Linear maps preserving the essential spectral radius

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Abstract

Let \( \mathcal{L}(\mathcal{H}) \) be the algebra of all bounded linear operators on an infinite dimensional complex Hilbert space \( \mathcal{H} \). We characterize linear maps from \( \mathcal{L}(\mathcal{H}) \) onto itself that preserve the essential spectral radius.

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1. Introduction

Throughout this note, \( \mathcal{H} \) will denote an infinite dimensional complex Hilbert space and \( \mathcal{L}(\mathcal{H}) \) will denote the algebra of all bounded linear operators on \( \mathcal{H} \). Recently, Mbekhta [8] characterized linear maps from \( \mathcal{L}(\mathcal{H}) \) onto itself that preserve the set of Fredholm operators in both directions. Further, in [7] Mbekhta and Šemrl characterized linear maps from \( \mathcal{L}(\mathcal{H}) \) onto itself preserving semi-Fredholm operators in both directions, and improved the recently obtained characterization of linear preservers of generalized invertibility that they obtained jointly with Rodman [9].

The goal of the present note is not only to extend the above mentioned results to more general settings but mainly to give shorter and simple proofs. We characterize linear maps from \( \mathcal{L}(\mathcal{H}) \) onto itself which preserve the essential spectral radius and recapture the main results of [5,7–9].

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2. Preliminaries

First we introduce some notation and terminology. An operator \( T \in \mathcal{L}(\mathcal{H}) \) is said to be left Fredholm (resp. right Fredholm) if range\( (T) \) is closed and \( \dim(\ker(T)) \) is finite (resp. \( \dim(\mathcal{H}/\text{range}(T)) \) is finite). We say that \( T \) is semi-Fredholm if it is left Fredholm or right Fredholm, and \( T \) is said to be Fredholm if it is both left and right Fredholm.

For every \( T \in \mathcal{L}(\mathcal{H}) \) we set

\[
\sigma_e(T) := \{ \lambda \in \mathbb{C} : (T - \lambda I) \text{ is not Fredholm} \},
\]

\[
\sigma_{le}(T) := \{ \lambda \in \mathbb{C} : (T - \lambda I) \text{ is not left Fredholm} \},
\]

\[
\sigma_{re}(T) := \{ \lambda \in \mathbb{C} : (T - \lambda I) \text{ is not right Fredholm} \},
\]

\[
\sigma_{SF}(T) := \{ \lambda \in \mathbb{C} : (T - \lambda I) \text{ is not semi-Fredholm} \}.
\]

These are called the essential spectrum, the left essential spectrum, the right essential spectrum, and the semi-Fredholm spectrum, respectively, of \( T \); see for instance [1].

For an operator \( T \in \mathcal{L}(\mathcal{H}) \), the essential norm is \( \|T\|_e := \text{dist}(T, \mathcal{C}(\mathcal{H})) \), and the essential spectral radius is defined by \( r_e(T) := \max\{|\lambda| : \lambda \in \sigma_e(T)\} \). It coincides with the limit of the convergent sequence \( \left(\|T^n\|^{1/n}\right)_{n \geq 1} \).

We say that a linear map \( \phi \) from \( \mathcal{L}(\mathcal{H}) \) into itself preserves the essential spectral radius if \( r_e(\phi(T)) = r_e(T) \) for all \( T \in \mathcal{L}(\mathcal{H}) \). Such maps are the main object of the paper. However, in the proof of the main result we will use a result concerning linear maps preserving the usual spectral radius, i.e., maps \( \phi \) satisfying \( r(\phi(T)) = r(T) \) where \( r(\cdot) \) denotes the spectral radius. This is a theorem by Lin and Mathieu [6, Corollary 2.6] that concerns surjective spectral radius preserving maps between two unital C*-algebras such that one of them is purely infinite with real rank zero. For our purposes, the only relevant example of an algebra having the latter properties is the Calkin algebra \( C(\mathcal{H}) := \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \); here \( \mathcal{K}(\mathcal{H}) \) of course denotes the closed ideal of \( \mathcal{L}(\mathcal{H}) \) consisting of all compact operators on \( \mathcal{H} \). Therefore we state the result by Lin and Mathieu only for them.

**Lemma 2.1.** Let \( \phi : C(\mathcal{H}) \to C(\mathcal{H}) \) be a surjective linear map preserving the spectral radius. Then there are \( \lambda \in \mathbb{C} \) with \( |\lambda| = 1 \) and an automorphism or an antiautomorphism \( J : C(\mathcal{H}) \to C(\mathcal{H}) \) such that \( \phi(a) = \lambda J(a) \) for all \( a \in C(\mathcal{H}) \).

A few comments must be added to this statement. In [6, Corollary 2.6] \( \lambda \) can be any central unitary element; however, since the center of \( C(\mathcal{H}) \) is trivial, \( \lambda \) must be a complex number of modulus 1 in our setting. Further, the conclusion of [6, Corollary 2.6] \( J \) is that \( J \) a Jordan isomorphism; since the algebra \( C(\mathcal{H}) \) is prime, a well known theorem of Herstein [4] tells us that \( J \) must be an automorphism or an antiautomorphism in our setting.

It should be noted that, unlike \( \mathcal{L}(\mathcal{H}) \), the Calkin algebra \( C(\mathcal{H}) \) has outer automorphisms as shown recently by Phillips and Weaver [3] who answered negatively a long-standing problem which asks whether every automorphism of \( C(\mathcal{H}) \) is inner.

The next lemma is a consequence of the semi-simplicity of \( C(\mathcal{H}) \) and Zemánek’s characterization of the radical, see e.g. [2, Theorem 5.3.1].

**Lemma 2.2.** The following are equivalent:

(i) \( K \in \mathcal{K}(\mathcal{H}) \).

(ii) \( r_e(T + K) = 0 \) for all \( T \in \mathcal{L}(\mathcal{H}) \) for which \( r_e(T) = 0 \).
Proof. Assume that \( K \in \mathcal{K}(\mathcal{H}) \), and let \( T \in \mathcal{L}(\mathcal{H}) \) be an operator such that \( r_e(T) = 0 \). Denote by \( \pi \) the quotient map from \( \mathcal{L}(\mathcal{H}) \) onto \( \mathcal{C}(\mathcal{H}) \), and note that \( \pi(K) = 0 \) and that
\[
\begin{align*}
  r_e(T + K) &= r(\pi(T + K)) \\
  &= r(\pi(T) + \pi(K)) \\
  &= r(\pi(T)) \\
  &= r_e(T) = 0.
\end{align*}
\]
This establishes the implication (i) \( \Rightarrow \) (ii).

Conversely, assume that \( K \in \mathcal{L}(\mathcal{H}) \) is an operator such that \( r_e(T + K) = 0 \) for all \( T \in \mathcal{L}(\mathcal{H}) \) for which \( r_e(T) = 0 \). Equivalently,
\[
\begin{align*}
  r(\pi(T) + \pi(K)) = 0
\end{align*}
\]
for all \( T \in \mathcal{H} \) for which \( r(\pi(T)) = 0 \). By the spectral Zemáněk’s characterization of the radical, we have \( \pi(K) \in \text{rad}(\mathcal{C}(\mathcal{H})) = \{0\} \), and \( K \) is a compact operator. \( \square \)

3. The main result

**Theorem 3.1.** Let \( \Phi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}) \) be a linear map surjective up to compact operators (i.e., \( \mathcal{L}(\mathcal{H}) = \text{range}(\Phi) + \mathcal{K}(\mathcal{H}) \)). Then \( \Phi \) preserves the essential spectral radius if and only if \( \Phi(\mathcal{K}(\mathcal{H})) \subset \mathcal{K}(\mathcal{H}) \), and the induced map \( \varphi : \mathcal{C}(\mathcal{H}) \to \mathcal{C}(\mathcal{H}) \) is either a continuous automorphism or a continuous antiautomorphism multiplied by a scalar of modulus 1.

**Proof.** Checking the ‘if’ part is straightforward, so we will only deal with the ‘only if’ part. So assume that \( \Phi \) preserves the essential spectral radius.

Our first goal is to show that \( \Phi \) leaves \( \mathcal{K}(\mathcal{H}) \) invariant. So pick a compact operator \( K \in \mathcal{K}(\mathcal{H}) \), and let us prove that \( \Phi(K) \) is compact as well. Let \( S \in \mathcal{L}(\mathcal{H}) \) be an operator for which \( r_e(S) = 0 \). Since \( \Phi \) is surjective up to compact operators, there exist \( T \in \mathcal{L}(\mathcal{H}) \) and \( K' \in \mathcal{K}(\mathcal{H}) \) such that \( S = \Phi(T) + K' \). We have \( r_e(T) = 0 \) and
\[
\begin{align*}
  r_e(\Phi(K) + S) &= r_e(\Phi(K) + \Phi(T) + K') \\
  &= r_e(\Phi(K) + \Phi(T)) \\
  &= r_e(\Phi(K + T)) \\
  &= r_e(K + T) \\
  &= r_e(T) = 0.
\end{align*}
\]
By Lemma 2.2, \( \Phi(K) \) is a compact operator.

Thus \( \Phi(\mathcal{K}(\mathcal{H})) \subset \mathcal{K}(\mathcal{H}) \), and so \( \Phi \) induces a surjective linear map
\[
\begin{align*}
  \varphi : \mathcal{C}(\mathcal{H}) \longrightarrow \mathcal{C}(\mathcal{H}) \\
  \pi(T) \longmapsto (\pi \circ \Phi)(T)
\end{align*}
\]
where \( \pi \) denotes the quotient map from \( \mathcal{L}(\mathcal{H}) \) onto \( \mathcal{C}(\mathcal{H}) = \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \). Clearly, \( \varphi \circ \pi = \pi \circ \Phi \) and for every \( T \in \mathcal{L}(\mathcal{H}) \), we have
\[
\begin{align*}
  r(\varphi(\pi(T))) &= r(\pi \circ \Phi(T)) \\
  &= r_e(\Phi(T)) \\
  &= r_e(T) \\
  &= r(\pi(T)).
\end{align*}
\]
In other words, the map \( \varphi \) preserves the spectral radius. Therefore Lemma 2.1 and [2, Theorem 5.2.2] tell us that \( \varphi \) is a continuous automorphism or a continuous antiautomorphism multiplied by a scalar of modulus 1. □

4. Corollaries

As corollaries, we recapture the following results: [7, Theorem 1.2] and [8, Theorems 2.1 and 3.2]; see also [5].

**Corollary 4.1** ([7, Theorem 1.2] and [8, Theorems 2.1 and 3.2]). Let \( \Phi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}) \) be a linear map surjective up to compact operators. Then \( \Phi \) preserves the essential spectrum (the semi-Fredholm spectrum) if and only if \( \Phi(\mathcal{K}(\mathcal{H})) \subset \mathcal{K}(\mathcal{H}) \), and the induced map \( \varphi : \mathcal{C}(\mathcal{H}) \to \mathcal{C}(\mathcal{H}) \) is either a continuous automorphism or a continuous antiautomorphism.

**Proof.** It suffices to prove the ‘only if’ part. Assume that \( \Phi \) preserves the essential spectrum. We first prove that \( \Phi \) is unital modulo a compact operator. Let \( T \in \mathcal{L}(\mathcal{H}) \) such that \( \text{re}(T) = 0 \). There are two operators \( S, K \in \mathcal{L}(\mathcal{H}) \) such that \( K \) is compact and \( T = \Phi(S) + K \). We have

\[
\sigma_e(T + \Phi(I) - I) = \sigma_e(T + \Phi(I)) - 1
\]

\[
= \sigma_e(\Phi(S) + K + \Phi(I)) - 1
\]

\[
= \sigma_e(\Phi(S) + \Phi(I)) - 1
\]

\[
= \sigma_e(S + I) - 1
\]

\[
= \sigma_e(S)
\]

\[
= \sigma_e(\Phi(S))
\]

\[
= \sigma_e(T)
\]

\[
= \{0\}.
\]

By Lemma 2.2 we see that \( \Phi(I) - I \) is a compact operator.

Note that, since \( \Phi \) preserves the essential spectrum (the semi-Fredholm spectrum), it preserves the essential spectral radius as well. It follows from Theorem 3.1 that \( \Phi(\mathcal{K}(\mathcal{H})) \subset \mathcal{K}(\mathcal{H}) \) and that there are a scalar \( \lambda \) of modulus 1 and either a continuous automorphism or a continuous antiautomorphism \( \psi \) such that \( \varphi = \lambda \psi \). What was shown above implies that \( \varphi \) is unital. As \( \psi \) is unital as well, we see that \( \lambda = 1 \) and \( \varphi \) is either a continuous automorphism or a continuous antiautomorphism. □

**Corollary 4.2.** Let \( \Phi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}) \) be a surjective linear map up to compact operators. Then \( \Phi \) preserves the left essential spectrum (or the right essential spectrum) if and only if \( \Phi(\mathcal{K}(\mathcal{H})) \subset \mathcal{K}(\mathcal{H}) \), and the induced map \( \varphi : \mathcal{C}(\mathcal{H}) \to \mathcal{C}(\mathcal{H}) \) is a continuous automorphism.

**Proof.** It suffices to prove the ‘only if’ part. Assume that \( \Phi \) preserves the left essential spectrum. Note that, since \( \partial \sigma_e(T) \subset \sigma_{le}(T) \) for all \( T \in \mathcal{L}(\mathcal{H}) \), we have \( r_e(T) = \max\{ |\lambda| : \lambda \in \sigma_{le}(T) \} \) for all \( T \in \mathcal{L}(\mathcal{H}) \). Thus just as for the proof of the above corollary, one can show that \( \varphi \) is either a continuous automorphism or a continuous antiautomorphism. It remains to show that \( \varphi \) can not be an antiautomorphism. Now, assume by the way of contradiction that \( \varphi \) is an antiautomorphism, and pick a non-essentially invertible operator \( T \in \mathcal{L}(\mathcal{H}) \) which is left essentially
invertible, i.e., $0 \in \sigma_e(T) \setminus \sigma_{le}(T)$. Therefore, there is $S \in \mathcal{L}(\mathcal{H})$ such that $\pi(S)\pi(T) = \pi(I)$ but $\pi(T)\pi(S) \neq \pi(I)$. We thus have

$$\pi(I) = \varphi(\pi(S)\pi(T)) = \varphi(\pi(T))\varphi(\pi(S)),$$

and $\varphi(\pi(T))$ is right invertible. Since $\pi(T)$ is left invertible, $\varphi(\pi(T))$ is left invertible as well. Thus, $\varphi(\pi(T))$ is in fact invertible and so is $\pi(T)$. This leads to a contradiction and shows that $\varphi$ is a continuous automorphism. □

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