Controllability of Second-Order Integrodifferential Evolution Systems in Banach Spaces

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Abstract—In this paper, we study the controllability of second-order nonlinear integrodifferential systems in Banach spaces. Further, we derive a set of sufficient conditions for the controllability of second-order nonlinear integrodifferential evolution systems in Banach spaces. The results are established by using the theory of strongly continuous cosine families of bounded linear operators and the Schaefer fixed-point theorem. © 2005 Elsevier Ltd. All rights reserved

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1. INTRODUCTION

Controllability of nonlinear systems represented by ordinary differential equations in infinite dimensional spaces has been extensively studied by several authors [1]. Tsujioka [2] investigated the controllability problem for second-order evolution systems in Hilbert spaces by converting it into a first-order system. The problem of controllability of second-order nonlinear systems in Banach spaces has received considerable attention in recent years. Park and Han [3] discussed controllability of second-order nonlinear systems in Banach spaces with the help of the Schauder fixed-point theorem. Balachandran and Marshal Anthoni [4] discussed the controllability of second-order semilinear differential systems in Banach spaces. Park and Han [5] established sufficient conditions for the approximate controllability of second-order integrodifferential systems in Banach spaces with the help of the Schauder fixed-point theorem. Balachandran et al. [6] discussed the controllability of second-order semilinear Volterra integrodifferential systems in

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Banach spaces and in [7], they studied delay integrodifferential systems. Recently, Balachandran and Marshal Anthoni [8] established sufficient conditions for the controllability of nonlinear second-order neutral systems in Banach spaces. The purpose of this paper is to study the controllability of second-order nonlinear integrodifferential systems and integrodifferential evolution systems in Banach spaces. The results are established by using the Schaefer fixed-point theorem.

2. PRELIMINARIES

The following basic results concerning strongly continuous cosine families have been established in [9,10].

**Definition 2.1.** A one-parameter family $C(t)$, $t \in \mathbb{R}$, of bounded linear operators mapping the Banach space $X$ into itself is called a strongly continuous cosine family if and only if

(i) $C(s + t) + C(s - t) = 2C(s)C(t)$ for all $s, t \in \mathbb{R}$;
(ii) $C(0) = I$;
(iii) $C(t)x$ is continuous in $t$ on $\mathbb{R}$ for each fixed $x \in X$.

If $C(t)$, $t \in \mathbb{R}$, is a strongly continuous cosine family in $X$, then $S(t)$, $t \in \mathbb{R}$, is the associated sine family of operators in $X$ defined by

$$S(t)x = \int_0^t C(s)x \, ds, \quad x \in X, \quad t \in \mathbb{R}.$$\hspace{1cm} (2.1)

**Proposition 2.1.** Let $C(t)$, $t \in \mathbb{R}$, be a strongly continuous cosine family in $X$. Then, the following are true.

(i) $C(t) = C(-t)$ for all $t \in \mathbb{R}$.
(ii) $C(s)$, $S(s)$, $C(t)$, and $S(t)$ commute for all $s, t \in \mathbb{R}$.
(iii) $S(t)x$ is continuous in $t$ on $\mathbb{R}$ for each fixed $x \in X$.
(iv) $S(s + t) + S(s - t) = 2S(s)C(t)$ for all $s, t \in \mathbb{R}$.
(v) $C(s + t) = S(s)C(t) + S(t)C(s)$ for all $s, t \in \mathbb{R}$.
(vi) $S(t) = -S(-t)$ for all $t \in \mathbb{R}$.
(vii) There exist constants $K \geq 1$ and $\omega \geq 0$, such that $|C(t)| \leq Ke^{\omega|t|}$.
(viii) $|S(t) - S(t')| \leq K \int_t^{t'} e^{\omega|s|} \, ds$, for all $t, t' \in \mathbb{R}$.

The infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, is the operator $A : X \to X$ defined by

$$Ax = \left. \frac{d^2}{dt^2} C(t)x \right|_{t=0}, \quad x \in D(A),$$

where $D(A) = \{x \in X : C(t)x$ is twice continuously differentiable in $t\}$. Define $E = \{x \in X : C(t)x$ is once continuously differentiable in $t\}$.

**Proposition 2.2.** Let $C(t)$, $t \in \mathbb{R}$, be a strongly continuous cosine family in $X$ with infinitesimal generator $A$. Then, the following are true.

(i) $D(A)$ is dense in $X$ and $A$ is a closed operator in $X$.
(ii) If $x \in X$ and $r, s \in \mathbb{R}$, then $x = \int_t^s S(t)x \, dt \in D(A)$ and $Ax = C(s)x - C(r)x$.
(iii) If $x \in X$ and $r, s \in \mathbb{R}$, then $x = \int_0^s \int_0^r C(t)C(\theta)x \, dt \, d\theta \in D(A)$ and $Ax = 2^{-1}(C(s + r)x - C(s - r)x)$.
(iv) If $x \in X$, then $C(t)x \in E$.
(v) If $x \in E$, then $S(t)x \in D(A)$ and $\frac{d}{dt} C(t)x = AS(t)x$.
(vi) If $x \in D(A)$, then $C(t)x \in D(A)$ and $\frac{d^2}{dt^2} C(t)x = AC(t)x = C(t)Ax$.
(vii) If $x \in E$, then $\lim_{t \to \infty} AS(t)x = 0$.
(viii) If $x \in E$, then $S(t)x \in D(A)$ and $\frac{d}{dt} S(t)x = AS(t)x$.
(ix) If $x \in D(A)$, then $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$.
(x) $C(t + s) - C(t - s) = 2AS(t)S(s)$ for all $s, t \in \mathbb{R}$.
Assume the following conditions on $A$.

$(H_1)$ $A$ is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, of bounded linear operators from $X$ into itself and the adjoint operator $A^*$ is densely defined, that is $\text{D}(A^*) = X^*$ (see [11]).

**Proposition 2.3.** (See [10]) Let $(H_1)$ hold, let $v : \mathbb{R} \rightarrow X$ be such that $v$ is continuously differentiable and let $q(t) = \int_0^t S(t-s)v(s)\,ds$. Then, $q$ is twice continuously differentiable, $q(t) \in \text{D}(A)$, for $t \in \mathbb{R}$, and

\[
q'(t) = \int_0^t C(t-s)v(s)\,ds
\]

and

\[
q''(t) = \int_0^t C(t-s)v'(s)\,ds + C(t)v(0) = Aq(t) + v(t).
\]

**Schaefer Theorem.** (See [12].) Let $E$ be a normed linear space. Let $F : E \rightarrow E$ be a completely continuous operator, that is, it is continuous and the image of any bounded set is contained in a compact set, and let

\[
\xi(F) = \{x \in E : x = \lambda Fx, \text{ for some } 0 < \lambda < 1\}.
\]

Then, either $\xi(F)$ is unbounded or $F$ has a fixed point.

### 3. Second-Order Delay Integrodifferential Systems

Consider the second-order delay integrodifferential control systems of the form,

\[
x''(t) = Ax(t) + f\left(t, x_t, \int_0^t g(t, s, x_s, x'(s))\,ds, x'(t)\right) + Bu(t), \quad t \in [0, T],
\]

\[
x_0 = \phi \in C, \quad x'(0) = y \in X,
\]

where $A$ is the infinitesimal generator of the strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$ of bounded linear operators in a Banach space $X$, $g$ is a function from $J \times J \times C \times X$ to $X$, $f$ is a nonlinear mapping from $J \times C \times X \times X$ to $X$, $B$ is a bounded linear operator from $U$ to $X$ and the control function $u(\cdot)$ is given in $L^2(J, U)$, a Banach space of admissible control functions, with $U$ as a Banach space. Here, $C = C([-r, 0] : X)$ is the Banach space of the continuous functions $\phi : [-r, 0] \rightarrow X$ endowed with the supremum norm,

\[
\|\phi\| = \sup \{|\phi(s)| : -r \leq s \leq 0\}.
\]

Also, for $x \in C([-r, T] : X)$, we have $x_t \in C$ for $t \in [0, T]$, $x_t(s) = x(t+s)$ for $s \in [-r, 0]$. We make the following assumptions.

$(H_2)$ $g(t, s, \cdot, \cdot) : C \times X \rightarrow X$ is continuous for each $t, s \in J$ and the function $g(\cdot, \cdot, x, y) : J \times J \rightarrow X$ is strongly measurable.

$(H_3)$ $f(t, \cdot, \cdot, \cdot) : C \times X \times X \rightarrow X$ is continuous for each $t \in J$ and the function $f(\cdot, x, y, z) : J \rightarrow X$ is strongly measurable for each $(x, y, z) \in C \times X \times X$.

$(H_4)$ For every positive constant $k$ there exists $\alpha_k \in L^1(J)$, such that

\[
\sup_{\|x\|,\|y\|,\|z\| \leq k} \|f(t, x, y, z)\| \leq \alpha_k(t), \quad \text{for } t \in J, \text{ a.e.}
\]
(H6) There exists an integrable function \( m : J \to [0, \infty) \), such that
\[
\| f(t, \phi, x, y) \| \leq m(t) \Omega \left( \| \phi \| + \| x \| + \| y \| \right), \quad t \in J, \quad \phi \in C, \quad x, y \in X,
\]
where \( \Omega : [0, \infty) \to (0, \infty) \) is a continuous nondecreasing function.

(H7) \( Bu(t) \) is continuous.

(H8) The linear operator \( W : L^2(J, U) \to X \) defined by
\[
Wu = \int_0^T S(T - s) Bu(s) \, ds
\]
has a bounded inverse operator \( W^{-1} : X \to L^2(J, U)/\ker W \).

(H9) \( C(t) \), \( t > 0 \) is compact.

Then, the system (1) has a mild solution of the form,
\[
x(t) = C(t) \phi(0) + S(t) y + \int_0^t S(t - s) Bu(s) \, ds
+ \int_0^t S(t - s) f \left( s, x_s, \int_0^s g(t, \tau, x_\tau, x'_\tau(\tau)) \, d\tau, x'_s(\tau) \right) \, ds,
\]
t \in J, \quad (2)

Let \( M = \sup \{ \| C(t) \| : t \in J \} \), \( M^* = \sup \{ \| AS(t) \| : t \in J \} \), \( \mu(t) = \sup \{ \| x(s) \| : s \in [-r, t] \} \), \( t \in J \), \( v(t) = \sup \{ \| x'(s) \| : s \in [0, t] \} \), \( t \in J \), and \( \hat{m}(t) = \max \{ M(T + 1) \mu(t), n(t, t) \} \). Let \( c = K_1 + K_2 \) where
\[
K_1 = M \| \phi \| + MT \| y \| + MT^2 \| B \| \left\| \frac{\hat{W}^{-1}}{\Omega} \right\| \left( \| x_1 \| + M \| \phi \| + MT \| y \| \right)
+ MT \int_0^T \hat{m}(s) \Omega \left( \| x_s \| + \int_0^s n(s, \tau) \Omega \left( \| x_\tau \| + \| x'_\tau(\tau) \| \right) \, d\tau + \| x'(s) \| \right) \, ds,
\]
\[
K_2 = M^* \| \phi \| + M \| y \| + MT \| B \| \left\| \frac{\hat{W}^{-1}}{\Omega} \right\| \left( \| x_1 \| + M \| \phi \| + MT \| y \| \right)
+ MT \int_0^T \hat{m}(s) \Omega \left( \| x_s \| + \int_0^s n(s, \tau) \Omega \left( \| x_\tau \| + \| x'_\tau(\tau) \| \right) \, d\tau + \| x'(s) \| \right) \, ds.
\]

DEFINITION 3.1. System (1) is said to be controllable on \( J \) if for every \( \phi \in C \) with \( \phi(0) \in D(A) \), \( y \in E \) and \( x_1 \in X \) there exists a control \( u \in L^2(J, U) \), such that the solution \( x(\cdot) \) of (1) satisfies \( x(T) = x_1 \).

THEOREM 3.1 Suppose (H1)-(H9) hold. Further if
\[
\int_0^T \hat{m}(s) \, ds < \int_c^\infty \frac{ds}{\Omega(s) + \Omega_0(s)},
\]
then the system (1) is controllable on \( J \).

PROOF. Consider the space \( Z = C([-r, T], X) \cap C^1(J, X) \) with the norm
\[
\| x \|^* = \max \{ \| x \|, \| x' \| \},
\]
where  
\[ \|x\|_r = \sup \{ |x(t)| : -r \leq t \leq T \}, \quad \|x\|_0 = \sup \{ |x'(t)| : 0 \leq t \leq T \}. \]

Using (Hs) for an arbitrary function \( x(\cdot) \), we define the control,

\[
 u(t) = \tilde{W}^{-1} \left[ x_1 - C(T) \phi(0) - S(T) y \right.
\]

\[
 - \int_0^T S(T-s) f \left( s, x_s, \int_0^s g(\tau, r, x_r, x_r') d\tau, x'(s) \right) ds \bigg] (t)
\]

Using this control, we will show that the operator defined by

\[
(Fx)(t) = C(t) \phi(0) + S(t) y + \int_0^t S(t-s) f \left( s, x_s, \int_0^s g(\tau, r, x_r, x_r') d\tau, x'(s) \right) ds 
\]

\[
+ \int_0^t S(t-s) B \tilde{W}^{-1} \left[ x_1 - C(T) \phi(0) - S(T) y \right.
\]

\[
 - \int_0^T S(T-\theta) f \left( \theta, x_\theta, \int_0^\theta g(\theta, r, x_r, x_r') d\tau, x'(\theta) \right) d\theta \bigg] (s) ds,
\]

\[ t \in J = \phi(t), \quad t \in [-r, 0], \]

has a fixed point. Then, this fixed point is a solution of equation (2).

Clearly, \((Fx)(T) = x_1\), which means that the control \( u \) steers the system from the initial function \( \phi \) to \( x_1 \) in time \( T \), provided we obtain a fixed point of the nonlinear operator \( F \).

In order to study the controllability problem for system (1), we have to apply the Schaefer theorem to the following nonlinear operator equation as in [13,14],

\[
x(t) = \lambda Fx(t), \quad \lambda \in (0, 1).
\]

That is,

\[
x(t) = \lambda \left( C(t) \phi(0) + S(t) y + \int_0^t S(t-s) f \left( s, x_s, \int_0^s g(\tau, r, x_r, x_r') d\tau, x'(s) \right) ds \right. 
\]

\[
+ \lambda \int_0^t S(t-s) B \tilde{W}^{-1} \left[ x_1 - C(T) \phi(0) - S(T) y \right.
\]

\[
 - \int_0^T S(T-\theta) f \left( \theta, x_\theta, \int_0^\theta g(\theta, r, x_r, x_r') d\tau, x'(\theta) \right) d\theta \bigg] (s) ds 
\]

\[ \|x(t)\| \leq M \|\phi\| + MT \|y\| + MT \int_0^t m(s) \Omega \left( \|x_s\| + \int_0^s n(s, \tau) \Omega_0 (\|x_\tau\| 
\]

\[ + \|x'(\tau)\| d\tau + \|x'(s)\| \right) ds + MT \|B\| \|\tilde{W}^{-1}\| \left[ \|x_1\| + M \|\phi\| + MT \|y\| 
\]

\[ + MT \int_0^T m(s) \Omega \left( \|x_s\| + \int_0^s n(s, \tau) \Omega_0 (\|x_\tau\| + \|x'(\tau)\|) d\tau + \|x'(s)\| \right) ds \right] 
\]

\[ = K_1 + MT \int_0^t m(s) \Omega \left( \|x_s\| + \int_0^s n(s, \tau) \Omega_0 (\|x_\tau\| + \|x'(\tau)\|) d\tau + \|x'(s)\| \right) ds. 
\]

Denoting by \( p(t) \) the right-hand side of the above inequality, we have

\[ p(0) = K_1, \quad \|x(t)\| \leq p(t), \quad t \in J, \]
and
\[ p'(t) = MT \left[ m(t) \Omega \left( \|x_t\| + \|x'(t)\| + \int_0^t n(t,s) \Omega_0 (\|x_s\| + \|x'(s)\|) ds \right) \right]. \]

However,
\[ x'(t) = \lambda [AS(t)x_0(t) + C(t)y(t)] + \lambda \int_0^t C(t-s) f \left( s, x_s, \int_0^s g(s, \tau, x_\tau, x'(\tau)) d\tau, x'(s) \right) ds \]
\[ + \lambda \int_0^t C(t-s) B\hat{W}^{-1} \left[ x_1 - C(T)x_0(t) - S(T)y(t) \right] \cdot \]
\[ - \int_0^T S(T-\theta) f \left( \theta, x_\theta, \int_0^\theta g(\theta, \tau, x_\tau, x'(\tau)) d\tau, x'(\theta) \right) d\theta \left( s \right) ds. \]

Thus, we have
\[
\|x'(t)\| \leq M^* \|x_0\| + M \|y\| + M \int_0^t m(s) \Omega \left( \|x_s\| + \int_0^s n(s, \tau) \Omega_0 (\|x_\tau\| + \|x'(\tau)\|) d\tau + \|x'(s)\| \right) ds + MT \|B\| \|\hat{W}^{-1}\| \left[ \|x_1\| + M \|x_0\| + MT \|y\| \right]
\[ + MT \int_0^T m(s) \Omega \left( \|x_s\| + \int_0^s n(s, \tau) \Omega_0 (\|x_\tau\| + \|x'(\tau)\|) d\tau + \|x'(s)\| \right) ds \]
\[ = K_2 + M \int_0^t m(s) \Omega \left( \|x_s\| + \int_0^s n(s, \tau) \Omega_0 (\|x_\tau\| + \|x'(\tau)\|) d\tau + \|x'(s)\| \right) ds. \]

Denoting by \( q(t) \) the right-hand side of the above inequality, we have
\[ q(0) = K_2, \quad \|x'(t)\| \leq q(t), \]
and
\[ \quad q'(t) = Mm(t) \Omega \left( \|x_t\| + \|x'(t)\| + \int_0^t n(t,s) \Omega_0 (\|x_s\| + \|x'(s)\|) ds \right), \quad t \in J. \]

Let
\[ w(t) = p(t) + q(t) + \int_0^t n(t,s) \Omega_0 (p(s) + q(s)) ds, \quad t \in J. \]

Then, \( w(0) = p(0) + q(0) = c \) and
\[
w'(t) = p'(t) + q'(t) + n(t,t) \Omega_0 (p(t) + q(t)) \
\leq MTm(t) \Omega (w(t)) + Mm(t) \Omega (w(t)) + n(t,t) \Omega_0 (w(t)) 
\leq M(T+1) m(t) \Omega (w(t)) + n(t,t) \Omega_0 (w(t)) 
\leq \hat{n}(t) (\Omega (w(t)) + \Omega_0 (w(t))), \quad t \in J. \]

This implies
\[
\int_{w(0)}^{w(t)} \frac{ds}{\Omega(s) + \Omega_0(s)} \leq \int_0^T \hat{n}(s) ds < \int_0^\infty \frac{ds}{\Omega(s) + \Omega_0(s)}. \]

This inequality implies that there is a constant \( K \), such that
\[ p(t) + q(t) \leq w(t) \leq K, \quad t \in J. \]

Then,
\[ \|x(t)\| \leq p(t), \quad \|x'(t)\| \leq q(t), \quad t \in J, \]
and hence,
\[
\|x\|^* \leq \|x(t)\| + \|x'(t)\| \leq p(t) + q(t) \leq K,
\]
where \(K\) depends only on \(T\) and on the functions \(m, n, \Omega, \Omega_0\). We shall now prove that the operator \(F : Z \to Z\) defined by
\[
(Fx)(t) = C(t) \phi(0) + S(t) y + \int_0^t S(t-s) f\left(s, x_s, \int_0^s g(s, \tau, x_\tau, x'_\tau(\tau)) \, d\tau, x'_s(s)\right) \, ds
\]
\[
+ \int_0^t S(t-s) B W^{-1} \left[ x_1 - C(T) \phi(0) - S(T) y \right]
\]
\[
- \int_0^T S(T-\theta) f\left(\theta, x_\theta, \int_0^\theta g(\theta, \tau, x_\tau, x'_\tau(\tau)) \, d\tau, x'_\theta(\theta)\right) \, d\theta \right) \, ds,
\]
\( t \in J, \)
\[
(Fx)(t) = \phi(t), \quad t \in [-r, 0],
\]
is a completely continuous operator. Let \(B_k = \{x \in Z : \|x\|^* \leq k\}\) for \(k \geq 1\). We first show that \(F\) maps \(B_k\) into an equicontinuous family. Let \(x \in B_k\) and \(t_1, t_2 \in J\). Then, if \(0 < t_1 < t_2 \leq T, \)
\[
\|(F(t_1) - (F(t_2)) \leq ||(C(t_1) - C(t_2)) \phi(0)|| + ||(S(t_1) - S(t_2)) y||
\]
\[
+ \left| \int_0^{t_1} [S(t_1-s) - S(t_2-s)] f\left(s, x_s, \int_0^s g(s, \tau, x_\tau, x'_\tau(\tau)) \, d\tau, x'_s(s)\right) \, ds \right|
\]
\[
+ \left| \int_{t_1}^{t_2} S(t_2-s) f\left(s, x_s, \int_0^s g(s, \tau, x_\tau, x'_\tau(\tau)) \, d\tau, x'_s(s)\right) \, ds \right|
\]
\[
+ \left| \int_0^{t_1} [S(t_1-s) - S(t_2-s)] B W^{-1} \left[ x_1 - C(T) \phi(0) - S(T) y \right]
\]
\[
- \int_0^T S(T-\theta) f\left(\theta, x_\theta, \int_0^\theta g(\theta, \tau, x_\tau, x'_\tau(\tau)) \, d\tau, x'_\theta(\theta)\right) \, d\theta \right) \, ds \right|
\]
\[
+ \left| \int_{t_1}^{t_2} S(t_2-s) B W^{-1} \left[ x_1 - C(T) \phi(0) - S(T) y \right]
\]
\[
- \int_0^T S(T-\theta) f\left(\theta, x_\theta, \int_0^\theta g(\theta, \tau, x_\tau, x'_\tau(\tau)) \, d\tau, x'_\theta(\theta)\right) \, d\theta \right) \, ds \right|
\]
\[
\leq ||(C(t_1) - C(t_2)) \phi(0)|| + ||(S(t_1) - S(t_2)) y||
\]
\[
+ \int_0^{t_1} \|S(t_1-s) - S(t_2-s)\| \alpha_k(s) \, ds + \int_{t_1}^{t_2} \|S(t_2-s)\| \alpha_k(s) \, ds
\]
\[
+ \int_0^{t_1} \|S(t_1-s) - S(t_2-s)\| B \left\| W^{-1} \right\| \left[ \|x_1\| + M \|\phi\|
\]
\[
+ MT \|y\| + MT \int_0^T \alpha_k(\theta) \, d\theta \right] \, ds + \int_{t_1}^{t_2} \|S(t_2-s)\| B \left\| W^{-1} \right\|
\]
\[
\times \left[ \|x_1\| + M \|\phi\| + MT \|y\| + MT \int_0^T \alpha_k(\theta) \, d\theta \right] \, ds.
\]
Similarly,
\[
\|(F(t_1))' - (F(t_2))'\| \leq \|[C'(t_1) - C'(t_2)] \phi(0)|| + \|[S'(t_1) - S'(t_2)] y||
\]
\[
+ \left[ C(t_1-s) - C(t_2-s) \right]
\]
\[ \times f \left( s, x_s, \int_0^s g \left( s, \tau, x_\tau, x'_\tau (\tau) \right) d\tau, x'(s) \right) ds \]
\[ + \left\| \int_{t_1}^{t_2} C(t_2 - s) f \left( s, x_s, \int_0^s g \left( s, \tau, x_\tau, x'_\tau (\tau) \right) d\tau, x'(s) \right) ds \right\| \]
\[ + \left\| \int_0^{t_1} \left[ C(t_1 - s) - C(t_2 - s) \right] BW^{-1} \left[ x_1 - C(T) \phi(0) \right. \right. \]
\[ \left. \left. - S(T) y - \int_0^T S(T - \theta) \right\| \right\| \times f \left( \theta, x_\theta, \int_0^\theta g \left( \theta, \tau, x_\tau, x'_\tau (\tau) \right) d\tau, x'_\theta (\theta) \right) d\theta \right\| (s) ds \]
\[ \leq \left\| A \left( S(t_1) - S(t_2) \right) \right\| \phi(0) \| + \left\| C(t_1) - C(t_2) \right\| y \|
\[ + \int_0^{t_1} \| C(t_1 - s) - C(t_2 - s) \| \alpha_k(s) ds \]
\[ + \int_{t_1}^{t_2} \| C(t_2 - s) \| \alpha_k(s) ds \]
\[ + \int_0^{t_1} \| C(t_1 - s) - C(t_2 - s) \| B \|
\[ \times \left\| W^{-1} \right\| \left[ \| x_1 \| + M \| \phi \| + MT \| y \| + MT \int_0^T \alpha_k(\theta) d\theta \right] ds \]
\[ + \int_{t_1}^{t_2} \| C(t_2 - s) \| B \| \left\| W^{-1} \right\| \]
\[ \times \left[ \| x_1 \| + M \| \phi \| + MT \| y \| + MT \int_0^T \alpha_k(\theta) d\theta \right] ds. \]

The right-hand sides of the above inequalities are independent of \( x \in B_k \) and tends to zero as \( t_2 \to t_1 \), since \( C(t), S(t) \) are uniformly continuous for \( t \in J \) and the compactness of \( C(t), S(t) \) for \( t > 0 \) imply the continuity in the uniform operator topology (see Remark in [15] and [16, p. 308]). The compactness of \( S(t) \) follows from that of \( C(t) \). Thus, \( F \) maps \( B_k \) into an equicontinuous family of functions.

The equicontinuity for the cases, \( t_1 < t_2 < 0 \) and \( t_1 < 0 < t_2 \), follows from the uniform continuity of \( \phi \) on \([-\tau, 0]\) and from the relation,
\[ \| (Fx)(t_1) - (Fx)(t_2) \| \leq \| \phi(t_1) - (Fx)(t_2) \| \leq \| (Fx)(t_2) - (Fx)(0) \| + \| \phi(0) - \phi(t_1) \| , \]
respectively. It is easy to see that the family \( FB_k \) is uniformly bounded. Next, we show \( \overline{FB_k} \) is compact. Since we have shown \( FB_k \) is an equicontinuous collection, it suffices by the Arzel-Ascoli theorem to show that \( F \) maps \( B_k \) into a precompact set in \( X \). Let \( 0 < t \leq T \) be fixed.
and \( \epsilon \), a real number satisfying \( 0 < \epsilon < t \). For \( x \in B_k \), we define

\[
(F \phi)(t) = C(t) \phi(0) + S(t) y + \int_0^t S(t-s) \left[ x_1 - C(T) \phi(0) - S(T) y \right] ds,
\]

where \( S(t-s) \) is the semigroup generated by \( A \). Since \( C(t) \) and \( S(t) \) are compact operators, the set \( Y_\epsilon(t) = \{(F \phi)(t) : x \in B_k\} \) is precompact in \( X \) for every \( \epsilon, 0 < \epsilon < t \). Moreover, for every \( x \in B_k \), we have

\[
\| (F \phi)'(t) - (F \phi)(t) \| \leq \int_0^t \| S(t-s) \left( \frac{\partial}{\partial t} \phi(s) \right) \| ds.
\]

Therefore, there are precompact sets arbitrarily close to the set \( \{(F \phi)(t) : x \in B_k\} \). Hence, the set \( \{(F \phi)(t) : x \in B_k\} \) is precompact in \( X \). For that consider the space \( C^0 = \{ x \in C([-r, T]; X) : x_0 = \phi = 0 \} \). Let \( \{x_n\}_{n=1}^\infty \subseteq C^0 \) with \( x_n \rightarrow x \) in \( C^0 \). Then, there is an integer \( \nu \), such that \( \|x_n(t)\| \leq \nu, \|x_n'(t)\| \leq \nu \) for all \( n \) and \( t \in J \), so \( \|x(t)\| \leq \nu, \|x'(t)\| \leq \nu \) and \( x, x' \in B_\nu \). By (H3),

\[
\int_0^t g(t, x_n(s), x'_n(s)) ds, x'_n(t) \rightarrow f(t, x_t, \int_0^t g(t, x_s, x'_s) ds, x'(t)),
\]

for each \( t \in J \) and since

\[
\| f(t, x_n, \int_0^t g(t, s, x_n, x'_n(s)) ds, x'_n(t)) - f(t, x_t, \int_0^t g(t, s, x, x'_s) ds, x'(t)) \| \leq 2\alpha(t),
\]
we have by dominated convergence theorem,

\[
\|F x_n - F x\| = \sup_{t \in J} \left\| \int_0^t S(t-s) \left[ f\left(s, x_{ns}, \int_0^s g(s, \tau, x_{nt}, x'_n(\tau)) \, d\tau, x'_n(s)\right) \right. \right.
\]

\[
\left. - f\left(s, x_s, \int_0^s g(s, \tau, x, x'_s(\tau)) \, d\tau, x'_s(s)\right) \right] ds
\]

\[
- \int_0^t S(t-s) B W^{-1} \int_0^T S(T-\theta) \left[ f\left(\theta, x_{n\theta}, \int_0^\theta g(\theta, \tau, x_{n\tau}, x'_n(\tau)) \, d\tau, x'_n(\theta)\right) \right.
\]

\[
\left. - f\left(\theta, x_\theta, \int_0^\theta g(\theta, \tau, x, x'_\tau(\tau)) \, d\tau, x'_\tau(\theta)\right) \right] d\theta \, ds
\]

\[
\left. \left\| F x_n - F x \right\| \leq \int_0^T \left\| S(t-s) \left[ f\left(s, x_{ns}, \int_0^s g(s, \tau, x_{nt}, x'_n(\tau)) \, d\tau, x'_n(s)\right) \right. \right.
\]

\[
\left. - f\left(s, x_s, \int_0^s g(s, \tau, x, x'_s(\tau)) \, d\tau, x'_s(s)\right) \right] \right\| ds
\]

\[
+ \int_0^T \left\| S(t-s) B W^{-1} \int_0^T S(T-\theta) \left[ f\left(\theta, x_{n\theta}, \int_0^\theta g(\theta, \tau, x_{n\tau}, x'_n(\tau)) \, d\tau, x'_n(\theta)\right) \right.
\]

\[
\left. - f\left(\theta, x_\theta, \int_0^\theta g(\theta, \tau, x, x'_\tau(\tau)) \, d\tau, x'_\tau(\theta)\right) \right] d\theta \right\| ds \to 0, \quad \text{as } n \to \infty,
\]

and

\[
\left\| (F x_n)' - (F x)' \right\| = \sup_{t \in J} \left\| \int_0^t C(t-s) \left[ f\left(s, x_{ns}, \int_0^s g(s, \tau, x_{nt}, x'_n(\tau)) \, d\tau, x'_n(s)\right) \right. \right.
\]

\[
\left. - f\left(s, x_s, \int_0^s g(s, \tau, x, x'_s(\tau)) \, d\tau, x'_s(s)\right) \right] ds
\]

\[
- \int_0^t C(t-s) B W^{-1} \int_0^T S(T-\theta)
\]

\[
\times \left[ f\left(\theta, x_{n\theta}, \int_0^\theta g(\theta, \tau, x_{n\tau}, x'_n(\tau)) \, d\tau, x'_n(\theta)\right) \right.
\]

\[
\left. - f\left(\theta, x_\theta, \int_0^\theta g(\theta, \tau, x, x'_\tau(\tau)) \, d\tau, x'_\tau(\theta)\right) \right] d\theta \, ds
\]

\[
\left. \left\| (F x_n)' - (F x)' \right\| \leq \int_0^T \left\| C(t-s) \left[ f\left(s, x_{ns}, \int_0^s g(s, \tau, x_{nt}, x'_n(\tau)) \, d\tau, x'_n(s)\right) \right. \right.
\]

\[
\left. - f\left(s, x_s, \int_0^s g(s, \tau, x, x'_s(\tau)) \, d\tau, x'_s(s)\right) \right] \right\| ds
\]

\[
+ \int_0^T \left\| C(t-s) B W^{-1} \int_0^T S(T-\theta)
\]

\[
\times \left[ f\left(\theta, x_{n\theta}, \int_0^\theta g(\theta, \tau, x_{n\tau}, x'_n(\tau)) \, d\tau, x'_n(\theta)\right) \right.
\]

\[
\left. - f\left(\theta, x_\theta, \int_0^\theta g(\theta, \tau, x, x'_\tau(\tau)) \, d\tau, x'_\tau(\theta)\right) \right] d\theta \right\| ds \to 0, \quad \text{as } n \to \infty.
\]

Thus, \( F \) is continuous. This completes the proof that \( F \) is completely continuous.

We have already proved that the set \( \mathcal{C}(F) = \{ x \in Z : x = \lambda F x, \lambda \in (0, 1) \} \) is bounded. Hence, by Schaefer's theorem, the operator \( F \) has a fixed point in \( Z \). This means that any fixed point
controllability of evolution systems

of \( F \) is a mild solution of (1) on \( J \) satisfying \((Fx)(t) = x(t)\). Thus, system (1) is controllable on \( J \).

**Example.** Consider the partial delay integrodifferential equation of the form,

\[
\begin{align*}
  z_{tt}(t, y) &= z_{yy}(t, y) + \mu(t, y) + \frac{z(t-h, y)}{t(1+t^{2})} \int_{0}^{t} e^{-z(s-h, y)} ds, \\
  z(t, 0) &= z(t, 1) = \phi(t), \quad t \in [-h, 0], \\
  z(y, 0) &= z_{0}(y), \quad z_{t}(y, 0) = z_{1}(y), \quad 0 < y < 1, \quad t \in J = [0, T],
\end{align*}
\]

where \( \mu : J \times (0, 1) \rightarrow J \) is continuous.

Let \( X = L^{2}[0, 1] \) and let \( A : X \rightarrow X \) be defined by \( Aw = w'' \), \( w \in D(A) \), where

\[
D(A) = \{ w \in X : w, w' \text{ are absolutely continuous, } w'' \in X, \ w(0) = w(1) = 0 \}
\]

Then,

\[
Aw = \sum_{n=1}^{\infty} n^{2} (w, w_{n}) w_{n}, \quad w \in D(A),
\]

where \( w_{n}(s) = \sqrt{2} \sin ns, n = 1, 2, 3 \ldots, \) is the orthogonal set of eigenvectors of \( A \). It can be easily shown that \( A \) is the infinitesimal generator of a strongly continuous cosine family \( C(t) \), \( t \in R \), in \( X \) given by

\[
C(t) w = \sum_{n=1}^{\infty} \cos nt (w, w_{n}) w_{n}, \quad w \in X,
\]

and that the associated sine family is given by

\[
S(t) w = \sum_{n=1}^{\infty} \frac{1}{n} \sin nt (w, w_{n}) w_{n}, \quad w \in X.
\]

Let

\[
\int_{0}^{t} g(t, s, z_{s}) (y) ds = \int_{0}^{t} e^{-z(s-h, y)} ds,
\]

\[
f(t, z_{t}, \int_{0}^{t} g(t, s, z_{s}) ds)(y) = \frac{1}{t(1+t^{2})} z(t-h, y) \int_{0}^{t} e^{-z(s-h, y)} ds.
\]

Further, we have

\[
\left| \frac{1}{t(1+t^{2})} z(t-h, y) \int_{0}^{t} e^{-z(s-h, y)} ds \right| \leq \frac{1}{1+t^{2}} |z|.
\]

Let \( Bu : J \rightarrow X \) be defined by

\[
(Bu)(t)(y) = \mu(t, y), \quad y \in (0, 1).
\]

With the choice of \( A, B, \) and \( f \), (1) is the abstract formulation of (4). Now, the linear operator \( W \) is given by

\[
(Wu)(y) = \sum_{n=1}^{\infty} \int_{0}^{1} \frac{1}{n} \sin ns (\mu(s, y), w_{n}) w_{n} ds, \quad y \in (0, 1).
\]

Assume that this operator has a bounded inverse operator \( \tilde{W}^{-1} \) in \( L^{2}(J, U)/\ker W \). Further, all other conditions of the theorem are satisfied. Hence, system (4) is controllable on \( J \).
4. SECOND-ORDER INTEGRODIFFERENTIAL EVOLUTION SYSTEMS

The main aim is to derive sufficient conditions for the controllability of the integrodifferential evolution system

\[ x''(t) = A(t)x(t) + Bu(t) + f(t,x(t),x'(t)) + \int_0^t g(t,s,x(s),x'(s)) \, ds, \]

\[ x(0) = x_0 \in X, \quad x'(0) = y_0 \in X, \quad t \in J = [0,T], \]

where the state \( x(\cdot) \) takes values in \( X \), \( A(t) : X \to X \) is a closed densely defined operator, \( f \) is a nonlinear mapping from \( J \times X \times X \) to \( X \), \( g \) is a nonlinear mapping from \( J \times J \times X \times X \) to \( X \), \( B \) is a bounded linear operator from a Banach space \( U \) to \( X \) and the control function \( u(\cdot) \) is given in \( L^2(J,U) \), a Banach space of admissible control functions. Let us assume that the domain of \( A(t) \) does not depend on \( t \in [0,T] \) and denote it by \( D(A(t)) \) (for each \( t \in [0,T] \), \( D(A(t)) = D(A) \)).

Now, we define the fundamental solution of a second-order equation.

Let \( X \) denote a real reflexive Banach space and, for each \( t \in [0,T] \), let \( A(t) : X \to X \) be a closed densely defined operator. The fundamental solution for the second-order evolution equation,

\[ x''(t) = A(t)x(t), \]

has been developed by Kozak [17] (see also [18]).

**Definition 4.1.** A family \( S \) of bounded linear operators \( S(t,s) : X \to X \), \( t,s \in [0,T] \), is called a fundamental solution of the second-order equation (6) if,

\[
\begin{align*}
&[Z_1] \text{ for each } x \in X, \text{ the mapping } [0,T] \times [0,T] \ni (t,s) \mapsto S(t,s)x \in X \text{ is of class } C^1 \text{ and } \\
&(i) \text{ for each } t \in [0,T], \ S(t,t) = 0, \\
&(ii) \text{ for all } t, s \in [0,T], \text{ and for each } x \in X, \\
&\left. \frac{\partial}{\partial t} S(t,s) \right|_{t=s} x = x, \quad \left. \frac{\partial}{\partial s} S(t,s) \right|_{t=s} x = -x;
\end{align*}
\]

\[
\begin{align*}
&[Z_2] \text{ for all } t, s \in [0,T], \text{ if } x \in D(A), \text{ then } S(t,s)x \in D(A), \text{ the mapping } [0,T] \times [0,T] \ni (t,s) \mapsto S(t,s)x \in X \text{ is of class } C^2 \text{ and } \\
&(i) \frac{\partial^2}{\partial t^2} S(t,s)x = A(t)S(t,s)x, \\
&(ii) \frac{\partial^2}{\partial s \partial t} S(t,s)x = S(t,s)A(s)x, \\
&(iii) \left. \frac{\partial}{\partial s} S(t,s) \right|_{t=s} x = 0; \\
&[Z_3] \text{ for all } t, s \in [0,T], \text{ if } x \in D(A), \text{ then } \frac{\partial}{\partial s} S(t,s)x \in D(A), \text{ there exist } \frac{\partial^2}{\partial s^2} \frac{\partial}{\partial s} S(t,s)x, \frac{\partial^2}{\partial s^2} S(t,s)x, \text{ and } \\
&(i) \frac{\partial^2}{\partial s^2} \frac{\partial}{\partial s} S(t,s)x = A(t) \frac{\partial}{\partial s} S(t,s)x, \\
&(ii) \frac{\partial^2}{\partial s \partial t} \frac{\partial}{\partial s} S(t,s)x = \frac{\partial}{\partial s} S(t,s)A(s)x, \text{ and the mapping } [0,T] \times [0,T] \ni (t,s) \mapsto A(t) \frac{\partial}{\partial s} S(t,s)x \text{ is continuous.}
\end{align*}
\]

**Definition 4.2.** Any continuous function \( x : [0,T] \to X \) is called a mild solution of problem (5) if \( x(t) \in D(A(t)) \), for each \( t \in [0,T] \) and if it satisfies the following integral equation,

\[
x(t) = \left. -\frac{\partial}{\partial s} S(t,s) \right|_{s=0} x_0 + S(t,0)y_0 + \int_0^t S(t,s)Bu(s) \, ds \\
+ \int_0^t S(t,s)f(s,x(s),x'(s)) \, ds + \int_0^t S(t,s)g(s,\tau,x(\tau),x'(\tau)) \, d\tau \, ds.
\]
DEFINITION 4.3. System (5) is said to be controllable on J if for every \( x_0, y_0 \in D(A) \) and \( x_1 \in X \) there exists a control \( u \in L^2(J, U) \), such that the solution \( x(\cdot) \) of (5) satisfies \( x(T) = x_1 \). To establish our main theorem we need the following assumptions.

1. \( x(t) \in D(A(t)) \), for each \( t \in [0, T] \).
2. There exists a fundamental solution \( S(t, s) \) of (6).
3. \( S(t, s) \) is compact for each \( t, s \in [0, T] \) and there exist positive constants \( M, M^* \) and \( N, N^* \), such that
   \[
   M = \sup \{ \| S(t, s) \| : t, s \in J \}, \quad M^* = \sup \{ \| \frac{\partial}{\partial t} S(t, s) \| : t, s \in J \},
   \]
   and
   \[
   N = \sup \{ \| \frac{\partial}{\partial s} S(t, s) \| : t, s \in J \}, \quad N^* = \sup \{ \| \frac{\partial^2}{\partial t \partial s} S(t, s) \| : t, s \in J \}, \text{ respectively.}
   \]
4. \( Bu(t) \) is continuous in \( t \) and \( \| B \| \leq M_1 \) for some constant \( M_1 > 0 \).
5. The linear operator \( W : L^2(J, U) \to X \) defined by
   \[
   W u = \int_0^T S(T, s) B u(s) \, ds
   \]
   induces a bounded invertible operator \( \hat{W} : L^2(J, U)/\ker W \to X \), such that \( \| \hat{W}^{-1} \| \leq M_2 \) for some constant \( M_2 > 0 \).
6. \( f(t, \cdot, \cdot) : X \times X \to X \) is continuous for each \( t \in J \) and the function \( f(\cdot, x, y) : J \to X \) is strongly measurable for each \( (x, y) \in X \times X \).
7. \( g(t, s, \cdot, \cdot) : X \times X \to X \) is continuous for each \( t, s \in J \) and the function \( g(\cdot, \cdot, x, y) : J \times J \to X \) is strongly measurable for each \( (x, y) \in X \times X \).
8. For every positive constant \( k \), there exists \( \alpha_k \in L^1(J) \), such that
   \[
   \sup_{\| x \|, \| y \| \leq k} \| f(t, x, y) \| \leq \alpha_k(t), \quad t \in J \text{ a.e.}
   \]
9. For every positive constant \( k \), there exists \( \beta_k \in L^1(J) \), such that
   \[
   \sup_{\| x \|, \| y \| \leq k} \left\| \int_0^t g(t, s, x, y) \, ds \right\| \leq \beta_k(t), \quad t \in J \text{ a.e.}
   \]
10. There exists an integrable function \( m : J \to [0, \infty) \), such that
    \[
    \| f(t, x, y) \| \leq m(t) \Omega(\| x \| + \| y \|), \quad t \in J, \ x, y \in X,
    \]
    where \( \Omega : [0, \infty) \to (0, \infty) \) is a continuous nondecreasing function.
11. There exists an integrable function \( n : J \to [0, \infty) \), such that
    \[
    \left\| \int_0^t g(t, s, x, y) \, ds \right\| \leq n(t) \Omega_0(\| x \| + \| y \|), \quad t \in J, \ x, y \in X,
    \]
    where \( \Omega_0 : [0, \infty) \to (0, \infty) \) is a continuous nondecreasing function and
    \[
    (M + N) \int_0^T q_0(s) \, ds < \int_0^\infty \frac{ds}{\Omega(s) + \Omega_0(s)},
    \]
    where \( q_0(s) = \max\{ m(t), n(t) \}, \ c = (M^* + N^*)\| x_0 \| + (M + N)\| y_0 \| + M_1 M_2 M_3 T \), and M_3 = \| x_1 \| + M^*\| x_0 \| + M\| y_0 \| + \int_0^T M m(s) \Omega(\| x(s) \| + \| x'(s) \|) \, ds + \int_0^T M n(s) \Omega_0(\| x(s) \| + \| x'(s) \|) \, ds.\]
THEOREM 4.1. If assumptions (C1)-(C11) hold, then system (5) is controllable on J.

PROOF. Consider the space \( Z = C^1(J, X) \) with norm \( \|x\| = \max\{\|x\|_0, \|x\|_1\} \) where \( \|x\|_0 = \sup\{\|x(t)\| : 0 \leq t \leq T\} \), \( \|x\|_1 = \sup\{\|x'(t)\| : 0 \leq t \leq T\} \). By (C6), for an arbitrary function \( x(\cdot) \), we define the control,

\[
 u(t) = W^{-1} \left[ x_0 + \frac{\partial}{\partial s} S(T, s) \int_{s=0}^{T} S(T, s) f(s, x(s), x'(s)) ds \right.

\[
- \int_{0}^{T} \int_{0}^{s} S(T, s) g(s, \tau, x(\tau), x'(\tau)) d\tau ds 
\]

Using this control, we will show that the operator \( F : Z \rightarrow Z \) defined by

\[
 (Fx)(t) = \frac{\partial}{\partial s} S(t, s) x_0 + S(t, 0) y_0 + \int_{0}^{t} S(t, s) BW^{-1} \left[ x_1 + \frac{\partial}{\partial \tau} S(T, \tau) \right]_\tau=0 \nonumber 

\[
- S(T, 0) y_0 - \int_{0}^{T} S(T, \tau) f(\tau, x(\tau), x'(\tau)) d\tau 
\]

has a fixed point. Clearly, \( (Fx)(T) = x_1 \), which means that the control \( u \) steers the system from the initial state \( x_0 \) to \( x_1 \) in time \( T \), provided we obtain a fixed point of the nonlinear operator \( F \). In order to study the controllability problem for system (5), we have to apply the Schaefer fixed-point theorem to the following operator equation,

\[
 x(t) = \lambda Fx(t), \quad \lambda \in (0, 1). \tag{9} 
\]

Then, from (8) and (9) we have

\[
 x(t) = -\lambda \frac{\partial}{\partial s} S(t, s) x_0 + \lambda S(t, 0) y_0 + \lambda \int_{0}^{t} S(t, s) BW^{-1} \left[ x_1 + \frac{\partial}{\partial \tau} S(T, \tau) \right]_\tau=0 \nonumber 

\[
- S(T, 0) y_0 - \int_{0}^{T} S(T, \tau) f(\tau, x(\tau), x'(\tau)) d\tau 
\]

So,

\[
 \|x(t)\| \leq M^* \|x_0\| + M \|y_0\| + MM_1M_2M_3T 
\]

\[
+ \int_{0}^{t} Mm(s) \Omega(\|x(s)\| + \|x'(s)\|) ds + \int_{0}^{T} Mm(s) \Omega_0(\|x(s)\| + \|x'(s)\|) ds. 
\]

Denoting by \( v(t) \) the right-hand side of the above inequality, we have

\[
\|x(t)\| \leq v(t), \quad t \in J, 
\]

\[
v(0) = M^* \|x_0\| + M \|y_0\| + MM_1M_2M_3T, 
\]

\[
v'(t) = Mm(t) \Omega(\|x(t)\| + \|x'(t)\|), \quad t \in J. 
\]
From (8), we have
\[ x'(t) = -\lambda \frac{\partial}{\partial t} S(t, s) \bigg|_{s=0} x_0 + \lambda \frac{\partial}{\partial t} S(t, 0) y_0 \]
\[ + \lambda \int_0^t \frac{\partial}{\partial t} S(t, s) B \bar{W}^{-1} \left[ x_1 + \frac{\partial}{\partial \tau} S(T, \tau) \right]_{\tau=0} x_0 - S(T, 0) y_0 \]
\[ - \int_0^T S(T, \tau) f(\tau, x(\tau), x'(\tau)) d\tau - \int_0^T \int_0^{\tau} S(T, \tau) g(\tau, \theta, x(\theta), x'(\theta)) d\theta d\tau \]
\[ + \lambda \int_0^t \frac{\partial}{\partial t} S(t, s) f(s, x(s), x'(s)) ds + \lambda \int_0^t \int_0^s \frac{\partial}{\partial t} S(t, s) g(s, \tau, x(\tau), x'(\tau)) d\tau ds \]
and
\[ \|x'(t)\| \leq N^* \|x_0\| + N \|y_0\| + NM_1 M_2 M_3 T \]
\[ + N \int_0^t m(s) \Omega(|x(s)| + \|x'(s)\|) ds + N \int_0^t n(s) \Omega_0 (|x(s)| + \|x'(s)\|) ds. \]

Denoting by \( r(t) \) the right-hand side of the above inequality, we get
\[ \|x'(t)\| \leq r(t), \quad t \in J, \]
\[ r(0) = N^* \|x_0\| + N \|y_0\| + NM_1 M_2 M_3 T, \]
\[ r'(t) = NM(t) \Omega(|x(t)| + \|x'(t)\|) + Nn(t) \Omega_0 (|x(t)| + \|x'(t)\|), \quad t \in J. \]

Let \( w(t) = v(t) + r(t), \) then, \( w(0) = v(0) + r(0) = c, \) and
\[ w'(t) = v'(t) + r'(t) \]
\[ \leq (M + N) [m(t) \Omega(w(t)) + n(t) \Omega_0 (w(t))] \]
\[ \leq (M + N) q_0(t) [\Omega(w(t)) + \Omega_0 (w(t))]. \]

This implies
\[ \int_{w(0)}^{w(t)} \frac{ds}{\Omega(s) + \Omega_0(s)} \leq (M + N) \int_0^t q_0(s) ds \leq (M + N) \int_0^T q_0(s) ds < \int_c^\infty \frac{ds}{\Omega(s) + \Omega_0(s)}. \]

This inequality implies that there is a constant \( K, \) such that
\[ w(t) = v(t) + r(t) \leq K, \quad t \in J. \]

Thus, \( \|x(t)\| \leq v(t), \|x'(t)\| \leq r(t), t \in J, \) and hence, \( \|x\| \leq K, \) where \( K \) depends only on \( T \) and on the functions \( m, n, \) and \( \Omega. \)

We shall now prove that the operator \( F \) defined by (8) is a completely continuous operator. Let
\[ B_k = \{ x \in Z : \|x\| \leq k \}\]
for some \( k \geq 1. \) We first show that \( F \) maps \( B_k \) into an equicontinuous family. Let \( x \in B_k \) and
Then, if $0 < t_1 < t_2 < T$, 

$$
\|{(F(x)}(t_1) - (F(x))(t_2))\| \leq \left\| \frac{\partial}{\partial s} [S(t_1, s) - S(t_2, s)] \right\|_{s=0} x_0 + \|S(t_1, 0) - S(t_2, 0)| v_0
$$

$$
+ \int_0^{t_1} \|S(t_1, s) - S(t_2, s)\| \|B\| \left\| \frac{\partial}{\partial t} S(T, \tau) \right\|_{\tau=0} x_0
$$

$$
+ \|S(T, 0) v_0\| + \int_0^{T} \|S(T, \tau)\| \alpha_k(\tau) \, d\tau + \int_0^{T} \|S(T, \tau)\| \beta_k(\tau) \, d\tau \right\| ds
$$

$$
+ \int_0^{t_1} \|S(t_2, s)\| \|B\| \left\| \frac{\partial}{\partial t} S(T, \tau) \right\|_{\tau=0} x_0 + \|S(T, 0) v_0\|
$$

$$
+ \int_0^{t_2} \|S(t_2, s)\| \|B\| \left\| \frac{\partial}{\partial t} S(T, \tau) \right\|_{\tau=0} x_0 + \int_0^{T} \|S(T, \tau)\| \alpha_k(\tau) \, d\tau + \int_0^{T} \|S(T, \tau)\| \beta_k(\tau) \, d\tau \right\| ds
$$

$$
+ \int_0^{t_1} \|S(t_1, s) - S(t_2, s)\| \|B\| \left\| \frac{\partial}{\partial t} S(T, \tau) \right\|_{\tau=0} x_0
$$

$$
+ \int_0^{t_2} \|S(t_2, s)\| \|B\| \left\| \frac{\partial}{\partial t} S(T, \tau) \right\|_{\tau=0} x_0 + \int_0^{T} \|S(T, \tau)\| \alpha_k(\tau) \, d\tau + \int_0^{T} \|S(T, \tau)\| \beta_k(\tau) \, d\tau \right\| ds
$$

$$
+ \int_0^{t_1} \|S(t_1, s) - S(t_2, s)\| \|B\| \left\| \frac{\partial}{\partial t} S(T, \tau) \right\|_{\tau=0} x_0 + \int_0^{T} \|S(t_1, s) - S(t_2, s)\| \|B\| \left\| \frac{\partial}{\partial t} S(T, \tau) \right\|_{\tau=0} x_0
$$

$$
+ \int_0^{t_2} \|S(t_2, s)\| \|B\| \left\| \frac{\partial}{\partial t} S(T, \tau) \right\|_{\tau=0} x_0 + \int_0^{T} \|S(t_1, s) - S(t_2, s)\| \|B\| \left\| \frac{\partial}{\partial t} S(T, \tau) \right\|_{\tau=0} x_0
$$

$$
+ \int_0^{t_1} \|S(t_1, s) - S(t_2, s)\| \|B\| \left\| \frac{\partial}{\partial t} S(T, \tau) \right\|_{\tau=0} x_0 + \int_0^{T} \|S(t_1, s) - S(t_2, s)\| \|B\| \left\| \frac{\partial}{\partial t} S(T, \tau) \right\|_{\tau=0} x_0
$$

$$
+ \int_0^{t_2} \|S(t_2, s)\| \|B\| \left\| \frac{\partial}{\partial t} S(T, \tau) \right\|_{\tau=0} x_0 + \int_0^{T} \|S(t_1, s) - S(t_2, s)\| \|B\| \left\| \frac{\partial}{\partial t} S(T, \tau) \right\|_{\tau=0} x_0
$$

$$
+ \int_0^{t_1} \|S(t_1, s) - S(t_2, s)\| \|B\| \left\| \frac{\partial}{\partial t} S(T, \tau) \right\|_{\tau=0} x_0 + \int_0^{T} \|S(t_1, s) - S(t_2, s)\| \|B\| \left\| \frac{\partial}{\partial t} S(T, \tau) \right\|_{\tau=0} x_0
$$

$$
+ \int_0^{t_2} \|S(t_2, s)\| \|B\| \left\| \frac{\partial}{\partial t} S(T, \tau) \right\|_{\tau=0} x_0 + \int_0^{T} \|S(t_1, s) - S(t_2, s)\| \|B\| \left\| \frac{\partial}{\partial t} S(T, \tau) \right\|_{\tau=0} x_0
$$

Thus, $F$ maps $B_k$ into an equicontinuous family of functions. It is easy to see that the family $FB_k$ is uniformly bounded.

Next, we show that $FB_k$ is compact. Since we have shown $FB_k$ is an equicontinuous collection, it suffices by the Arzela-Ascoli theorem to show that $F$ maps $B_k$ into a precompact set in $X$. Let $0 < t \leq T$.
be fixed and $\epsilon$ a real number satisfying $0 < \epsilon < t$. For $x \in B_k$, we define

$$
(F\epsilon x)(t) = -\frac{\partial}{\partial s} S(t,s) \bigg|_{s=0} x_0 + S(t,0) y_0 
+ \int_0^{t-\epsilon} S(t,s) B \tilde{W}^{-1} \left[ x_1 + \frac{\partial}{\partial \tau} S(T,\tau) \bigg|_{\tau=0} x_0 - S(T,0) y_0 
- \int_0^T S(T,\tau) f(\tau, x(\tau), x'(\tau)) \, d\tau 
- \int_0^T \int_0^T S(T,\tau) g(\tau, \theta, x(\theta), x'(\theta)) \, d\theta \, d\tau \right] (s) \, ds 
+ \int_0^{t-\epsilon} S(t,s) f(s, x(s), x'(s)) \, ds 
+ \int_0^{t-\epsilon} S(t,s) g(s, \tau, x(\tau), x'(\tau)) \, d\tau \, ds, \quad t \in J.
$$

Since $S(t,s)$ is a compact operator, the set $Y_\epsilon(t) = \{(F\epsilon x)(t) : x \in B_k\}$ is precompact in $X$ for every $\epsilon$, $0 < \epsilon < t$. Moreover, for every $x \in B_k$, we have

$$
\| (F\epsilon x)(t) - (F\epsilon \tilde{x})(t) \| \leq \int_{t-\epsilon}^t I_1 + \int_{t-\epsilon}^t S(t,s) \| \alpha_k(s) \| ds + \int_{t-\epsilon}^t S(t,s) \| \beta_k(s) \| ds \to 0, \quad \text{as } \epsilon \to 0,
$$

and

$$
\| (F\epsilon x)'(t) - (F\epsilon \tilde{x})'(t) \| \leq \int_{t-\epsilon}^t I_2 + \int_{t-\epsilon}^t S(t,s) \| \alpha_k(s) \| ds + \int_{t-\epsilon}^t S(t,s) \| \beta_k(s) \| ds \to 0, \quad \text{as } \epsilon \to 0.
$$

Therefore, there are precompact sets arbitrarily close to the set $\{(F\epsilon x)(t) : x \in B_k\}$. Hence, the set $\{(F\epsilon x)(t) : x \in B_k\}$ is precompact in $X$.

It remains to show that $F : Z \to Z$ is continuous. Let $\{x_n\}_{n=0}^\infty \subseteq Z$ with $x_n \to x$ in $Z$. Then, there is an integer $\nu$, such that $\|x_n(t)\| \leq \nu$, $\|x_n'(t)\| \leq \nu$ for all $n$, and $t \in J$, so $\|x'(t)\| \leq \nu$, and $x, x' \in B_\nu$. By (C6),

$$
f(s, x_n(s), x_n'(s)) \to f(s, x(s), x'(s)), \quad g(t, s, x_n(s), x_n'(s)) \to g(t, s, x(s), x'(s)),$$

for each $t, s \in J$ and since

$$
\| f(s, x_n(s), x_n'(s)) - f(s, x(s), x'(s)) \| \leq 2\alpha_\nu(s)
$$

and

$$
\| \int_0^t [g(t, s, x_n(s), x_n'(s)) - g(t, s, x(s), x'(s))] \, ds \| \leq 2\beta_\nu(s),
$$
we have by dominated convergence theorem,

\[
\|F_{x_n} - Fx\| = \sup_{t \in J} \left| \int_0^t S(t, s) B\dot{W}^{-1} \left[ x_1 + \frac{\partial}{\partial \tau} S(T, \tau) \right]_{\tau = 0} x_0 - S(T, 0) y_0 
- \int_0^T S(T, \tau) f(\tau, x_n(\tau), x'_n(\tau)) \, d\tau 
- \int_0^T \int_0^\tau S(T, \tau) g(\tau, \theta, x_n(\theta), x'_n(\theta)) \, d\theta \, d\tau \right] (s) \, ds 
- \int_0^t S(t, s) B\dot{W}^{-1} \left[ x_1 + \frac{\partial}{\partial \tau} S(T, \tau) \right]_{\tau = 0} x_0 - S(T, 0) y_0 
- \int_0^T S(T, \tau) f(\tau, x(\tau), x'(\tau)) \, d\tau 
- \int_0^T \int_0^\tau S(T, \tau) g(\tau, \theta, x(\theta), x'(\theta)) \, d\theta \, d\tau \right] (s) \, ds 
+ \int_0^t S(t, s) f(s, x_n(s), x'_n(s)) \, ds - \int_0^t S(t, s) f(s, x(s), x'(s)) \, ds 
+ \int_0^t \int_0^s S(t, s) g(s, \tau, x_n(s), x'_n(s)) \, d\tau \, ds 
- \int_0^t \int_0^s S(t, s) g(s, \tau, x(s), x'(s)) \, d\tau \, ds 
\leq \int_0^t \left\| S(t, s) B\dot{W}^{-1} \left[ \int_0^T S(T, \tau) [f(\tau, x_n(\tau), x'_n(\tau)) - f(\tau, x(\tau), x'(\tau))] \, d\tau 
+ \int_0^T \int_0^\tau S(T, \tau) [g(\tau, \theta, x_n(\theta), x'_n(\theta)) - g(\tau, \theta, x(\theta), x'(\theta))] \, d\theta \, d\tau \right] (s) \right\| \, ds 
+ \int_0^t \|S(t, s) [f(s, x_n(s), x'_n(s)) - f(s, x(s), x'(s))]\| \, ds 
+ \int_0^t \int_0^s \|S(t, s) [g(s, \tau, x_n(s), x'_n(s)) - g(s, \tau, x(s), x'(s))]\| \, d\tau \, ds \rightarrow 0, \quad \text{as } n \rightarrow \infty,
\]
\[
+ \int_0^t \frac{\partial}{\partial t} S(t,s) f(s, x_n(s), x'_n(s)) \, ds \\
- \int_0^t \frac{\partial}{\partial t} S(t,s) f(s, x(s), x'(s)) \, ds \\
+ \int_0^t \int_0^s \frac{\partial}{\partial t} S(t,s) g(s, \tau, x_n(\tau), x'_n(\tau)) \, d\tau \, ds \\
- \int_0^t \int_0^s \frac{\partial}{\partial t} S(t,s) g(s, \tau, x(\tau), x'(\tau)) \, d\tau \, ds \\
\leq \int_0^t \left| \frac{\partial}{\partial t} S(t,s) B W^{-1} \left[ \int_0^T S(T,\tau) \\
\times [f(\tau, x_n(\tau), x'_n(\tau)) - f(\tau, x(\tau), x'(\tau))] \, d\tau + \int_0^T \int_0^\tau S(T,\tau) \\
\times \left[ g(\tau, \theta, x_n(\theta), x'_n(\theta)) - g(\tau, \theta, x(\theta), x'(\theta)) \right] \, d\theta \, d\tau \right] (s) \right| \, ds \\
+ \int_0^t \left| \frac{\partial}{\partial t} S(t,s) [f(s, x_n(s), x'_n(s)) - f(s, x(s), x'(s))] \right| \, ds \\
+ \int_0^t \int_0^s \left| \frac{\partial}{\partial t} S(t,s) [g(s, \tau, x_n(\tau), x'_n(\tau)) - g(s, \tau, x(\tau), x'(\tau))] \right| \, d\tau \, ds \to 0, \quad \text{as } n \to \infty.
\]

Thus, \( F \) is continuous. This completes the proof that \( F \) is completely continuous. We have already proved that the set \( \zeta(F) = \{ x \in Z : x = \lambda Fx, \lambda \in (0,1) \} \) is bounded. Hence, by the Schaefer fixed-point theorem, the operator \( F \) has a fixed point in \( Z \). This means that any fixed point of \( F \) is a mild solution of (5) on \( J \) satisfying \( (Fx)(t) = x(t) \). Thus, system (5) is controllable on \( J \).

REFERENCES