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A comparative study of the numerical approximation of the random Airy differential equation

J.-C. Cortés *, L. Jódar, J.-V. Romero, M.-D. Roselló

Instituto Universitario de Matemática Multidisciplinar, Universitat Politècnica de València, Camino de Vera s/n, Building 8G, access C, 2nd floor, 46022, Valencia, Spain

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ABSTRACT

The aim of this paper is twofold. First, we deal with the extension to the random framework of the piecewise Fröbenius method to solve Airy differential equations. This extension is based on mean square stochastic calculus. Second, we want to explore the capability to provide not only reliable approximations for both the average and the standard deviation functions associated to the solution stochastic process, but also to save computational time as it happens in dealing with the analogous problem in the deterministic scenario. This includes a comparison of the numerical results with respect to those obtained by other commonly used operational methods such as polynomial chaos and Monte Carlo simulations. To conduct this comparative study, we have chosen the Airy random differential equation because it has highly oscillatory solutions. This feature allows us to emphasize differences between all the considered approaches.

1. Introduction

In the deterministic scenario, Airy differential equations appear in a variety of applications to mathematical physics such as the description of the solution of Schrödinger equation for a particle confined within a triangular potential, in the solution for one-dimensional motion of a quantum particle affected by a constant force, or in the theory of diffraction of radio waves around the earth's surface [1,2]. From these few examples, the introduction of randomness in the Airy differential equation seems to be quite natural. This can be straightforwardly justified from two perspectives: first, because in practice the calibration of both the initial conditions and the coefficient require exhaustive measurements that usually contain some sort of error; second, because the inherent complexity of the phenomena under study justifies that it be more coherent to consider the information that determines the model as random variables rather than deterministic values. These types of argument lead to consideration of the Airy random differential equation to be of great interest in physics and related areas.

Solutions to deterministic Airy differential equations are highly oscillatory, and they have claimed the attention of numerical analysts in comparing the effectiveness of different computational methods [3]. Thus, it seems to be a good example to check the numerical capacity of different techniques to compute the corresponding solution stochastic process of the random Airy differential equation. Recently, in [4], some of the authors have studied the random Airy differential equation

$$\ddot{X}(t) + AtX(t) = 0, \qquad -\infty < t < \infty, \qquad X(0) = Y_0, \qquad \dot{X}(0) = Y_1, \tag{1}$$

* Corresponding author. Tel.: +34 963879144.
E-mail addresses: jccortes@imm.upv.es, jccortes@mat.upv.es (J.-C. Cortés), ljodar@imm.upv.es (L. Jódar), jvromero@imm.upv.es (J.-V. Romero), drosello@imm.upv.es (M.-D. Roselló).

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where A, Y_0, and Y_1 are random variables. This study is based on an extension of the deterministic Fröbenius method to the random framework by applying mean square calculus; see [5]. In that paper, it is assumed that statistical absolute moments with respect to the origin of random input A grow at the most exponentially, i.e., there exist a nonnegative integer n_0 and positive constants H and M such that

$$\mathrm{E}\left[|A|^n\right] \leq H\, M^n < +\infty, \quad \forall\, n \geq n_0. \tag{2}$$

This allows one to obtain an approximate solution stochastic process of the Airy model as well as its main statistical functions such as average and variance by truncating a random power series solution. In [4], it is shown that every random variable A whose codomain or support is bounded satisfies such a condition; otherwise (as happens, for example, when A is a Gaussian random variable), we proposed to truncate the support to take advantage of that approach. The truncation can be done in such a way that the censured support contains most of the values of the random variable (for instance, if A is a Gaussian random variable with mean μ_A and standard deviation σ_A, then the interval $[\mu_A - 3\sigma_A, \mu_A + 3\sigma_A]$ contains on average 99.7% of its values). Although a priori this truncation may mean a loss of accuracy that could affect computations related to relevant statistical information about the solution stochastic process, such as its average and standard deviation functions, this potential inconvenience can be easily overcome by enlarging the length of the censured interval. Furthermore, in practice, this strategy does not entail any significant increase of computational cost.

In the deterministic scenario, a modification of the Fröbenius method has been successfully developed by some of the authors in order to save computational time when dealing with problems like (1) [6,7]. This motivates the two goals of this paper: first, to explore whether this modification of the Fröbenius method works in the random framework and also speeds up computations, and, second, to compare results obtained by this new approach with respect to those provided by other available methods, including the polynomial chaos technique.

The application of the homogeneous and generalized polynomial chaos method to the solution of random differential equations has already been tested successfully by some authors [8–10], although other contributions have highlighted its current limitations in dealing with random differential models appearing in some engineering applications [11]. In addition, we emphasize that interesting alternative methods to deal with random inputs are those based on Wiener–Hermite expansion, which can be regarded as its continuous counterpart [12].

The paper is organized as follows. Based on the deterministic approach shown in [7], Section 2 is devoted to presenting a modification of the random Fröbenius method developed in paper [4]. Section 3 introduces the polynomial chaos method, including its application to model (1). In Section 4, through an illustrative example, we compare the numerical results obtained by the modified random Fröbenius method to approximate the average and standard deviation functions with respect to the corresponding ones computed by polynomial chaos, Monte Carlo simulations, and the random Fröbenius method presented in [4]. Conclusions are presented in Section 5.

2. Developing a piecewise random Fröbenius method

In the recent paper [4], an extension of the deterministic Fröbenius method to deal with the random Airy differential equation (1) is presented. The method is based on the construction of a mean square convergent random infinite power series solution centered at the origin $t = 0$ which is truncated in order to obtain approximations of the average and variance of the solution stochastic process to (1). To apply the method, condition (2) is assumed to be satisfied by random input A. Although the computation time required in the numerical experiments presented in [4] showed themselves to be competitive with respect to other approaches, as we have pointed out above, in this paper we are also interested in comparing it with respect to other strategies. More precisely, in this section we want to adapt to the random framework a piecewise version of the Fröbenius method that some of the authors have tested to be more advantageous in the deterministic scenario [7].

The method consists of dividing the t-interval where we want to construct the approximate solution, say $[0, T]$, into K subintervals of length $a = T/K$, denoted respectively by $[(j-1)a, ja]$, $1 \leq j \leq K$, where $K = [T/a]$, $[\cdot]$ being the integer part function. Then, following [4], in the first step we construct the solution $X_K^1(t)$ on the interval $[0, a]$ using a random power series centered at $t_0 = 0$ and considering the random initial data $X_K^1(0) = Y_0$, $\dot{X}_K^1(0) = Y_1$. Taking as initial conditions $X_K^2(a) = X_K^1(a)$ and $\dot{X}_K^2(a) = \dot{X}_K^1(a)$, in the second step, we then construct an approximate random power series solution $X_K^2(t)$ centered at the point $t_1 = a$ on the interval $[a, 2a]$. In general, in the j-th step, we construct an approximate random power series $X_K^j(t)$ centered at the point $t_{j-1} = (j-1)a$ on the interval $[(j-1)a, ja]$, taking as initial conditions $X_K^j((j-1)a) = X_K^{j-1}((j-1)a)$ and $\dot{X}_K^j((j-1)a) = \dot{X}_K^{j-1}((j-1)a)$. The procedure continues until K approximate random power series solutions have been defined on each subinterval, respectively, covering the total domain $[0, T]$. Then a piecewise random power series solution $X_K(t)$ is defined on interval $[0, T]$ through $\left\{X_K^j(t) : t \in [(j-1)a, ja],\ 1 \leq j \leq K\right\}$.

Based on the previous exposition, and following an analogous development to that shown in [4], the approximate random power series solution centered at the point t_{j-1} is constructed on the interval $[(j-1)a, ja]$ as follows:

$$X_K^j(t) = \sum_{n\geq 0} X_n^j (t - t_{j-1})^n, \qquad t_{j-1} = (j-1)a, \quad 1 \leq j \leq K, \tag{3}$$

where coefficients X_n^j satisfy the following recurrence relationship:

$$X_2^j = -\frac{At_{j-1}X_0^j}{2},$$
$$X_{n+2}^j = -\frac{A\left(X_{n-1}^j + t_{j-1}X_n^j\right)}{(n+2)(n+1)}, \quad n \geq 1,\ 1 \leq j \leq K, \tag{4}$$

for given initial conditions X_0^j and X_1^j. These coefficients become those given in expression (17) in [4] when $K = 1$ (and so $j = K = 1$). Note that, with a fixed value $j:\ 1 \leq j \leq K$, recurrence (4) starts from $X_0^j = X_K^{j-1}(t_{j-1})$ and $X_1^j = \dot{X}_K^{j-1}(t_{j-1})$. Setting a truncation order of series (3), say N, one obtains the following approximate random power series solution on the interval $[(j-1)a, ja]$ which is centered at the point t_{j-1}:

$$X_{K,N}^j(t) = \sum_{n=0}^{N} X_n^j (t - t_{j-1})^n, \qquad t_{j-1} = (j-1)a, \quad 1 \leq j \leq K.$$

This allows us to define the following approximation for the average of the approximate solution stochastic process on the interval $[(j-1)a, ja]$:

$$\mathrm{E}\left[X_{K,N}^j(t)\right] = \sum_{n=0}^{N} \mathrm{E}\left[X_n^j\right](t - t_{j-1})^n, \quad t_{j-1} = (j-1)a,\ 1 \leq j \leq K. \tag{5}$$

In order to compute an approximation of the variance, we must take into account that

$$\mathrm{Var}\left[X_{K,N}^j(t)\right] = \mathrm{E}\left[\left(X_{K,N}^j(t)\right)^2\right] - \left(\mathrm{E}\left[X_{K,N}^j(t)\right]\right)^2, \tag{6}$$

together with

$$\mathrm{E}\left[\left(X_{K,N}^j(t)\right)^2\right] = \sum_{n=0}^{N} \mathrm{E}\left[(X_n^j)^2\right](t - t_{j-1})^{2n} + 2\sum_{n=1}^{N}\sum_{m=0}^{n-1} \mathrm{E}\left[X_n^j X_m^j\right](t - t_{j-1})^{n+m}, \quad 1 \leq j \leq K.$$

In this way, approximate average and standard deviation functions of piecewise random power series $X_K(t)$ are defined. In the following, these approximations will be denoted by $\mu_{X_{K,N}}^{\mathrm{F}}(t)$ and $\sigma_{X_{K,N}}^{\mathrm{F}}(t)$, respectively.

3. Applying the polynomial chaos method

This section is concerned with introducing the polynomial chaos method, including its application to construct an approximate solution stochastic process for problem (1). Henceforth, we shall assume that coefficient A is a random variable (r.v.) defined on a sample space Ω of a certain probability space $(\Omega, \mathcal{F}, P)$ [13, part I]. Thus, r.v. A depends on an outcome $\omega \in \Omega$, i.e., $A = A(\omega)$. As a consequence, the solution $X(t) = X(t;\omega)$ to problem (1) becomes a stochastic process (s.p.).

The polynomial chaos method was first introduced by Wiener, who called it *homogeneous chaos* [14]. In this context, if L_2 denotes the set of all r.v.'s χ whose statistical second-order moments with respect to the origin are finite, i.e., r.v.'s such that $\langle \chi^2 \rangle < +\infty$ (and as a consequence its variance is also finite), then every $\chi \in L_2$ can be represented in the form

$$\chi(\omega) = \chi_0 H_0 + \sum_{i_1=1}^{\infty} \chi_{i_1} H_1(\xi_{i_1}(\omega)) + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1} \chi_{i_1 i_2} H_2(\xi_{i_1}(\omega), \xi_{i_2}(\omega)) + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} \chi_{i_1 i_2 i_3} H_3(\xi_{i_1}(\omega), \xi_{i_2}(\omega), \xi_{i_3}(\omega)) + \cdots.$$

In this representation, $H_n = H_n(\boldsymbol{\xi})$ are Hermite polynomials in terms of vector $\boldsymbol{\xi}^{\mathrm{T}} = \left(\xi_{i_1}, \ldots, \xi_{i_n}\right)$ whose components are n independent standard Gaussian r.v.'s. An explicit formula to generate these polynomials is given by

$$H_n(\xi_{i_1}(\omega), \ldots, \xi_{i_n}(\omega)) = \exp\left(\frac{1}{2}\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\xi}\right)(-1)^n \frac{\partial^n}{\partial\xi_{i_1}\cdots\partial\xi_{i_n}}\left(-\frac{1}{2}\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\xi}\right).$$

$H_n(\cdot)$ is usually referred to as the *n-th order homogeneous chaos*. As a consequence, the two first terms in the representation (7) related to H_0 and H_1 can be interpreted as the Gaussian part of r.v. χ. For convenience, this representation can be arranged through a certain polynomials basis $\{\Phi_i\}$ as

$$\chi(\omega) = \sum_{i=0}^{\infty} \chi_i \Phi_i(\boldsymbol{\xi}(\omega)), \tag{7}$$

Table 1
One-dimensional polynomial chaos values and their variances for $n = 1$ (taken from Table 2.1. [15, p. 52]).

i	p, order of the polynomial chaos	i-th polynomial chaos Φ_i	$\langle(\Phi_i)^2\rangle$
0	$p = 0$	1	1
1	$p = 1$	ξ_1	1
2	$p = 2$	$(\xi_1)^2 - 1$	2
3	$p = 3$	$(\xi_1)^3 - 3\xi_1$	6
4	$p = 4$	$(\xi_1)^4 - 6(\xi_1)^2 + 3$	24

since there is a one-to-one correspondence between $\Phi_i(\cdot)$ and $H_i(\cdot)$. The number of r.v.'s in $\boldsymbol{\xi}$ represents the *dimension of the chaos*. $\{\Phi_i\}$ constitutes a complete set of statistically orthogonal s.p.'s of the Hilbert space L_2 with respect to the inner product, i.e., $\langle\Phi_i, \Phi_j\rangle = \delta_{ij}\langle\Phi_i, \Phi_i\rangle$, where $\langle\cdot\rangle$ denotes the following average:

$$\begin{aligned}\langle f(\boldsymbol{\xi}), g(\boldsymbol{\xi})\rangle &= \int_{\mathbb{R}^n} f(\boldsymbol{\xi})g(\boldsymbol{\xi})W(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi},\\ W(\boldsymbol{\xi}) &= \frac{1}{\sqrt{(2\pi)^n}}\exp\left(-\frac{1}{2}\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\xi}\right),\end{aligned} \tag{8}$$

and δ_{ij} is the Kronecker delta function. In addition, for $i \geq 1$, these polynomials are centered at the origin, i.e., $\langle\Phi_i\rangle = 0$, $i \geq 1$, and $\Phi_0 = 1$. As a consequence, from (7), the expectation and variance of r.v. χ can be computed in terms of coefficients χ_i in the following way:

$$\langle\chi(\omega)\rangle = \chi_0, \qquad \mathrm{Var}[\chi(\omega)] = \sum_{i=1}^{\infty}(\chi_i)^2\left\langle(\Phi_i(\boldsymbol{\xi}(\omega)))^2\right\rangle; \tag{9}$$

see [15] for further details.

In operational practice, the infinite summation (7) needs to be truncated at a finite term, say P. In our case, this leads to the following expansion of both the input r.v. $A(\omega)$ and the solution s.p. $X(t;\omega)$:

$$A(\omega) = \sum_{i=0}^{P} A_i\Phi_i(\boldsymbol{\xi}(\omega)), \qquad X(t;\omega) = \sum_{i=0}^{P} X_i(t)\Phi_i(\boldsymbol{\xi}(\omega)). \tag{10}$$

In these expansions, the total number of terms is $P+1$. This value is fixed by the dimension of the chaos, i.e., n (the number of components of vector $\boldsymbol{\xi}$) and the highest order p of the polynomial basis $\{\Phi_i\}$ in the following way: $P+1 = (n+p)!/(n!p!)$. Since we are going to consider A as the only input r.v. in problem (1), we will take $n = 1$, so $p = P$. In practice, the value of truncation index P is obtained after observing the stabilization of numerical solution. Purely as an illustrative example, if we fix $p = 4$, then this implies that r.v. A is going to be expanded by means of the one-dimensional polynomial chaos whose functionals $\{\Phi_i\}$ are just the Hermite polynomials of degree $0, 1, \ldots, 4$ which depend on r.v. ξ_1 (see Table 1). In this particular case, $P = 4$ in the spectral representation given by (10).

Now, we are ready to explain how the polynomial chaos operational methodology works in model (1). First, we impose that the truncated polynomial chaos series given by (10) satisfies random Airy differential equation (1)

$$\sum_{i=0}^{P}\ddot{X}_i(t)\Phi_i(\xi_1(\omega)) + t\sum_{i=0}^{P}\sum_{j=0}^{P}A_iX_j(t)\Phi_i(\xi_1(\omega))\Phi_j(\xi_1(\omega)) = 0.$$

A Galerkin projection of the previous equation onto each polynomial basis $\{\Phi_i\}$ is then conducted in order to ensure that the error is orthogonal to the functional space spanned by the finite-dimensional basis $\{\Phi_i\}$:

$$\sum_{i=0}^{P}\ddot{X}_i(t)\langle\Phi_i(\xi_1(\omega)), \Phi_l(\xi_1(\omega))\rangle + t\sum_{i=0}^{P}\sum_{j=0}^{P}A_iX_j(t)\left\langle\Phi_i(\xi_1(\omega))\Phi_j(\xi_1(\omega)), \Phi_l(\xi_1(\omega))\right\rangle = 0, \quad l = 0, 1, \ldots, P.$$

Now, taking advantage of the orthogonality properties of polynomial basis $\{\Phi_i\}$, one obtains the following coupled second-order system of deterministic differential equations:

$$\ddot{X}_l(t) = -\frac{t}{e_l}\sum_{i=0}^{P}\sum_{j=0}^{P}e_{ijl}A_iX_j(t), \quad l = 0, 1, \ldots, P, \tag{11}$$

where

$$\begin{aligned}e_{ijl} &= \langle\Phi_i(\xi_1(\omega))\Phi_j(\xi_1(\omega)), \Phi_l(\xi_1(\omega))\rangle, \quad 0 \leq i, j, l \leq P,\\ e_l &= \left\langle(\Phi_l(\xi_1(\omega)))^2\right\rangle, \qquad A_i = \frac{\langle A, \Phi_i(\xi_1(\omega))\rangle}{\langle(\Phi_i(\xi_1(\omega)))^2\rangle}, \quad l, i = 0, 1, \ldots, P.\end{aligned} \tag{12}$$

Note that the coefficients e_l and e_{ijl} can be computed directly from expression (8). More precisely, in the illustrative case previously introduced where $n = 1$ and $p = 4$, to compute the coefficients e_l and e_{ijl}, we just need to use expression (8) and the two last columns of Table 1. In the significant case where A is also a Gaussian r.v., the coefficients A_i can still be computed in the same way as e_l and e_{ijl}. However, if A is a non-Gaussian r.v., the computation of the numerator of coefficients A_i requires both the r.v.'s involved, A and ξ_1, to be transformed to the same uniformly distributed r.v. u using the inverse transformation method. This can be done as follows:

$$\langle A, \Phi_i(\xi_1(\omega))\rangle = \int_0^1 F_A^{-1}(u)\Phi_i(F_{\xi_1}^{-1}(u))\mathrm{d}u, \quad i = 0, 1, \ldots, P, \tag{13}$$

where $F_A^{-1}(\cdot)$ and $F_{\xi_1}^{-1}(\cdot)$ denote the inverse probability distribution functions of r.v.'s A and ξ_1, respectively.

4. Comparing the modified random Fröbenius method with other techniques: An illustrative example

This section compares the modified random Fröbenius method presented in Section 2 with other available methods. On the one hand, as we have explained in Section 1, we have selected the random Airy differential equation to conduct this comparative study because it has highly oscillating solutions, so it is expected to be an adequate model that is capable of showing discrepancies between the different methods. On the other hand, a random variable with unbounded domain will be considered to play the role of coefficient A in model (1) in order to require the truncation of its codomain to deal with the random Fröbenius method. This will allow us to show the differences between the considered approaches, including polynomial chaos and Monte Carlo methods [16], in a better way. Specifically, this numerical comparative study will be made by computing approximations of the average and standard deviation functions of the solution s.p. to problem (1).

From (11), note that we first need to compute the coefficients e_l, e_{ijl}, and A_i. As we have already pointed out, the coefficients e_l and e_{ijl} do not depend on r.v. A; therefore these computations can be stored for reusability with independence of the r.v. A involved. Note also that the coefficients A_i given by (12) neither depend on the form of the random differential equation to be solved nor on the initial conditions.

Let us assume the frequent case where A is a Gaussian r.v. with mean μ_A and standard deviation $\sigma_A > 0$, i.e., $A \sim \mathrm{N}(\mu_A; \sigma_A)$. Taking into account that A can be written as $A = \mu_A + \sigma_A\xi_1$, $\xi_1 \sim \mathrm{N}(0; 1)$, then, from (12), it is straightforward to see that

$$A_0 = \mu_A, \qquad A_1 = \sigma_A, \qquad A_i = 0, \quad i = 2, 3, 4, \ldots, P.$$

In the context of the illustrative example introduced above, that is, for $n = 1$ and $p = P = 4$, the deterministic coupled linear differential system (11) becomes

$$\left.\begin{aligned}
\ddot{X}_0(t) &= -t\left(\mu_A X_0(t) + \sigma_A X_1(t)\right),\\
\ddot{X}_1(t) &= -t\left(\sigma_A X_0(t) + \mu_A X_1(t) + 2\sigma_A X_2(t)\right),\\
\ddot{X}_2(t) &= -\frac{t}{2}\left(2\sigma_A X_1(t) + 2\mu_A X_2(t) + 6\sigma_A X_3(t)\right),\\
\ddot{X}_3(t) &= -\frac{t}{6}\left(6\sigma_A X_2(t) + 6\mu_A X_3(t) + 24\sigma_A X_4(t)\right),\\
\ddot{X}_4(t) &= -\frac{t}{24}\left(24\sigma_A X_3(t) + 24\mu_A X_4(t)\right).
\end{aligned}\right\} \tag{14}$$

In order to establish the corresponding initial conditions, let us assume that $Y_0 \sim \mathrm{N}(\mu_{Y_0}; \sigma_{Y_0})$ and $Y_1 \sim \mathrm{N}(\mu_{Y_1}; \sigma_{Y_1})$; hence $Y_0 = \mu_{Y_0} + \sigma_{Y_0}\xi_1$, $Y_1 = \mu_{Y_1} + \sigma_{Y_1}\xi_1$. We now multiply by Φ_i, $0 \le i \le P$ expression (10) for $X(t;\omega)$ and $\dot{X}(t;\omega)$ with $t = 0$. Then we apply the expectation operator, and finally we take advantage of orthogonality of the polynomial basis $\{\Phi_i\}$; thus we obtain

$$\left.\begin{aligned}
X_0(0) &= \mu_{Y_0}; \qquad X_1(0) = \sigma_{Y_0}; \qquad X_i(0) = 0, \quad i = 2, 3, \ldots, P,\\
\dot{X}_0(0) &= \mu_{Y_1}; \qquad \dot{X}_1(0) = \sigma_{Y_1}; \qquad \dot{X}_i(0) = 0, \quad i = 2, 3, \ldots, P.
\end{aligned}\right\} \tag{15}$$

Since we are only interested in comparing the piecewise random Fröbenius method with other approaches considering the case that random input A has codomain unbounded, for readability, we have taken deterministic initial conditions: $Y_0 = 3$ and $Y_1 = 1$ (then $\mu_{Y_0} = 3$, $\mu_{Y_1} = 1$, $\sigma_{Y_0} = \sigma_{Y_1} = 0$) and $A \sim \mathrm{N}(\mu_A = 2; \sigma_A = 0.5)$. For the computations, we have taken $[-6, 10]$ as the censured interval for random variable A, in order to apply the random Fröbenius method.

Tables 2 and 3 collect, respectively, the numerical approximations at several points for the average (μ) and standard deviation (σ) by using the following techniques: the random Fröbenius approach and its piecewise modification introduced in Section 2, Monte Carlo simulations, and the polynomial chaos method. Columns $\mu_{X_N}^{\mathrm{F}}(t)$ and $\sigma_{X_N}^{\mathrm{F}}(t)$ have been computed by applying the random Fröbenius method with truncation order N. Columns $\mu_{X_{K,N}}^{\mathrm{F}}(t)$ and $\sigma_{X_{K,N}}^{\mathrm{F}}(t)$ have been computed by applying the piecewise random Fröbenius method with truncation order N and splitting interval $[0, 5]$ into K subintervals of the same length. Note that expressions (5)–(7) have been used to deal with computations. Columns $\mu_{X_m}^{\mathrm{MC}}(t)$ and $\sigma_{X_m}^{\mathrm{MC}}(t)$ have been obtained by applying the Monte Carlo technique with m simulations and, finally, columns $\mu_{X_p}^{\mathrm{pc}}(t)$, $\sigma_{X_p}^{\mathrm{pc}}(t)$ have

Table 2
Comparison of the average obtained by using the random Fröbenius method, piecewise random Fröbenius method, Monte Carlo simulations, and the polynomial chaos approach for $A \sim \mathrm{N}(\mu_A = 2; \sigma_A = 0.5)$, $Y_0 = 3$ and $Y_1 = 1$.

t	$\mu^{\mathrm{F}}_{X_{69}}(t)$, $\mu^{\mathrm{F}}_{X_{2,35}}(t)$, $\mu^{\mathrm{F}}_{X_{5,17}}(t)$	$\mu^{\mathrm{F}}_{X_{200,4}}(t)$	$\mu^{\mathrm{MC}}_{X_m}(t)$ $m = 100\,000$	$\mu^{\mathrm{pc}}_{X_6}(t)$
0.00	3.00000	3.00000	3.00000	3.00000
1.00	2.91023	2.91023	2.90932	2.91023
2.00	−1.22508	−1.22508	−1.228	−1.22508
3.00	−0.759985	−0.759962	−0.755933	−0.759985
4.00	1.07227	1.07223	1.06919	1.07227
5.00	−0.705977	−0.705934	−0.702016	−0.705977

Table 3
Comparison of the standard deviation obtained by using the random Fröbenius method, piecewise random Fröbenius method, Monte Carlo simulations, and the polynomial chaos approach for $A \sim \mathrm{N}(\mu_A = 2; \sigma_A = 0.5)$, $Y_0 = 3$ and $Y_1 = 1$.

t	$\sigma^{F}_{X_{63}}(t)$, $\sigma^{F}_{X_{2,32}}(t)$, $\sigma^{F}_{X_{5,17}}(t)$	$\sigma^{F}_{X_{200,4}}(t)$	$\sigma^{MC}_{X_m}(t)$ $m = 100\,000$	$\sigma^{pc}_{X_{12}}(t)$
0.00	0	0	0	0
1.00	0.256018	0.25602	0.255866	0.256018
2.00	0.816923	0.81692	0.815421	0.816923
3.00	1.19504	1.19504	1.19562	1.19504
4.00	1.18406	1.18404	1.18305	1.18406
5.00	1.39934	1.39927	1.38065	1.39934

been calculated by formulae (9), respectively, following the polynomial chaos approach previously presented. In column 2, for both the random Fröbenius methods of order N and $\{N, K\}$, respectively, and also for polynomial chaos of degree P (last column), these numerical computations were performed until they reached stabilization of six significant digits of the numerical values with respect to N and P. *Mathematica*©instruction NDSolve was used to obtain a numerical solution of the corresponding analogous systems to (14) together with the initial conditions (15). In accordance with Tables 2 and 3, at this point, we stress that in order to obtain numerical stabilization of the average, a nonlinear coupled system like (14) but with seven equations was solved while the corresponding one for the standard deviation has 13 equations.

From the previous example, we have observed that both versions of the random Fröbenius method as well as the polynomial chaos technique achieve stabilization of the numerical results. We stress that, by increasing the number K of subintervals associated to the piecewise random Fröbenius method, we can reduce the degree N of the approximate finite series, which is just a polynomial. In practice, the smaller the degree N of the polynomial is, the lower the computational cost will be. However, this cost will increase as the number K of subintervals increases. Comments on this issue are added later.

From a computational standpoint, unlike what happens in the deterministic scenario, the piecewise random Fröbenius method takes more computational time than the random Fröbenius method. This is expected, because splitting the whole domain into subintervals entails a considerable increase of handling of the algebraic expressions involved in order to deal with the average and standard deviation functions.

In practice, a balance between K and N must be sought. In our case, we found that the time taken until numerical stabilization was reduced by applying the random Fröbenius method (corresponding to $N = 69$), while the time taken for computations using polynomial chaos ($P = 6$) and the piecewise random Fröbenius method (with $K = 5$, $N = 17$) were similar.

As the numerical approximations obtained by the differential approaches practically coincide, for the sake of clarity in the presentation, Fig. 1 shows the average (given by $X_0(t)$) and the standard deviation (denoted by $\sigma_X(t)$) approximations on the interval [0, 5]. Note that the standard deviation shape is justified by the oscillatory behavior of the average.

5. Conclusions

Based on the results obtained by some of the authors through a piecewise Fröbenius method in the deterministic case [6,7], in this paper we have explored whether we can also take advantage of this approach by extending it to the random scenario. For this study, we have chosen Airy differential equations because in the deterministic framework their solutions are highly oscillatory. Thus, this seems to be a good example to test the quality of this piecewise random Fröbenius method, and compare it with other techniques commonly used in the study of random differential equations, including the polynomial chaos method.

The formulation in the random framework of the piecewise random Fröbenius method has taken as a starting point a previous paper by some of the authors [4]. In that former contribution, the input random variable A was assumed to satisfy condition (2). Note that a flexible and wide family of random variables satisfying this condition is defined by random

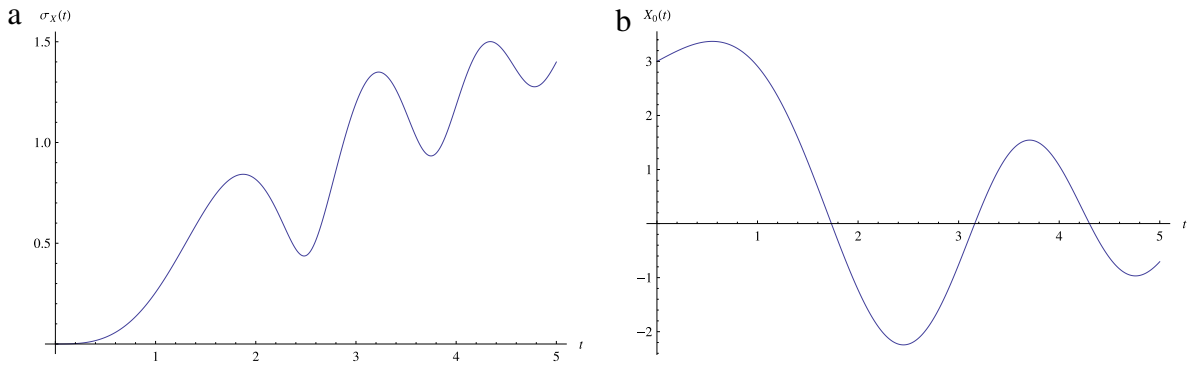

Fig. 1. Representation of the mean approximation $X_0(t)$ (left) and the standard deviation approximation $\sigma_X(t)$ (right) on the interval [0, 5] when $A \sim \mathrm{N}(\mu_A = 2; \sigma_A = 0.5)$, $Y_0 = 3$, and $Y_1 = 1$.

variables having bounded codomain. Otherwise, the truncation method allows us to deal with unbounded random variables, although in this case a loss of quality of the approximations is expected. As we also want to consider this feature, we have chosen a Gaussian random variable to play the role of the random input A.

As a remarkable difference with respect to it occurs in the deterministic case, and based on the illustrative example that we have considered, we realized that the piecewise random Fröbenius method requires more computational time than its former version developed in [4]. As far as the piecewise random Fröbenius method and polynomial chaos are concerned, we have seen that both require similar computational time whenever an appropriate balance between parameters $\{N, K\}$ is kept. Although it is one of the most popular methods to deal with random differential equations, in this example the Monte Carlo technique is not competitive with both the random Fröbenius and the polynomial chaos approaches. This feature is highlighted in our test example as being likely due to the highly oscillatory behavior of the solutions.

Although it is not the case for the study of random Airy differential equation (1), mean square analyticity of the coefficients is, in general, required in order to apply the random Fröbenius method. Nevertheless, we want to stress that useful characterizations of mean square analyticity can be found in terms of a correlation function [5, p. 99]. We point out that another significant advantage of the Fröbenius method with respect to polynomial chaos is that the Fröbenius method provides us with a series representation of the solution stochastic process *directly* in terms of the random input, say A, rather than as a function of standard Gaussian random variables which constitutes the cornerstone of the homogeneous polynomial chaos-type representation. This feature can become of prime importance in order to deduce, for instance, the true statistical distribution of the solution stochastic process.

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