\|\pi_A \cdot \pi_B(x)\|\},$$

where the supremum is taken over all C^*-homomorphisms $\pi_A : A \to B(H)$ and $\pi_B : B \to B(H)$ with commuting ranges, i.e. we have

$$\pi_A(a)\pi_B(b) = \pi_B(b)\pi_A(a)$$

for all $a \in A$ and $b \in B$. The readers are referred to Takesaki's book [41] for details.

Motivated by this, we may define the *maximal* tensor product $\otimes^{t\max}$ for TROs. Given TROs V and W, there is a canonical triple product on the algebraic tensor product $V \otimes W$ given by

$$(v_1 \otimes w_1)(v_2 \otimes w_2)^*(v_3 \otimes w_3) = v_1 v_2^* v_3 \otimes w_1 w_2^* w_3.$$

If we are given TRO-homomorphisms $\theta_V : V \to B(H)$ and $\theta_W : W \to B(H)$ such that

$$\theta_V(v)\theta_W(w) = \theta_W(w)\theta_V(v) \text{ and } \theta_V(v)\theta_W(w)^* = \theta_W(w)^*\theta_V(v) \qquad (5.1)$$

for all $v \in W$ and $w \in W$, then we can define a linear map $\theta_V \cdot \theta_W : V \otimes W \to B(H)$ by letting

$$\theta_V \cdot \theta_W(x) = \sum_i \theta_V(v_i)\theta_W(w_i)$$

for $x \in V \otimes W$ with $x = \sum_i v_i \otimes w_i$. This is a well-defined linear map which preserves the triple product from $V \otimes W$ into $B(H)$, and extends to a contraction from $V \hat{\otimes} W$ into $B(H)$. Then for every $x \in V \otimes W$, we can define

$$\|x\|_{t\max} = \sup\{\|\theta_V \cdot \theta_W(x)\|\} \leqslant \|x\|_{V\hat{\otimes} W} < \infty,$$

where the supremum is taken over all TRO-homomorphisms $\theta_V : V \to B(H)$ and $\theta_W : W \to B(H)$ satisfying (5.1). This is a well-defined (operator space) cross norm on $V \otimes W$ since

$$\|x\|_\vee \leqslant \|x\|_{t\max} \leqslant \|x\|_\wedge \tag{5.2}$$

for every $x \in V \otimes W$. We let $V \otimes^{t\max} W$ denote the norm completion of $V \otimes W$ with respect to this tensor norm. There is a canonical completely isometric triple product preserving inclusion

$$\theta = \bigoplus_{\{\theta_V, \theta_W, H\}} \theta_V \cdot \theta_W : V \overset{t\max}{\otimes} W \to \Pi_{\{\theta_V, \theta_W, H\}} B(H)$$

given by $\theta(u) = \oplus_{\{\theta_V, \theta_W, H\}} \theta_V \cdot \theta_W(u)$ for all $u \in V \otimes W$, where $\{\theta_V, \theta_W, H\}$ are taken over all TRO-homomorphisms of V and W satisfying (5.1). Therefore, $V \otimes^{t\max} W$ is a TRO, which can be identified with the norm closure of $\theta(V \otimes W)$ in $\Pi_{\{\theta_V, \theta_W, H\}} B(H)$.

LEMMA 5.1. *Let B be a C^*-algebra. Then every TRO-homomorphism $\theta : B \to B(H)$ has the form $\theta = v\pi$, where $\pi : B \to B(H)$ is a C^*-homomorphism and v is a partial isometry in $B(H)$.*

Proof. Let us first assume that B is a unital C^*-algebra. We let $1 \in B$ denote the unital element. If $\theta : B \to B(H)$ is a TRO-homomorphism, then $v = \theta(1)$ is a partial isometry in $B(H)$ since it is a contractive operator satisfying

$$vv^*v = \theta(1)\theta(1)^*\theta(1) = \theta(11^*1) = v.$$

Let $\pi : B \to B(H)$ be the complete contraction given by

$$\pi(a) = v^*\theta(a)$$

for all $a \in B$. Then $p = v^*v = \pi(1)$ is the orthogonal projection from H onto $pH = v^*H$, and π is a unital completely contractive algebraic homomorphism from B into $B(pH)$ since

$$\pi(a) = v^*\theta(a) = (v^*vv^*)(\theta(a)\theta(1)^*\theta(1)) = p\pi(a)p$$

and

$$\pi(ab) = v^*\theta(a1^*b) = v^*\theta(a)\theta(1)^*\theta(b) = \pi(a)\pi(b)$$

for all $a, b \in B$. It follows (see [31]) that π is a unital C^*-homomorphism from B into $B(pH)$ such that

$$\theta(a) = \theta(1)\theta(1)^*\theta(a) = v\pi(a).$$

If B is non-unital, we may pass the argument to its second dual B^{**}. It is known from the TRO analogue of Kaplansky's density theorem that every TRO-homomorphism $\theta : B \to B(H)$ extends uniquely to a weak* continuous TRO-homomorphism $\tilde{\theta} : B^{**} \to B(H)$. Since B^{**} is a von Neumann algebra and thus unital, we may apply the unital case to $\tilde{\theta}$ and thus obtain the result for θ. ∎

If $W = B$ is a C^*-algebra and $\pi : B \to B(H)$ is a C^*-homomorphism, then the first commuting condition in (5.1) implies the second one. In this case, we can simply require that θ and π have commuting ranges. It is clear that for every $x \in V \otimes B$, we have

$$\sup\{\|\theta \cdot \pi(x)\|\} \leqslant \|x\|_{t\max},$$

where the supremum is taken over all TRO-homomorphisms $\theta : V \to B(H)$ and C^*-homomorphisms $\pi : B \to B(H)$ with commuting ranges. On the other hand, it follows from Lemma 5.1 that

$$\begin{aligned} \|x\|_{t\max} &= \sup\{\|\theta_V \cdot \theta_B(x)\|\} \\ &= \sup\{\|\theta_V \cdot v\pi(x)\|\} \\ &\leqslant \sup\{\|\theta_V \cdot \pi(x)\|\}, \end{aligned}$$

where θ_V and θ_B are TRO-homomorphisms satisfying the commuting conditions in (5.1), and θ_V and π have the commuting ranges. This shows that we actually have the following result.

PROPOSITION 5.2. *Let V be a TRO and B a C^*-algebra. Then for every $x \in V \otimes B$, we have*

$$\|x\|_{t\max} = \sup\{\|\theta \cdot \pi(x)\|\}, \tag{5.3}$$

where the supremum is taken over all TRO-homomorphisms $\theta : V \to B(H)$ and C^*-homomorphisms $\pi : B \to B(H)$ with commuting ranges.

Similarly, we may obtain the following result for C^*-algebras.

PROPOSITION 5.3. *Let A and B be C^*-algebras. For any $x \in A \otimes B$, we have*

$$\|x\|_{t\max} = \sup\{\|\pi_A \cdot \pi_B(x)\|\} = \|x\|_{\max}, \tag{5.4}$$

where the supremum is taken over all C^*-homomorphisms $\pi_A : A \to B(H)$ and $\pi_B : B \to B(H)$ with commuting ranges.

Therefore, we have the TRO-isomorphism $A \otimes^{t\max} B = A \otimes^{\max} B$.

Given a TRO V and a C^*-algebra B, we let $\imath_V(V)\bar{\otimes}^{\max} B$ denote the norm closure of $\imath_V(V) \otimes B$ in $A(V) \otimes^{\max} B$. Then $\imath_V(V)\bar{\otimes}^{\max} B$ is a TRO.

THEOREM 5.4. *The canonical map $\imath_V \otimes id_B : V \otimes B \to \imath_V(V) \otimes B$ extends to a TRO-isomorphism from $V \otimes^{t\max} B$ onto $\imath_V(V)\bar{\otimes}^{\max} B$, and we have the C^*-isomorphism*

$$A(V \overset{t\max}{\otimes} B) = A(V) \overset{\max}{\otimes} B.$$

Proof. If $\tau : A(V) \to B(H)$ and $\pi : B \to B(H)$ are commuting C^*-homomorphisms, then $\theta = \tau \circ \imath_V$ is a TRO-homomorphism from V into $B(H)$ with commuting range with $\pi(B)$. For any $x \in V \otimes B$, we have

$$\|\tau \cdot \pi(\imath_V \otimes id_B(x))\| = \|\theta \cdot \pi(x)\| \leqslant \|x\|_{V \overset{t\max}{\otimes} B}.$$

This shows that

$$\|\imath_V \otimes id_B(x))\|_{\imath_V(V)\bar{\otimes}^{\max} B} \leqslant \|x\|_{V \overset{t\max}{\otimes} B}.$$

On the other hand, if $\theta : V \to B(H)$ is a TRO-homomorphism, then we can obtain C^*-homomorphisms $\varphi : C \to B(H)$ and $\psi : D \to B(H)$ as given in (2.4) and (2.5) such that

$$\pi_\theta = \begin{bmatrix} \varphi & \theta \\ \theta^* & \psi \end{bmatrix} : A(V) \to M_2(B(H)) = B(H \oplus H)$$

is a well-defined C^*-homomorphism. If $\pi : B \to B(H)$ is a C^*-homomorphism having the commuting range with $\theta(V)$, then $\pi \oplus \pi : B \to B(H \oplus H)$ is a C^*-homomorphism with the commuting range with $\pi_\theta(A(V))$. It follows that for every $x \in V \otimes B$,

$$\|\theta \cdot \pi(x)\| = \|\pi_\theta \cdot (\pi \oplus \pi)(\imath_V \otimes id_B(x))\| \leqslant \|\imath_V \otimes id_B(x))\|_{A(V) \overset{\max}{\otimes} B}.$$

Then we have

$$\|x\|_{V \overset{t\max}{\otimes} B} \leqslant \|\imath_V \otimes id_B(x))\|_{\imath_V(V)\bar{\otimes}^{\max} B}.$$

This shows that the canonical map $\imath_V \otimes id_B$ induces an isometric TRO-isomorphism from $V \otimes^{t\max} B$ onto $\imath_V(V)\bar{\otimes}^{\max} B$.

Let us assume that $\imath_C(C)\bar{\otimes}^{\max} B$ and $\imath_D(D)\bar{\otimes}^{\max} B$ denote the norm closure of $\imath_C(C) \otimes B$ and $\imath_D(D) \otimes B$ in $A(V) \otimes^{\max} B$, respectively. It is easy to show that we have $C(\imath_V(V)\bar{\otimes}^{\max} B) = \imath_C(C)\bar{\otimes}^{\max} B$ and $D(\imath_V(V)\bar{\otimes}^{\max} B) = \imath_D(D)\bar{\otimes}^{\max} B$. If we identify $V \otimes^{t\max} B$ with $V\bar{\otimes}^{\max} B$ in $A(V) \otimes^{\max} B$, we obtain the C^*-isomorphisms

$$A(V \overset{t\max}{\otimes} B) \cong A(\imath_V(V)\bar{\otimes}^{\max} B) = \begin{bmatrix} \imath_C(C)\bar{\otimes}^{\max} B & \imath_V(V)\bar{\otimes}^{\max} B \\ \imath_{\bar{V}}(\bar{V})\bar{\otimes}^{\max} B & \imath_D(D)\bar{\otimes}^{\max} B \end{bmatrix}$$
$$= A(V) \overset{\max}{\otimes} B. \quad \blacksquare$$

The following result is an immediate consequence of Theorem 5.4.

THEOREM 5.5. *Let V be a TRO and B a C^*-algebra. We have $V\check{\otimes}B = V \otimes^{t\max} B$ if and only if $A(V)\check{\otimes}B = A(V) \otimes^{\max} B$.*

Proof. It is easy to see that $A(V)\check{\otimes}B = A(V) \otimes^{\max} B$ implies $V\check{\otimes}B = V \otimes^{t\max} B$.

On the other hand, if $V\check{\otimes}B = V \otimes^{t\max} B$, then we have from Proposition 3.1 and Theorem 5.4 that

$$A(V)\check{\otimes}B = A(V\check{\otimes}B) = A(V \overset{t\max}{\otimes} B) = A(V) \overset{\max}{\otimes} B. \quad \blacksquare$$

Remark 5.6. If V and W are two TROs, then we have the canonical isometric TRO-inclusions

$$V \check{\otimes} W \hookrightarrow V \check{\otimes} A(W) \qquad \text{and} \qquad V \overset{t\max}{\otimes} W \hookrightarrow V \overset{t\max}{\otimes} A(W).$$

Using a similar argument as above, we can show that $V \check{\otimes} W = V \otimes^{t\max} W$ if and only if $V \check{\otimes} A(W) = V \otimes^{t\max} A(W)$. We leave the details to the readers.

6. NUCLEARITY AND INJECTIVITY FOR TROS

In [29] Lance introduced the notion of *nuclearity* for C^*-algebras. We recall that a C^*-algebra A is said to be *nuclear* (or *Lance-nuclear*) if for every C^*-algebra B, there is a unique C^*-algebra tensor norm on $A \otimes B$, i.e. we have $A \check{\otimes} B = A \otimes^{\max} B$. Motivated by this, we say that a TRO V is *Lance-nuclear* if for every C^*-algebra B, there is a unique TRO tensor norm on $V \otimes B$, i.e. we have $V \check{\otimes} B = V \otimes^{t\max} B$. We use the notion of 'Lance-nuclearity' for TROs in this paper to avoid the confusion with another notion 'λ-nuclearity' defined below. The following result is an immediate consequence of Theorem 5.5.

THEOREM 6.1. *A TRO V is Lance-nuclear if and only if its linking C^*-algebra $A(V)$ is nuclear.*

Nuclear C^*-algebras have many nice properties. One of the most important (equivalent) properties for nuclear C^*-algebras is that the identity map on a nuclear C^*-algebra can be approximated by completely bounded finite rank maps which can be factored through matrix algebras. Operator algebraists have used the notion of *λ-nuclearity* for this approximation property. The equivalence was first proved by Choi–Effros [3] for completely positive contractions (see Kirchberg [24] for another proof). It was generalized to the general case by Smith [38] and Pisier [34]. Our goal of this section is to investigate the equivalence between Lance-nuclearity and λ-nuclearity for TROs.

Let us first recall that an operator space V is said to be *λ-nuclear* for some $\lambda \geqslant 1$ if there exist diagrams of completely bounded maps

$$\begin{array}{ccccc} & & M_{n(\alpha)} & & \\ & {}^{\varphi_\alpha}\nearrow & & \searrow{}^{\psi_\alpha} & \\ V & & \xrightarrow{\;id\;} & & V \end{array} \tag{6.1}$$

such that $\|\psi_\alpha\|_{cb}\|\varphi_\alpha\|_{cb} \leqslant \lambda$ and $\psi_\alpha \circ \varphi_\alpha \to id_V$ in the point-norm topology.

If a TRO V is Lance-nuclear, then its linking C^*-algebra $A(V)$ is nuclear by Theorem 6.1, and thus has the approximation by finite rank completely positive contractions. Since V can be identified with off-diagonal corner $\iota_V(V) = eA(V)e^\perp$ of $A(V)$, we may obtain diagrams of complete contractions

$$\begin{array}{ccccc} & & M_{n(\alpha)} & & \\ & \varphi_\alpha \nearrow & & \searrow \psi_\alpha & \\ V \hookrightarrow A(V) & & \xrightarrow{id} & & A(V) \to V, \end{array}$$

which approximately commute in the point-norm topology. This shows that if a TRO V is Lance-nuclear, then it is 1-nuclear and thus λ-nuclear for every $\lambda \geqslant 1$. We will show in the following that λ-nuclearity implies Lance-nuclearity for TROs. To obtain this result, we have to pass to the second dual and show that V^{**} and thus $A(V)^{**}$ are injective. Then using the well-known C^*-algebra result, we can conclude that $A(V)$ is nuclear and thus V is Lance-nuclear by Theorem 6.1. Therefore, all corresponding equivalent conditions still hold for TROs (see Theorem 6.5).

To begin with, let us recall that an operator space V is said to be λ-*injective* if for any operator spaces $W_1 \subseteq W_2$, every complete contraction $\varphi : W_1 \to V$ has a completely bounded extension $\tilde{\varphi} : W_2 \to V$ with $\|\tilde{\varphi}\|_{cb} \leqslant \lambda$. If an operator space is 1-injective, we simply say that it is *injective*. It is known from the Arveson–Wittstock–Hahn–Banach theorem that $B(H)$ is injective. Therefore, an operator space is λ-injective if and only if it is λ-completely complemented in some $B(H)$.

For general operator spaces (even for C^*-algebras), λ-injectivity does not imply injectivity. Surprisingly, Pisier [34] and Christensen–Sinclair [8] independently proved that for von Neumann algebras, λ-injectivity is equivalent to injectivity. It is known from [12, 43] that every W^*-TRO has the form $V = eRe^\perp$ for some von Neumann algebra R and projection $e \in R$. Furthermore, we may assume that the central cover $C_e = C_{e^\perp} = 1$ in R (otherwise, we may replace R by pR with $p = C_e C_{e^\perp}$). Then the following lemma shows that a W^*-TRO is λ-injective if and only if it is injective. We omit the proof since it can be obtained by applying an argument similar to that given in the proof of [12, Theorem 1.3] and by applying the Pisier and Christensen–Sinclair result for λ-injective von Neumann algebras.

LEMMA 6.2. *Let R be an von Neumann algebra and let e and f be projections in R with central covers $C_e = C_f = 1$. If the W^*-TRO $V = eRf$ is λ-injective* (*for some $\lambda \geqslant 1$*), *then R is injective and thus V is injective.*

It was shown in [12] that if V is an injective W^*-TRO, then V is the off-diagonal corner of some injective von Neumann algebra R. In the following theorem, we show that if V is an injective W^*-TRO, then its linking von Neumann algebra $R(V)$ is also injective.

THEOREM 6.3. *Let V be a W^*-TRO. Then V is λ-injective for some $\lambda \geqslant 1$ if and only if $R(V)$ is injective.*

Proof. Let us assume that V is a (non-degenerate) W^*-TRO contained in $B(K, H)$. Then $R(V) = A(V)''$ is a (non-degenerate) von Neumann subalgebra of $B(H \oplus K)$ and we can write $V = eR(V)e^{\perp}$ (see (2.9)). It is obvious that if $R(V)$ is injective, then so is V.

On the other hand, we let $\Omega = V_1$ denote the closed unit ball V_1 of V. Then Ω is a compact convex set with respect to the weak* topology on V. We have from the Krein–Milman theorem that the set $Ext(\Omega)$ of all *extreme points* of Ω is non-empty and satisfies

$$\Omega = \overline{co(Ext(\Omega))}^{\text{weak}^*}.$$

Given any $v \in Ext(\Omega)$, it is known from Zettl [43] that v (identified with $\imath_V(v)$) is a *partial isometry* in V, i.e. it satisfies $vv^*v = v$. Then v is a partial isometry in the von Neumann algebra $R(V)$, and we obtain projections $e_v = vv^* \in eR(V)e = M(C) = C''$ and $f_v = v^*v \in e^{\perp}R(V)e^{\perp} = M(D) = D''$. We let p_v denote the *central cover* of e_v in $R(V)$. Since f_v is equivalent to e_v in $R(V)$, p_v is also the central cover of f_v (see [22, p. 410]). If we let $V_v = e_v V f_v$ and $R_v = p_v R(V)$, then we have

$$V_v = e_v(eR(V)e^{\perp})f_v = e_v R(V) f_v = e_v(p_v R(V))f_v = e_v R_v f_v, \tag{6.2}$$

where the central covers of e_v and f_v are equal to 1 in R_v. It follows from Lemma 6.2 that R_v is an injective von Neumann algebra.

Now let us assume that $p = \bigvee_v p_v$ be the projection in $R(V)$ spanned by all p_v with $v \in Ext(\Omega)$. Then p is a central projection in $R(V)$ such that $pv = v$ for all $v \in Ext(\Omega)$. It follows that we have $px = x$ for every $x \in V$. Since $R(V)$ is a von Neumann subalgebra of $B(H \oplus K)$ generated by V, we can conclude that $px = x$ for every $x \in R(V)$. This shows that $pR(V) = R(V)$. Since for each $v \in Ext(\Omega)$, $p_v R(V)$ is an injective von Neumann algebra, we can conclude from Effros–Lance [11] that $R(V) = pR(V)$ is also injective. ∎

It was shown by Haagerup [15] that a von Neumann algebra R is injective if and only if there exists a constant $\lambda \geqslant 1$ such that for any $n \in \mathbb{N}$ and any complete contraction $T : \ell_\infty(n) \to R$, there exist completely positive maps

$S_i : \ell_\infty(n) \to R$ such that $\max\{\|S_1\|, \|S_2\|\} \leqslant \lambda$ and the induced map

$$\Phi = \begin{bmatrix} S_1 & T \\ T^* & S_2 \end{bmatrix} : x \in \ell_\infty(n) \mapsto \begin{bmatrix} S_1(x) & T(x) \\ T(x)^* & S_2(x) \end{bmatrix} \in M_2(R)$$

is completely positive. For W^*-TROs we may obtain the following result, which will be useful in Section 7.

THEOREM 6.4. *Let V be a W^*-TRO. Then V is injective if and only if there exists a constant $\lambda \geqslant 1$ such that for any $n \in \mathbb{N}$ and any complete contraction $T : \ell_\infty(n) \to V$, there exist completely positive maps $S_1 : \ell_\infty(n) \to M(C)$ and $S_2 : \ell_\infty(n) \to M(D)$ such that* $\max\{\|S_1\|, \|S_2\|\} \leqslant \lambda$ *and the induced map*

$$\Phi = \begin{bmatrix} S_1 & T \\ T^* & S_2 \end{bmatrix} : x \in \ell_\infty(n) \mapsto \begin{bmatrix} S_1(x) & T(x) \\ T(x)^* & S_2(x) \end{bmatrix} \in R(V)$$

is completely positive.

Proof. ($\Rightarrow$) If V is injective, then $R(V)$ is an injective von Neumann algebra by Theorem 6.3, and thus we may easily obtain the result for $\lambda = 1$ (and thus for every $\lambda \geqslant 1$) by applying Paulsen's off-diagonal trick (see [31]).

($\Leftarrow$) We need some notions and arguments developed in the proof of Theorem 6.3. Let us first recall that if V is a (non-degenerate) W^*-TRO contained in $B(K, H)$, then we have $M(C) = C''$, $M(D) = D''$ and

$$R(V) = A(V)'' = \begin{bmatrix} C'' & V \\ V^\sharp & D'' \end{bmatrix}.$$

For every partial isometry $v \in Ext(\Omega)$, where $\Omega = V_1$ is the closed unit ball of V, we obtain two projections $e_v = vv^* \in C''$ and $f_v = v^*v \in D''$, respectively. It is known from Zettl [43] that $V_v = e_v V f_v$ is a von Neumann algebra with multiplication and involution given by

$$x \cdot y = xv^*y \qquad \text{and} \qquad x^\sharp = vx^*v$$

for all $x, y \in V_v$. There is a complete isometry φ_v from V_v onto $e_v C'' e_v$ given by

$$\varphi_v : x \in V_v \to xv^* \in e_v C'' e_v$$

which sends the unital element $v \in V_v$ to the unital element e_v in $e_v C'' e_v$. Then φ_v is a unital complete order isomorphism and thus is a unital (spatial) $*$-isomorphism from V_v onto the von Neumann algebra $e_v C'' e_v$. Similarly, the map

$$\psi_v : x \in V_v \to v^* x \in f_v D'' f_v$$

is a unital (spatial) $*$-isomorphism from V_v onto the von Neumann algebra $f_v D'' f_v$.

If we are given a complete contraction $T : \ell_\infty(n) \to V_v$, then it is known from the hypothesis that there exist completely positive maps $S_1 : \ell_\infty(n) \to C''$ and $S_2 : \ell_\infty(n) \to D''$ such that $\max\{\|S_1\|, \|S_2\|\} \leqslant \lambda$ and the induced map

$$\Phi = \begin{bmatrix} S_1 & T \\ T^* & S_2 \end{bmatrix} : x \in \ell_\infty(n) \mapsto \begin{bmatrix} S_1(x) & T(x) \\ T(x)^* & S_2(x) \end{bmatrix} \in R(V)$$

is completely positive. Then $S_{e_v} = e_v S_1 e_v : \ell_\infty(n) \to e_v C'' e_v$ and $S_{f_v} = f_v S_2 f_v : \ell_\infty(n) \to f_v D'' f_v$ are completely positive maps such that $\max\{\|S_{e_v}\|, \|S_{f_v}\|\} \leqslant \lambda$ and the induced map

$$\Phi_v = \begin{bmatrix} S_{e_v} & T \\ T^* & S_{f_e} \end{bmatrix} : x \in \ell_\infty(n) \mapsto \begin{bmatrix} S_{e_v}(x) & T(x) \\ T(x)^* & S_{f_v}(x) \end{bmatrix} \in \begin{bmatrix} e_v C'' e_v & V_v \\ V_v^\sharp & f_v D'' f_v \end{bmatrix}$$

is completely positive since $\Phi_v = (e_v \oplus f_v)\Phi(e_v \oplus f_v)$. Then we can conclude from Haagerup's result that V_v is an injective von Neumann algebra. Let p_v be the central cover of e_v and f_v. It follows from (6.2) and Lemma 6.2 that $R_v = p_v R(V)$ is an injective von Neumann algebra. Using the same argument as that given in Theorem 6.3, we can conclude that $R(V)$ is an injective von Neumann algebra, and thus V is an injective W^*-TRO. ∎

Summarizing our results, we can obtain the following theorem.

THEOREM 6.5. *Let V be a TRO. Then the following are equivalent*:

(1) *V is Lance-nuclear*,
(2) *V is* 1-*nuclear* (*or λ-nuclear for some $\lambda \geqslant 1$*),
(3) *V^{**} is injective* (*or λ-injective for some $\lambda \geqslant 1$*),
(4) *$A(V)^{**}$ is injective*,
(5) *$A(V)$ is nuclear*.

Proof. We have discussed (1) $\Rightarrow$ (2) after Theorem 6.1. If V is a λ-nuclear TRO, then it is λ-exact and thus 1-exact by Theorem 4.4. This implies that V is locally reflexive (see [12, Sect. 4]). We may use the same argument as that given in [12, Theorem 4.5(i) $\Rightarrow$ (ii)] to show that V^{**} is λ-injective. This proves (2) $\Rightarrow$ (3). It follows from Lemma 6.2 that the injectivity is equivalent to λ-injectivity on W^*-TROs. To prove (3) $\Rightarrow$ (4), we can assume that V^{**} is an injective W^*-TRO. Since we have

$A(V)^{**} = R(V^{**})$ by Proposition 2.4, we can conclude from Theorem 6.3 that $A(V)^{**}$ is an injective von Neumann algebra. Using Connes' deep work [7], Choi and Effros proved (4) $\Leftrightarrow$ (5) for C^*-algebras in [4, 5]. Finally, we can obtain (5) $\Rightarrow$ (1) from Theorem 6.1. ∎

An intriguing aspect for TROs is that, like C^*-algebras, one does not need to assume the local reflexivity to prove (3) $\Rightarrow$ (2) in Theorem 6.5. Kirchberg also observed this in [25, Sect. 6] for C^*-spaces, where he indicated that an operator space V is 1-nuclear if and only if it is a C^*-space and its second dual V^{**} is injective. However, the local reflexivity is a necessary condition for general operator spaces (see [12]) since there exist examples of operator spaces V, for which V^{**} are 1-injective, but V are not 1-nuclear (see [13, 26]).

A dual operator space V is said to be λ-*semidiscrete* (or *semidiscrete* if $\lambda = 1$) if there exist diagrams of weak* continuous completely bounded maps

$$\begin{array}{ccccc} & & M_{n(\alpha)} & & \\ & \varphi_\alpha \nearrow & & \searrow \psi_\alpha & \\ V & & \xrightarrow{id} & & V \end{array}$$

such that $\|\varphi_\alpha\|_{\mathrm{cb}}\|\psi_\alpha\|_{\mathrm{cb}} \leqslant \lambda$ and $\psi_\alpha \circ \varphi_\alpha \to id_V$ in the point-weak* topology. It is known from [12, Proposition 3.1] that a dual operator space is injective if and only if it is semidiscrete. In this case, we can conclude that V is weak* homeomorphic and completely isometrically isomorphic to an (injective) W^*-TRO. For $\lambda > 1$, it is still an open question whether every λ-injective (or λ-semidiscrete) dual operator space is completely isomorphic to some injective (or semidiscrete) W^*-TRO. As a consequence of Theorem 6.3, we may obtain the following corollary for W^*-TROs.

Corollary 6.6. *Let V be a W^*-TRO. Then the following are equivalent*:

(1) *V is λ-injective for some $\lambda \geqslant 1$* (*or equivalently*, *V* is injective),
(2) *V is λ-semidiscrete for some $\lambda \geqslant 1$* (*or equivalently*, *V is semidiscrete*).

7. DECOMPOSABLE MAPS AND PISIER'S δ-NORM ON TROS

In [15] Haagerup introduced the notion of decomposable maps between C^*-algebras. We recall that given C^*-algebras A and B, a linear map $T : A \to B$ is called *decomposable* if it can be written as

$$T = (T_1 - T_2) + i(T_3 - T_4),$$

where $T_i : A \to B$ are completely positive maps. In this case, there exist completely positive maps $S_i : A \to B$ such that the map

$$\Phi = \begin{bmatrix} S_1 & T \\ T^* & S_2 \end{bmatrix} : a \in A \mapsto \begin{bmatrix} S_1(a) & T(a) \\ T(a)^* & S_2(a) \end{bmatrix} \in M_2(B)$$

is completely positive, or equivalently, the corresponding map

$$\Psi = \begin{bmatrix} S_1 & T \\ T^* & S_2 \end{bmatrix} : \begin{bmatrix} a_1 & a_2 \\ a_3^* & a_4 \end{bmatrix} \in M_2(A) \mapsto \begin{bmatrix} S_1(a_1) & T(a_2) \\ T(a_3)^* & S_2(a_4) \end{bmatrix} \in M_2(B) \qquad (7.1)$$

is completely positive. Then $D(A, B)$, the space of all decomposable maps from A to B, is a Banach space with the norm given by

$$\|T\|_{\text{dec}} = \inf\{\max\{\|S_1\|, \|S_2\|\}\}.$$

Motivated by (7.1), we can define decomposable maps between TROs as follows. Let V and W be TROs. A linear map $T : V \to W$ is called *decomposable* if there exists completely positive maps $S_1 : C(V) \to C(W)$ and $S_2 : D(V) \to D(W)$ such that the map $\Psi : A(V) \to A(W)$ given by

$$\Psi = \begin{bmatrix} S_1 & T \\ T^* & S_2 \end{bmatrix} : \begin{bmatrix} c & v \\ w^* & d \end{bmatrix} \in A(V) \mapsto \begin{bmatrix} S_1(c) & T(v) \\ T(w)^* & S_2(d) \end{bmatrix} \in A(W) \qquad (7.2)$$

is completely positive. In this case, we let

$$\|T\|_{\text{dec}} = \inf\{\max\{\|S_1\|, \|S_2\|\}\},$$

where the infimum is taken over all completely positive maps S_i in (7.2). If $V = B$ and $W = C$ are C^*-algebras, we have $A(V) = M_2(B)$ and $A(W) = M_2(C)$. Then it is clear that our definition is a natural generalization of decomposable maps on C^*-algebras.

It is easy to see from the definition that if $T_1 : V \to W$ and $T_2 : W \to Z$ are decomposable maps, then $T_2 \circ T_1 : V \to Z$ is decomposable with

$$\|T_2 \circ T_1\|_{\text{dec}} \leqslant \|T_2\|_{\text{dec}} \|T_1\|_{\text{dec}}. \qquad (7.3)$$

If V is a TRO, then the canonical inclusion $\imath_V : V \hookrightarrow A(V)$ is a decomposable map with $\|\imath_V\|_{\text{dec}} = 1$. We may obtain this by considering the canonical inclusions $S_1 = \imath_C$ and $S_2 = \imath_D$ of $C(V)$ and $D(V)$ into $A(V)$. Similarly, it is easy to see that the canonical projection $P_V : A(V) \to V$ is a decomposable map with $\|P_V\|_{\text{dec}} = 1$. If $T : V \to W$ is a completely bounded map between TRO's V and W, then $\tilde{T} = \imath_W \circ T \circ P_V : A(V) \to A(W)$ is a completely bounded map between the linking C^*-algebras $A(V)$

and $A(W)$. Since $T = P_W \circ \tilde{T} \circ \iota_V$, we actually have $\|T\|_{cb} = \|\tilde{T}\|_{cb}$. Moreover, we may conclude from (7.3) that $T : V \to W$ is decomposable if and only if $\tilde{T} : A(V) \to A(W)$ is decomposable with $\|T\|_{dec} = \|\tilde{T}\|_{dec}$. Therefore, we may identify $D(V, W)$, the space of all decomposable maps from V into W, with a norm closed subspace of $D(A(V), A(W))$. We can also deduce the inequality

$$\|T\|_{cb} = \|\tilde{T}\|_{cb} \leqslant \|\tilde{T}\|_{dec} = \|T\|_{dec}. \tag{7.4}$$

Let $T : V \to W$ be a decomposable map. Then for any C^*-algebra B, $T \otimes id_B$ extends to a decomposable map from $V \otimes^{t\max} B$ into $W \otimes^{t\max} B$ with

$$\|T \otimes id_B\|_{dec} \leqslant \|T\|_{dec}. \tag{7.5}$$

This follows from the fact that if $\Psi = \begin{bmatrix} S_1 & T \\ T^* & S_2 \end{bmatrix} : A(V) \to A(W)$ is completely positive, then $\Psi \otimes id_B$ extends to a completely positive map from $A(V) \otimes^{\max} B$ into $A(W) \otimes^{\max} B$. As a consequence of (7.3)–(7.5), we obtain

$$\|T_1 \otimes T_2\|_{cb} \leqslant \|T_1 \otimes T_2\|_{dec} \leqslant \|T_1\|_{dec}\|T_2\|_{dec} \tag{7.6}$$

for any decomposable maps $T_1 : V \to W$ and $T_2 : B_1 \to B_2$.

In [35], Pisier introduced a δ norm on the tensor product of a C^*-algebra A and an operator space E. We recall that for any $y \in A \otimes E$, the δ norm is defined by

$$\delta(y) = \sup\{\|\pi \cdot \sigma(y)\|\},$$

where the supremum is taken over all C^*-homomorphisms $\pi : A \to B(H)$ and complete contractions $\sigma : E \to B(H)$ with commuting ranges. Pisier proved that the δ norm can also be expressed in the following form:

$$\delta(y) = \inf\left\{ \|x\|_{M_n(E)} \left\| \sum_{i=1}^{n} a_i a_i^* \right\|^{\frac{1}{2}} \left\| \sum_{j=1}^{n} b_j^* b_j \right\|^{\frac{1}{2}} \right\}, \tag{7.7}$$

where the infimum runs over all possible representations of y with the form

$$y = \sum_{ij} a_i b_j \otimes x_{ij}.$$

Motivated by Pisier's result, we can define a $\tilde{\delta}$ norm on the tensor product $V \otimes E$ for a TRO V and an operator space E by letting

$$\tilde{\delta}(y) = \sup\{\|\theta \cdot \sigma(y)\|\}$$

for $y \in V \otimes E$, where the supremum is taken over all TRO-homomorphisms $\theta : V \to B(H)$ and complete contractions $\sigma : E \to B(H)$ which satisfy the commuting condition

$$\theta(v)\sigma(x) = \sigma(x)\theta(v) \qquad \text{and} \qquad \theta(v)\sigma(x)^* = \sigma(x)^*\theta(v) \tag{7.8}$$

for all $v \in V$ and $x \in E$. We note that if $V = A$ is a C^*-algebra, then we can conclude from Lemma 5.1 that

$$\tilde{\delta}(y) = \delta(y)$$

for all $y \in A \otimes E$.

We let $V \otimes^{\tilde{\delta}} E$ denote the completion of $V \otimes E$ with respect to the $\tilde{\delta}$ norm. If we let $C^*\langle E\rangle$ be the free C^*-algebra generated by E, then every complete contraction $\sigma : E \to B(H)$ extends (uniquely) to a C^*-homomorphism $\pi_\sigma : C^*\langle E\rangle \to B(H)$. The commuting condition (7.8) implies that θ and π_σ have the commuting ranges. Therefore, we may isometrically identify $V \otimes^{\tilde{\delta}} E$ with a norm closed subspace of $V \otimes^{t\max} C^*\langle E\rangle$. As a consequence of this fact and (7.6), we may easily obtain the following result.

Corollary 7.1. *Let $T_1 \in D(V, W)$ be a decomposable map between TRO's V and W, and let $T_2 \in CB(E, F)$ be a completely bounded map between operator spaces E and F. Then for any $y \in V \otimes E$, we have*

$$\tilde{\delta}((T_1 \otimes T_2)(y)) \leqslant \|T_1\|_{\text{dec}}\|T_2\|_{\text{cb}}\tilde{\delta}(y). \tag{7.9}$$

The following proposition shows that there is a close connection between $V \otimes^{\tilde{\delta}} E$ and $A(V) \otimes^{\delta} E$.

Proposition 7.2. *Let V be a TRO and E an operator space. Then we have the diagram of isometric inclusions*

$$\begin{array}{ccc} V \overset{\text{tmax}}{\otimes} C^*\langle E\rangle & \xrightarrow{\iota_V \otimes id_{C^*\langle E\rangle}} & A(V) \overset{\max}{\otimes} C^*\langle E\rangle \\ \uparrow & & \uparrow \\ V \otimes^{\tilde{\delta}} E & \xrightarrow{\iota_V \otimes id_E} & A(V) \otimes^{\delta} E \end{array} . \tag{7.10}$$

Proof. It is obvious that the column inclusions are isometries. It is also known from Theorem 5.4 that we may identify $V \otimes^{t\max} C^*\langle E\rangle$ with the off-diagonal corner $\iota_V(V)\bar{\otimes}^{\max} C^*\langle E\rangle$ in $A(V) \otimes^{\max} C^*\langle E\rangle$, and thus the map on the top is an isometric TRO-inclusion. This implies that the map on the bottom is an isometric inclusion. ∎

It was shown in [35] that $A(V) \otimes^{\delta} E$ can be identified with a quotient of the Haagerup tensor product $A(V) \otimes^h E \otimes^h A(V)$ with the quotient map q given by

$$q(a \otimes x \otimes b) = ab \otimes x.$$

Since we may identify $V \otimes^{t\max} C^*\langle E\rangle$ with the off-diagonal corner

$$\iota_V(V)\bar{\otimes}^{\max} C^*\langle E\rangle = (e \otimes 1)(A(V) \overset{\max}{\otimes} C^*\langle E\rangle)(e^{\perp} \otimes 1)$$

in $A(V) \otimes^{\max} C^*\langle E\rangle$, the map q restricts to a quotient map

$$\tilde{q} : [C(V)V] \otimes^h E \otimes^h \begin{bmatrix} V \\ D(V) \end{bmatrix} = eA(V) \otimes^h E \otimes^h A(V)e^{\perp} \to V \otimes^{\tilde{\delta}} E.$$

Then we may obtain the following expression for $\tilde{\delta}$ from Proposition 7.2 and (7.7). For every $y \in V \otimes E$,

$$\tilde{\delta}(y) = \inf\left\{ \|x\|_{M_n(E)} \|\sum_{i=1}^{n} c_i c_i^* + v_i v_i^*\|_C^{\frac{1}{2}} \|\sum_{j=1}^{n} w_j^* w_j + d_j^* d_j\|_D^{\frac{1}{2}} \right\}, \qquad (7.11)$$

where the infimum runs over all possible representations

$$y = \sum_{ij} (c_i w_j + v_i d_j) \otimes x_{ij}.$$

If $E = F^*$ is a dual operator space, then every element $y \in V \otimes E$ is one-to-one correspondent to a finite rank map $T_y : F \to V$ given by $T_y(f) = (id_V \otimes f)(y)$. We can obtain the following TRO analogue of Pisier's results in [35, Sect. 12].

Lemma 7.3. *Let V be a TRO.*

(1) *If $E = M_n^*$ (or $E = \ell_1(n)$), then for every $y \in V \otimes E$, we have*

$$\tilde{\delta}(y) = \|T_y\|_{\text{dec}}.$$

(2) *If $E = F^*$ is a dual operator space and $y \in V \otimes^{\tilde{\delta}} E$, then we have $\tilde{\delta}(y) < \lambda$ if and only if there exists a diagram of completely bounded*

maps

$$\begin{array}{ccccc} & & M_n & & \\ & \varphi\nearrow & & \searrow\psi & \\ F & & \xrightarrow{T_y} & & V \end{array}$$

such that $\|\varphi\|_{\mathrm{cb}}\|\psi\|_{\mathrm{dec}} < \lambda$.

Therefore, we can obtain

$$\tilde{\delta}(y) = \inf\{\|\varphi\|_{\mathrm{cb}}\|\psi\|_{\mathrm{dec}}\},$$

where the infimum is taken over all decompositions $T_y = \psi \circ \varphi$.

Proof. Let us first recall that the canonical inclusion $\imath_V : V \hookrightarrow A(V)$ and the canonical projection $P_V : A(V) \to V$ are decomposable maps with $\|\imath_V\|_{\mathrm{dec}} = \|P_V\|_{\mathrm{dec}} = 1$. Since $P_V \circ \imath_V = id_V$, it is easy to see that $T_y : F \to V$ is decomposable if and only if $\imath_V \circ T_y : F \to A(V)$ is decomposable. In this case, we have

$$\|T_y\|_{\mathrm{dec}} = \|\imath_V \circ T_y\|_{\mathrm{dec}}.$$

Since $\imath_V \circ T_y : F \to A(V)$ corresponds to the element $(\imath_V \otimes id_E)(y) \in A(V) \otimes E$, we obtain

$$\|\imath_V \circ T_y\|_{\mathrm{dec}} = \delta((\imath_V \otimes id_E)(y))$$

from Pisier [35]. It follows from Proposition 7.2 that

$$\tilde{\delta}(y) = \delta((\imath_V \otimes id_E)(y)) = \|T_y\|_{\mathrm{dec}}.$$

This proves (1). Similarly, we may obtain (2) by applying Pisier's result to the induced map $\imath_V \circ T_y : F \to A(V)$ and the fact that $T_y = P_V \circ (\imath_V \circ T_y)$. ∎

The following proposition is a TRO analogue of [20, Sect. 2].[3]

PROPOSITION 7.4. *Let* $T : V \to W$ *be a finite rank map between TRO's* V *and* W. *Then for every* $\varepsilon > 0$, *there exists a diagram of completely bounded maps*

[3] We wish to thank M. Junge for pointing out this simple argument to us.

$$\begin{array}{ccccc} & & M_n & & \\ & \varphi\nearrow & & \searrow\psi & \\ V & & \xrightarrow{T_y} & & W \end{array}$$

such that $\|\varphi\|_{cb}\|\psi\|_{dec} < (1+\varepsilon)\|T\|_{dec}$.

If we let $y \in W \otimes V^*$ *denote the element corresponding to* T, *then we have*

$$\tilde{\delta}(y) = \|T\|_{dec}. \tag{7.12}$$

Proof. If V and W are C^*-algebras, the result is known by Junge and Le Merdy [20, Sect. 2]. This can be easily generalized to TRO case since $T : V \to W$ is decomposable if and only if the induced map $\tilde{T} = \iota_W \circ T \circ P_V : A(V) \to A(W)$ is decomposable with $\|T\|_{dec} = \|\tilde{T}\|_{dec}$. Then we may obtain the TRO result by simply considering $\tilde{T}$ and the fact that $T = P_W \circ \tilde{T} \circ \iota_V$. ∎

8. MORE EQUIVALENT CONDITIONS FOR NUCLEARITY

Motivated by Kirchberg [27], Pisier [35], and Smith and William [40], we discuss some more conditions equivalent to nuclearity for TRO's.

THEOREM 8.1. *Let* V *be a TRO. Then the following are equivalent*:

(1) V *is Lance-nuclear*,

(2) *for every* (*finite-dimensional*) *operator spaces* E, *we have*

$$V \overset{t\max}{\otimes} C^*\langle E\rangle = V \check{\otimes} C^*\langle E\rangle,$$

(3) *for every* (*finite-dimensional*) *operator space* E, *we have the isometry*

$$V \otimes^{\tilde{\delta}} E = V \check{\otimes} E,$$

(4) *for some* (*or for every*) $\lambda > 1$, *there exists a net of finite rank maps* $T_i : V \to V$ *such that* $\|T_i\|_{dec} \leqslant \lambda$ *and* $T_i \to id_V$ *in the point-norm topology*.

Proof. It is obvious that (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

To prove (3) $\Rightarrow$ (4), we let E be an arbitrary finite-dimensional subspace of V and let $\iota_E : E \hookrightarrow V$ be the inclusion map from E into V. Then condition (3) implies that $\tilde{\delta}(\iota_E) = \|\iota_E\|_{cb} = 1$. It follows from Lemma 7.3 that for any $\lambda > 1$, there exists an integer n and a diagram of completely bounded maps

$$\begin{array}{ccc} & M_n & \\ \varphi\nearrow & & \searrow\psi \\ E & \xrightarrow{\iota_E} & V \end{array} \tag{8.1}$$

such that $\|\varphi\|_{cb} \leqslant 1$ and $\|\psi\|_{dec} < \lambda$. Since M_n is injective, φ has a completely contractive extension $\varphi_E : V \to M_n$ which satisfies $\|\varphi_E\|_{dec} = \|\varphi_E\|_{cb} \leqslant 1$. Let $\psi_E = \psi$. Then we obtain a net of finite rank maps $T_E = \psi_E \circ \varphi_E$ on V such that

$$\|T_E\|_{dec} \leqslant \|\psi_E\|_{dec}\|\varphi_E\|_{dec} < \lambda,$$

and T_E converges to id_V in the point-norm topology. This proves (4).

Finally, let us prove (4) $\Rightarrow$ (1). Let B be an arbitrary C^*-algebra, and let $T_i : V \to V$ be a net of finite rank maps satisfying condition (4). It follows from (7.5) that each $T_i \otimes id_B$ extends to a decomposable map on $V \otimes^{tmax} B$ with

$$\|T_i \otimes id_B\|_{dec} \leqslant \|T_i\|_{dec} \leqslant \lambda.$$

Let $\pi_q : V \otimes^{tmax} B \to V \check{\otimes} B$ denote the canonical quotient map from $V \otimes^{tmax} B$ onto $V \check{\otimes} B$. Since T_i are finite rank maps, we must have $T_i \otimes id_B(ker\pi_q) = \{0\}$, and thus deduce a net of bounded maps

$$T_i \widetilde{\otimes id_B} : V \check{\otimes} B = V \overset{tmax}{\otimes} B/ker\pi_q \to V \overset{tmax}{\otimes} B$$

such that $\|T_i \widetilde{\otimes id_B}\| \leqslant \lambda$. Since $T_i \to id_V$ in the point-norm topology, we can conclude that $T_i \widetilde{\otimes id_B}$ converges to a bounded map on $V \check{\otimes} B$, which extends the identity map on $V \otimes B$. This shows that we must have $V \check{\otimes} B = V \otimes^{tmax} B$, and thus V is Lance-nuclear. ∎

We may weaken the condition (2) in Theorem by considering the TRO-isomorphisms

$$V \overset{tmax}{\otimes} C^*(\mathbb{F}) = V \check{\otimes} C^*(\mathbb{F})$$

for all free groups (with finite or infinite generators). Then we may obtain the following TRO analogue of Lance [29, Theorem 3.3] (also see [11, Theorem 6.3]) and Kirchberg [27, Theorem 1.1 (iii)] for C^*-algebras.

PROPOSITION 8.2. *Let V be a TRO. Then the following are equivalent*:

(1) *For any TRO-inclusion* $\imath : V \hookrightarrow W$ *and any* C^*-algebra B, *the induced map*

$$\imath \otimes id_B : V \overset{tmax}{\otimes} B \to W \overset{tmax}{\otimes} B$$

is an isometric TRO-homomorphism,

(2) *for any free group* $\mathbb{F}$, *we have the TRO-isomorphism*

$$V \overset{tmax}{\otimes} C^*(\mathbb{F}) = V \check{\otimes} C^*(\mathbb{F}),$$

(3) *the linking* C^**-algebra* $A(V)$ *has the WEP.*

Proof. Let V be a TRO non-degenerately contained in $B(K, H)$. It is known from Kirchberg [27] that for every free group $\mathbb{F}$, we have the C^*-isomorphism

$$B(H \oplus K) \overset{\max}{\otimes} C^*(\mathbb{F}) = B(H \oplus K) \check{\otimes} C^*(\mathbb{F}).$$

Taking the off-diagonal corners, we obtain

$$B(K, H) \overset{tmax}{\otimes} C^*(\mathbb{F}) = B(K, H) \check{\otimes} C^*(\mathbb{F}).$$

If we have (1), then we may obtain the following commutative diagram:

$$\begin{array}{ccc} B(K,H) \overset{\mathrm{tmax}}{\otimes} C^*(\mathbb{F}) & = & B(K,H) \check{\otimes} C^*(\mathbb{F}) \\ \uparrow & & \uparrow \\ V \overset{\mathrm{tmax}}{\otimes} C^*(\mathbb{F}) & \to & V \check{\otimes} C^*(\mathbb{F}) \end{array},$$

where the column maps are isometric TRO-inclusions. This implies that the bottom line must be an isometric TRO-isomorphism, i.e. we must have $V \otimes^{tmax} C^*(\mathbb{F}) = V \check{\otimes} C^*(\mathbb{F})$.

If we have (2), then for any free group $\mathbb{F}$ we may obtain the C^*-isomorphism

$$A(V) \overset{\max}{\otimes} C^*(\mathbb{F}) = A(V) \check{\otimes} C^*(\mathbb{F})$$

from Theorem 5.5. This implies that $A(V)$ has WEP by Kirchberg [27].

To see (3) $\Rightarrow$ (1), let us assume that $A(V)$ has WEP. If W is a TRO containing V as a sub-TRO, then it is clear that $A(V)$ is a C^*-subalgebra of

$A(W)$. For any C^*-algebra B, we may obtain the isometric C^*-inclusion

$$A(V) \overset{\max}{\otimes} B \hookrightarrow A(W) \overset{\max}{\otimes} B$$

from Lance [29, Theorem 3.3] (also see [11, Theorem 6.3]). Taking the off-diagonal corners, we can obtain the isometric TRO-inclusion

$$V \overset{t\max}{\otimes} B \hookrightarrow W \overset{t\max}{\otimes} B.$$

Let us recall that a C^*-algebra A is said to have the *weak expectation property* (or simply, *WEP*) of Lance [29] if for every (non-degenerate) faithful representation $\iota : A \hookrightarrow B(H)$ there exists a completely positive contraction $P : B(H) \to \bar{A}^{\text{weak}^*}$ such that $P(x) = x$ for all $x \in A$. We can analogously consider this property for TROs. Following Lance's definition, we say that a TRO V has WEP if for every isometric TRO-inclusion $\iota : V \hookrightarrow B(K, H)$, there exists a complete contraction $P : B(K, H) \to \overline{\iota(V)}^{\text{weak}^*}$ such that $P(v) = v$ for all $v \in V$. Equivalently, we may simply replace $\overline{\iota(V)}^{\text{weak}^*}$ by V^{**} and assume that there exists a complete contraction P from $B(K, H)$ into V^{**} such that $P(v) = v$ for all $v \in V$. To see the equivalence, we may apply the Kaplanski's density theorem for dual TROs (see [43]) to show that every isometric TRO-inclusion $\iota : V \hookrightarrow \bar{V}^{\text{weak}^*}$ can be extended to a weak* continuous TRO-homomorphism $\tilde{\iota} : V^{**} \to \bar{V}^{\text{weak}^*}$.

It is clear that the WEP of $A(V)$ (and thus any of the equivalent conditions in Proposition 8.2) implies the WEP of V. However, we cannot prove that they are equivalent at this moment. The difficulty is that we do not know if we can extend the (completely contractive) weak expectation P from $B(K, H)$ to V^{**} to a completely positive weak expectation $\tilde{P}$ from $B(H \oplus K)$ into $A(V)^{**}$. However, if we add the local reflexivity to V, then we may obtain the following result, which is a TRO analogue of Effros–Haagerup [10, Proposition 5.4] and Robertson–Smith [40, Theorem 2.1].

Corollary 8.3. *Let V be a TRO. Then the following are equivalent*:

(1) *V is Lance-nuclear*,

(2) *V is locally reflexive and for any free group $\mathbb{F}$ we have the TRO-isomorphism*

$$V \overset{t\max}{\otimes} C^*(\mathbb{F}) = V \check{\otimes} C^*(\mathbb{F}),$$

(3) *V is locally reflexive and for any $n \in \mathbb{N}$, we have the isometry*

$$V \otimes^{\tilde{\delta}} \ell_1(n) = V \check{\otimes} \ell_1(n),$$

(3′) *V is locally reflexive and for any* $n \in \mathbb{N}$ *and any* $T : \ell_\infty(n) \to V$, *we have*

$$\|T\|_{\mathrm{dec}} = \|T\|_{\mathrm{cb}},$$

(4) *V is locally reflexive and V has WEP.*

Proof. It is easy to see that we can obtain: (1) ⇒ (2) by Theorem 8.1, (2) ⇒ (3) since for every $n \in \mathbb{N}$ $\ell_1(n)$ is completely isometric to the operator subspace E_n (spanned by the generators) of $C^*(\mathbb{F}_n)$ and the free C^*-algebra $C^*\langle \ell_1(n) \rangle$ generated by $\ell_1(n)$ is equal to the full free group C^*-algebra $C^*(\mathbb{F}_n)$, (3) ⇔ (3′) by Lemma 7.3, and (2) ⇒ (4) by Proposition 8.2. Moreover, (4) ⇒ (1) can be obtained by using the same argument as that given in [10, Proposition 5.4], i.e. we need to show that V^{**} is injective. The readers are referred to [10] for details. Here we only need to prove (3′) ⇒ (1).

Suppose that we have (3′). Then by the local reflexivity, for any $T : \ell_\infty(n) \to V^{**}$ with $\|T\|_{\mathrm{cb}} = 1$, there exists a net of complete contractions $T_\alpha : \ell_\infty(n) \to V$ such that T_α converges to T in the point-weak* topology. Since for every α we have $\|T_\alpha\|_{\mathrm{dec}} = \|T_\alpha\|_{\mathrm{cb}} = 1$, for any given $\lambda > 1$ there exist completely positive maps $S_1^\alpha : \ell_\infty(n) \to C(V)$ and $S_2^\alpha : \ell_\infty(n) \to D(V)$ such that $\max\{\|S_1^\alpha\|, \|S_2^\alpha\|\} < \lambda$ and the map

$$\Psi_\alpha = \begin{bmatrix} S_1^\alpha & T_\alpha \\ T_\alpha^* & S_2^\alpha \end{bmatrix} : M_2(\ell_\infty(n)) \mapsto A(V)$$

defined in (7.1) is completely positive. Passing to subnets if necessary, we may assume that S_1^α and S_2^α converges to completely positive maps $S_1 : \ell_\infty(n) \to C(V)^{**}$ and $S_2 : \ell_\infty(n) \to D(V)^{**}$ in the weak* topology. Then it is easy to see that $\max\{\|S_1\|, \|S_2\|\} \leqslant \lambda$ and that induced map

$$\Psi = \begin{bmatrix} S_1 & T \\ T^* & S_2 \end{bmatrix} : M_2(\ell_\infty(n)) \mapsto A(V)^{**} = R(V^{**})$$

is completely positive. Since $\tau : \ell_\infty(n) \to M_2(\ell_\infty(n))$ given by

$$\tau(x) = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$$

is completely positive, the composition map

$$\Phi = \Psi \circ \tau : x \in \ell_\infty(n) \to \begin{bmatrix} S_1(x) & T(x) \\ T^*(x) & S_2(x) \end{bmatrix} \in R(V^{**}) = A(V)^{**}$$

is completely positive. Then we can conclude from Theorem 6.4 that V^{**} is an injective W^*-TRO. Therefore, V is Lance-nuclear by Theorem 6.5. ∎

Finally we wish to end this section with the following remarks.

Remark 8.4. It has been shown that some local properties of a TRO V are closely related to the corresponding local properties of its linking C^*-algebras $A(V)$. However, this is not true for injectivity. For example, if $V = B(\mathbb{C}, \ell_2)$ is the *column Hilbert space*, then V is an injective TRO. In this case, we have $C = K(\ell_2)$ and $D = \mathbb{C}$. The linking C^*-algebra $A(V) = K(\ell_2 \oplus \mathbb{C})$ is not injective.

To preserve the injectivity, we need to consider $R(V)$ defined in (1.2). If V is a TRO, we may consider Paulsen's operator system

$$\mathscr{L}_V = \begin{bmatrix} \mathbb{C} & V \\ V^{\sharp} & \mathbb{C} \end{bmatrix}.$$

It is known from Hamana [17] and Ruan [37] that the injective envelope $I(\mathscr{L}_V)$ of $\mathscr{L}_V$ has the form

$$I(\mathscr{L}_V) = \begin{bmatrix} I_{11} & I(V) \\ I(V^{\sharp}) & I_{22} \end{bmatrix},$$

where I_{11} and I_{22} are injective C^*-algebras and $I(V)$ and $I(V^{\sharp})$ are the (operator space) injective envelopes of V and $V^{\sharp}$, respectively. Moreover, it was shown by Blecher and Paulsen [2, Sect. 1] that $I_{11} = M(C(I(V)))$ and $I_{22} = M(D(I(V)))$. From this we can conclude that a TRO V is injective (we have $I(V) = V$ in this case) if and only if $R(V) = I(\mathscr{L}_V)$ is an injective C^*-algebra. In particular, if V is a W^*-TRO it is known from Theorem 6.3 that V is injective if and only if $R(V)$ is an injective von Neumann algebra.

Remark 8.5. In his talk [28], Kirchberg discussed weakly decomposable maps on TROs. Let us recall that a map $T : V \to W$ is called *weakly decomposable* if there exist completely positive maps $\tilde{S}_1 : C(V) \to C(W)^{**}$ and $\tilde{S}_2 : D(V) \to D(W)^{**}$ such that the induced map

$$\tilde{\Psi} : \begin{bmatrix} c & v \\ w^* & d \end{bmatrix} \in A(V) \mapsto \begin{bmatrix} \tilde{S}_1(c) & j_W \circ T(v) \\ j_W \circ T(w)^* & \tilde{S}_2(d) \end{bmatrix} \in A(W)^{**} \tag{8.2}$$

is completely positive, where we let $j_W : W \hookrightarrow W^{**}$ denote the canonical inclusion. If T is weakly decomposable, we let

$$\|T\|_{\mathrm{wdec}} = \inf\{\max\{\|\tilde{S}_1\|, \|\tilde{S}_2\|\}\}$$

to denote the *weakly decomposable norm* of T, where the infimum is taken over all completely positive maps $\tilde{S}_i$ in (8.2).

It is clear from the definition that if $T : V \to W$ is decomposable, then $j_W \circ T : V \to W^{**}$ is decomposable, and thus T is weakly decomposable with

$$\|T\|_{\text{wdec}} \leqslant \|j_W \circ T\|_{\text{dec}} \leqslant \|T\|_{\text{dec}}.$$

There are examples of weakly decomposable maps on TROs (even on C^*-algebras) which are not decomposable (see [20, Sect. 2]). If T is a finite rank map between TROs, then T is decomposable (and thus weakly decomposable), and we may obtain

$$\|T\|_{\text{wdec}} = \|T\|_{\text{dec}}, \tag{8.3}$$

which is the TRO analogue of Junge–Le Merdy [20, Corollary 2.6]. To see this, we may apply a similar discussion as that given in Section 7, i.e. we can first show that a map $T : V \to W$ is weakly decomposable if and only if $\imath_W \circ T : V \to A(W)$ is weakly decomposable. Since $A(W)$ is a C^*-algebra, we have

$$A(A(W))^{**} = M_2(A(W))^{**} = M_2(A(W)^{**}) = A(A(W)^{**}),$$

and thus can obtain the equalities

$$\|T\|_{\text{wdec}} = \|\imath_W \circ T\|_{\text{wdec}} = \|j_{A(W)} \circ \imath_W \circ T\|_{\text{dec}}.$$

Let $y \in W \otimes V^*$ denote the element corresponding to T. Then we can obtain the equalities

$$\begin{aligned}\|j_{A(W)} \circ \imath_W \circ T\|_{\text{dec}} &= \tilde{\delta}(j_{A(W)} \circ \imath_W \otimes id_{V^*}(y)) = \tilde{\delta}(\imath_W \otimes id_{V^*}(y)) \\ &= \|\imath_W \circ T\|_{\text{dec}} = \|T\|_{\text{dec}}\end{aligned}$$

from Proposition 7.4 and the fact that

$$j_W \otimes id_{V^*} : A(W) \otimes^{\delta} V^* \to A(W)^{**} \otimes^{\delta} V^*$$

is an isometric inclusion. This proves (8.3).

APPENDIX

In this section, we will construct an operator space which is λ-locally reflexive for some $1 < \lambda < \infty$. More precisely, we will prove the following theorem.

THEOREM A.1. *For $n > 2$, there exists an operator space V such that*

(1) *V is $(n+1)$-locally reflexive,*
(2) *if V is λ-locally reflexive, then we must have $\lambda \geqslant \frac{n}{2\sqrt{n-1}}$.*

Proof. To prove the theorem, we will use the operator space constructed in [13, Theorem 14.5.6]. Let us first recall from Pisier [33] that for $n > 2$, $\ell_1(n)$ with the MAX operator space matrix norm is not 1-exact. More precisely, Pisier proved that if we let $\pi : B(\ell_2) \to Q(\ell_2)$ denote the canonical quotient map, then the contractive linear map

$$id_{\ell_1(n)} \otimes \pi : \ell_1(n)\check{\otimes} B(\ell_2) \to \ell_1(n)\check{\otimes} Q(\ell_2)$$

induces a contractive linear isomorphism

$$T : \ell_1(n)\check{\otimes} B(\ell_2)/\ell_1(n)\check{\otimes} K(\ell_2) \to \ell_1(n)\check{\otimes} Q(\ell_2)$$

with $\|T^{-1}\| \geqslant \frac{n}{2\sqrt{n-1}}$.

Then for every $\varepsilon > 0$ (sufficiently small), there exists a contractive element $v \in \ell_1(n)\check{\otimes} Q(\ell_2)$ such that

$$\|T^{-1}(v)\| > \frac{n}{2\sqrt{n-1}} - \varepsilon. \tag{A.1}$$

Since $\ell_1(n)\check{\otimes} Q(\ell_2) = CB(\ell_\infty(n), Q(\ell_2))$, we may let $\varphi : \ell_\infty(n) \to Q(\ell_2)$ denote the complete contraction corresponding to v. Then we can deduce from (A.1) that any completely bounded lifting $\psi : \ell_1(n) \to B(\ell_2)$ of φ (with $\pi \circ \psi = \varphi$) must have the cb-norm $\|\psi\|_{\text{cb}} > \frac{n}{2\sqrt{n-1}} - \varepsilon$. Moreover, we may assume that $\varphi : \ell_\infty(n) \to Q(\ell_2)$ is a complete isometry (see the discussion given in [13, Theorem 14.5.6]). If we let $L = \varphi(\ell_\infty(n)) \subseteq Q(\ell_2)$ be the image space of φ in $Q(\ell_2)$, then $V = \pi^{-1}(L) \subseteq B(\ell_2)$ is the operator space we wish to construct for the theorem.

Let us first prove (1). Since the second adjoint $\pi^{**} : B(\ell_2)^{**} \to Q(\ell_2)^{**}$ is a weak* continuous quotient map from $B(\ell_2)^{**}$ onto $Q(\ell_2)^{**}$, there exists a central projection $e \in B(\ell_2)^{**}$ such that $eB(\ell_2)^{**} = ker\pi^{**} = \overline{K(\ell_2)}^{\text{weak}^*}$ and $(1-e)B(\ell_2)^{**} \cong Q(\ell_2)^{**}$. It follows that we have the ℓ_∞-decomposition

$$B(\ell_2)^{**} = eB(\ell_2)^{**} + (1-e)B(\ell_2)^{**} \cong \overline{K(\ell_2)}^{\text{weak}^*} \oplus_\infty Q(\ell_2)^{**}.$$

We let $p : Q(\ell_2)^{**} \to ((1-e)B(\ell_2)^{**}$ denote the canonical (completely isometric) $*$-isomorphism from $Q(\ell_2)^{**}$ onto $((1-e)B(\ell_2)^{**}$, which satisfies

$\pi^{**} \circ p = id_{Q(\ell_2)^{**}}$. It follows from the commutative diagram

$$\begin{array}{ccccc} V & \subseteq & V^{**} & \subseteq & B(\ell_2)^{**} \\ \pi\downarrow & & \pi^{**}\downarrow\uparrow p & & \pi^{**}\downarrow\uparrow p \\ L & = & L^{**} & \subseteq & Q(\ell_2)^{**} \end{array}$$

that we have $V^{**} = (\pi^{**})^{-1}(L^{**})$ and p restricted to L^{**} is a complete isometry from L^{**} onto $p(L^{**}) = (1-e)V^{**}$. Then we may obtain the ℓ_∞-decomposition

$$V^{**} = (\pi^{**})^{-1}(L^{**}) = \overline{K(\ell_2)}^{\text{weak}^*} \oplus_\infty p(L^{**}).$$

For every finite-dimensional subspace $E \subseteq V^{**}$, we can decompose E into

$$E = E_1 \oplus_\infty E_2,$$

where $E_1 \subseteq \overline{K(\ell_2)}^{\text{weak}^*} \cong K(\ell_2)^{**}$ and $E_2 \subseteq p(L^{**})$. Since $K(\ell_2)$ is locally reflexive, there exists a net of complete contractions $\psi_\alpha^1 : E_1 \to K(\ell_2) \subseteq V$ such that ψ_α^1 converges to $\imath_{E_1}$ in the point-weak* topology in V^{**}. On the other hand, we note that the space V is always locally reflexive in Banach space sense. Then there exists a net of contractions $\psi_\alpha^2 : E_2 \to V$ such that ψ_α^2 converges to $\imath_{E_2}$ in the point-weak* topology in V^{**}. Since E_2 is a finite-dimensional subspace of V^{**} with $dim\, E_2 \leqslant dim\, L = n$, we can conclude from [10] that $\|\psi_\alpha^2\|_{cb} \leqslant n\|\psi_\alpha^2\| \leqslant n$. It follows that $\psi_\alpha : E = E_1 \oplus_\infty E_2 \to V$ given by

$$\psi_\alpha(x_1 + x_2) = \psi_\alpha^1(x_1) + \psi_\alpha^2(x_2)$$

is a net of completely bounded maps such that $\|\psi_\alpha\|_{cb} \leqslant n+1$ and $\psi_\alpha \to \imath_E$ in the point-weak* topology. This shows that V is $(n+1)$-locally reflexive.

To prove (2), let us identify $L = L^{**}$ and let $\imath : L \hookrightarrow L^{**}$ denote the canonical embedding of L onto L^{**}. Then

$$\tilde{\psi} = p \circ \varphi : \ell_\infty(n) \to V^{**}$$

is a completely contractive lifting of the complete isometry

$$\tilde{\varphi} = \imath \circ \varphi : \ell_\infty(n) \to L^{**}.$$

Since we have the isometry $CB(\ell_\infty(n), V^{**}) = \ell_1(n) \check{\otimes} V^{**}$, $\tilde{\psi} \in CB(\ell_\infty(n), V^{**})$ corresponds to a contractive element $u_{\tilde{\psi}} \in \ell_1(n) \check{\otimes} V^{**}$. If V is λ-locally reflexive, then it satisfies condition C''_λ and thus the canonical map

$$\phi^* : (\ell_1(n) \check{\otimes} V)^{**} \to \ell_1(n) \check{\otimes} V^{**}$$

is a contractive linear isomorphism with $\|\phi^{*-1}\| \leqslant \lambda$. It follows that $u'' = \phi^{*-1}(u_{\tilde{\psi}})$ is an element in $(\ell_1(n)\check{\otimes} V)^{**}$ with $\|u''\| \leqslant \lambda$. It is easy to see that

$$id \otimes \pi : \ell_1(n)\check{\otimes} V \to \ell_1(n)\check{\otimes} L$$

is a (complete) contraction from $\ell_1(n)\check{\otimes} V$ onto $\ell_1(n)\check{\otimes} L$, and

$$F = (id \otimes \pi)^*((\ell_1(n)\check{\otimes} L)^*) = (id \otimes \pi^*)(\ell_\infty(n)\hat{\otimes} L^*)$$

is a finite-dimensional subspace of $(\ell_1(n)\check{\otimes} V)^*$. It follows from Helly's lemma (see [9, p. 73]) that for the given $\varepsilon > 0$, there exists an element $u \in \ell_1(n)\check{\otimes} V$ such that

$$\|u\| \leqslant (1+\varepsilon)\|u''\| \leqslant (1+\varepsilon)\lambda \tag{A.2}$$

and

$$\langle u, x \otimes \pi^*(f)\rangle = \langle u'', x \otimes \pi^*(f)\rangle \tag{A.3}$$

for all $x \otimes \pi^*(f) \in F = (id \otimes \pi^*)(\ell_\infty(n)\hat{\otimes} L^*)$. If we let $\psi \in CB(\ell_\infty(n), V)$ denote the map corresponding to u, then can deduce from (A.2) and (A.3) that

$$\|\psi\|_{\rm cb} \leqslant (1+\varepsilon)\|\psi\|_{\rm cb} \leqslant (1+\varepsilon)\lambda,$$

and $\pi \circ \psi = \varphi$. This shows $\psi : \ell_\infty(n) \to V \subseteq B(\ell_2)$ is a completely bounded lifting of φ and thus must satisfy

$$\frac{n}{2\sqrt{n-1}} - \varepsilon \leqslant \|\psi\|_{\rm cb} \leqslant (1+\varepsilon)\|\tilde{\psi}\|_{\rm cb} \leqslant (1+\varepsilon)\lambda.$$

Letting $\varepsilon \to 0$, we obtain $\frac{n}{2\sqrt{n-1}} \leqslant \lambda$. ∎

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