Lengths of tours and permutations on a vertex set of a convex polygon

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Abstract

Let \( x_0, x_1, \ldots, x_{n-1} \) be vertices of a convex \( n \)-gon in the plane (each internal angle may be
equal to \( \pi \)), where, \((x_0, x_1), (x_1, x_2), \ldots, (x_{n-2}, x_{n-1})\), and \((x_{n-1}, x_0)\) are edges of the \( n \)-gon. De-
ote the length of the line segment \( x_ixj \) by \( d(i, j) \). Let \( \sigma \) be a permutation on \( \{0, 1, \ldots, n-1\} \).

Define a length of \( \sigma \) as \( S(\sigma) = \sum_{i=0}^{n-1} d(i, \sigma(i)) \). Further, define \( \sigma_p \) as \( \sigma_p(i) = i + p \mod n \) for all \( i \in \{0, 1, \ldots, n-1\} \). This paper shows that \( S(\sigma_p) \) is a strictly concave and strictly increasing function for \( 1 \leq p \leq \lfloor n/2 \rfloor \). It is also shown that \( \sigma_{\lfloor n/2 \rfloor} \) and \( \sigma_{\lceil n/2 \rceil} \) are longest permutations and \( \sigma_1 \) and \( \sigma_{n-1} \) are shortest permutations under some restriction. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let \( x_0, x_1, \ldots, x_{n-1} \) be vertices of a convex \( n \)-gon in the plane (each internal angle may be
equal to \( \pi \)), where, \((x_0, x_1), (x_1, x_2), \ldots, (x_{n-2}, x_{n-1})\), and \((x_{n-1}, x_0)\) are edges of the \( n \)-gon. If all internal angles are less than \( \pi \), the polygon is called strictly convex. Denote the length of the line segment \( x_ixj \) by \( d(i, j) \). \( i \mod n \) denotes \( i' \) such that \( i \equiv i' (\mod n) \) and \( 0 \leq i' \leq n-1 \). \( d(i \mod n, j \mod n) \) may be written as \( d(i, j) \) for notational simplicity. Let \( \sigma \) be a permutation on \( \{0, 1, \ldots, n-1\} \), i.e., \( \sigma(i) \in \{0, 1, \ldots, n-1\} \) and \( \sigma(i) \neq \sigma(j) \) if \( i \neq j \). If a permutation \( \sigma \) on \( \{0, 1, \ldots, n-1\} \) satisfies that “for any proper subset \( N \subset \{0, 1, \ldots, n-1\} \), there is \( i \in N \) such that \( \sigma(i) \notin N' \), then \( \sigma \) is called a tour. In this paper, a permutation on \( \{0, 1, \ldots, n-1\} \) and a tour on \( \{0, 1, \ldots, n-1\} \) is simply called a permutation and a tour, respectively.

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For $i, j \in \{0, 1, \ldots, n-1\}$, define
\[ N[i,j] := \begin{cases} \{i, i+1, \ldots, j\} & \text{if } i \leq j, \\ \{i, i+1, \ldots, n-1, 0, 1, \ldots, j\} & \text{if } i > j. \end{cases} \]

If a permutation $\sigma$ satisfies that “for any pair $i, j \in \{0, 1, \ldots, n-1\}$, there is $k \in N[i,j]$ such that $\sigma(k) \not\in N[i,j]$”, then $\sigma$ is called a connected permutation. Clearly, a tour is a connected permutation. Especially, if $\sigma(i) = i + p \mod n$ for a fixed integer $0 \leq p \leq n-1$, then the permutation is denoted by $\sigma_p$. Note that $S(\sigma_p) = S(\sigma_{n-p})$, and so we may assume that $0 \leq p \leq \lfloor n/2 \rfloor$. $\sigma_p$ is a tour if $p$ and $n$ are relatively prime.

For a permutation $\sigma$, define a length of $\sigma$ as
\[ S(\sigma) = \sum_{i=0}^{n-1} d(i, \sigma(i)). \]

$S(\sigma_p)$ may be written as $S_p$ for notational simplicity.

The following conjecture was presented by Jorge Urrutia in the Open Problem Session of Japan Conference on Discrete and Computational Geometry’98 (JCDCG’98) held in Tokai University, Tokyo, December 9–12, 1998.

**Conjecture A (See Fig. 1).** $S_p$ is an increasing function for $1 \leq p \leq \lfloor n/2 \rfloor$ for any convex polygon.

If the polygon is not convex, the conjecture does not hold. For example, $S(1) > S(2)$ in the concave polygon of Fig. 2. If the convex polygon is regular, then Conjecture A is clear, because $d(i, i+p) < d(i, i+q)$ for every $1 \leq p < q \leq \lfloor n/2 \rfloor$. However, if the polygon is not regular, there is a case that $d(i, i+p) > d(i, i+q)$ for some $1 \leq p < q \leq \lfloor n/2 \rfloor$. For example, $d(0,2) > d(0,3)$ in the polygon in Fig. 1. However in such a case, Conjecture A describes that the sum becomes an increasing function. This paper will prove the conjecture and moreover shows the concavity of $S_p$ as follows:
Theorem 1. $S_p$ is a strictly concave and strictly increasing function for $1 \leq p \leq \lfloor n/2 \rfloor$ for any convex polygon.

Moreover, this paper shows that

Theorem 2. (1) $\sigma_1$ is a shortest connected permutation for any convex polygon and is a unique shortest connected permutation for any strictly convex polygon.

(2) $\sigma_{\lfloor n/2 \rfloor}$ is a longest permutation for any convex polygon and is a unique longest permutation for any strictly convex polygon.

Note that if connectivity is not necessary, the shortest permutation is trivially $\sigma_0$.

The following notations are used.

$$N(i, j) := N[i, j] \setminus \{i, j\},$$
$$N(i, j) := N[i, j] \setminus \{i\},$$
$$N[i, j] := N[i, j] \setminus \{j\}.$$

2. A proof of Theorem 1

Proposition 1. For four distinct vertices $x_h, x_i, x_j,$ and $x_k$ of the convex polygon lying counterclockwise in this order, it follows that

$$d(h, j) + d(i, k) \geq d(i, j) + d(k, h).$$
The equality holds if and only if $x_i$ and $x_j$ are on a line segment $x_k x_h$ or $x_k$ and $x_h$ are on a line segment $x_j x_i$.

This proposition can be directly obtained from the triangle inequality and is an elementary observation, hence the proof is omitted here.

Now we prove Theorem 1. First, we prove the following inequality:

$$S_{\lfloor n/2 \rfloor} > S_{\lfloor n/2 \rfloor - 1}. \quad (1)$$

Consider the case that $n$ is even, i.e., $n = 2m$ for an integer $m \geq 2$ (note that $m = \lfloor n/2 \rfloor$). By Proposition 1,

$$d(i, i + m) + d(i + 1, i + 1 + m) \geq d(i + 1, i + m) + d(i + 1 + m, i) \quad (2)$$

for all $i \in \{0, 1, \ldots, n - 1\}$. Moreover, there is at least one $i \in \{0, 1, \ldots, n - 1\}$ satisfying

$$d(i, i + m) + d(i + 1, i + 1 + m) > d(i + 1, i + m) + d(i + 1 + m, i). \quad (3)$$

By summing these inequalities from $i = 0$ to $n - 1$,

$$2S_m > 2S_{m-1}. \quad (4)$$

Thus, (1) is obtained in this case.

Consider the case that $n$ is odd, i.e., $n = 2m + 1$ for a natural number $m$ (note that $m = \lfloor n/2 \rfloor$). (2) also holds for every $i \in \{0, 1, \ldots, n - 1\}$. Moreover, there is at least one $i \in \{0, 1, \ldots, n - 1\}$ satisfying (3). By summing these inequalities from $i = 0$ to $n - 1$,

$$2S_m > S_m + S_{m-1}. \quad (5)$$

Thus, (1) is also obtained in this case. Therefore, (1) is proved for every $n$.

**Lemma 1.** For an integer $p \geq 2$, $S_p - S_{p-1} > S_{p+1} - S_p$.

**Proof.** By Proposition 1,

$$d(i, i + p) + d(i + 1, i + p + 1) \geq d(i + 1, i + p) + d(i, i + p + 1) \quad (4)$$

for all $i \in \{0, 1, \ldots, n - 1\}$. Moreover, there is at least one $i \in \{0, 1, \ldots, n - 1\}$ satisfying

$$d(i, i + p) + d(i + 1, i + p + 1) > d(i + 1, i + p) + d(i, i + p + 1). \quad (5)$$

By summing these inequalities from $i = 0$ to $n - 1$,

$$2S_p > S_{p+1} + S_{p-1}. \quad (6)$$

It is equivalent to $S_p - S_{p-1} > S_{p+1} - S_p$. □

**Proof of Theorem 1.** By using (1) and Lemma 1, it is proved by induction. □
3. A proof of Theorem 2

For proving Theorem 2, some notations are introduced. If two distinct \( i, j \in \{0, 1, \ldots, n-1\} \) satisfy that \( j \in N(i, \sigma(i)) \) and \( \sigma(j) \in N(\sigma(i), i) \), then it is said that \( i \) and \( j \) are crossing. If two distinct \( i, j \in \{0, 1, \ldots, n-1\} \) satisfy that \( j, \sigma(j) \in N(i, \sigma(i)) \) or \( j, \sigma(j) \in N(\sigma(i), i) \), then it is said that \( i \) and \( j \) are separated.

Lemma 2. There is a longest permutation including no separated pair. If the polygon is strictly convex, any longest permutation doesn’t include any separated pair.

Proof. Let \( \sigma \) be a longest permutation such that the number of separated pairs is minimum. Assume that there exist a separated pair \( i, j \). Without loss of generality, we may assume that \( j, \sigma(j) \in N(i, \sigma(i)) \). Thus, \( \sigma(j) \in N[j, \sigma(i)] \) or \( \sigma(j) \in N(i, j) \).

First suppose that \( \sigma(j) \in N[j, \sigma(i)] \). Consider a new permutation \( \sigma' \) such that \( \sigma'(i) = \sigma(j) \), \( \sigma'(j) = \sigma(i) \), and \( \sigma'(k) = \sigma(k) \) for other \( k \neq i, j \). By Proposition 1,

\[
S(\sigma') \geq S(\sigma),
\]

hence, \( \sigma' \) is also a longest permutation. Moreover, the number of separated pairs of \( \sigma' \) is less than one of \( \sigma \). This is a contradiction.

We next assume that \( \sigma(j) \in N(i, j) \). Consider a sequence of integers \( Q = \langle \sigma(j) = j_0, j_1, \ldots, j_{k-1} = j \rangle \) such that \( j_p = \sigma(j_{p-1}) \) for \( p = 1, 2, \ldots, k-1 \). Assume that \( i \) is not contained in \( Q \) (see Fig. 3). Consider a new permutation \( \sigma' \) such that \( \sigma'(i) = j \), \( \sigma'(\sigma(j)) = i \), \( \sigma'(j_q) = \sigma(j_{q-1}) \) for \( q=1, 2, \ldots, k-1 \), and \( \sigma'(q) = \sigma(q) \) for other \( q \notin Q \cup \{i\} \). By Proposition 1, (6) holds, hence, \( \sigma' \) is also a longest permutation. Moreover, the number of separated pairs of \( \sigma' \) is less than one of \( \sigma \). This is a contradiction. Thus we may assume that \( i \) is contained in \( Q \) (see Fig. 4). Hence, \( Q = \langle \sigma(j) = j_0, j_1, \ldots, j_{h-1} = i, j_h = \sigma(i), j_{h+1}, \ldots, j_{k-1} = j \rangle \). Let \( Q' = \langle \sigma(j) = j_0, j_1, \ldots, j_{h-1} = i \rangle \). Consider a new permutation \( \sigma' \) such that \( \sigma'(j) = i \), \( \sigma'(\sigma(j)) = \sigma(i), \sigma'(j_q) = \sigma(j_{q-1}) \) for \( q=1, 2, \ldots, h-1 \), and \( \sigma'(q) = \sigma(q) \) for other \( q \notin Q' \cup \{j\} \). By Proposition 1, (6) is obtained, hence, \( \sigma' \) is also a longest permutation. Moreover, the number of separated pairs of \( \sigma' \) is less than one of \( \sigma \). This is a contradiction.

When \( j, \sigma(j) \in N(\sigma(i), i) \), a similar discussion can be applied, hence it is omitted.

Next, we consider the case that the polygon is strictly convex. Let \( \sigma \) be a longest permutation. We assume that \( \sigma \) includes at least one separated pair. The equality of
Proposition 1 does not hold. Thus by using the same discussion, $S(\sigma') > S(\sigma)$ is obtained in place of (6). It contradicts that $\sigma$ is a longest permutation. Therefore $\sigma$ never includes a separated pair. □

**Lemma 3.** There is a shortest connected permutation including no crossing pair. If the polygon is strictly convex, any shortest permutation does not include any crossing pair.

Let $\sigma$ be a shortest connected permutation such that the number of crossing pairs is minimum. Assume that there exist a crossing pair $i,j$, i.e., $j \in N(i, \sigma(i))$ and $\sigma(j) \in N(\sigma(i), i)$. Consider a sequence of integers $Q = \langle \sigma(j) = j_0, j_1, \ldots, j_{k-1} = j \rangle$ such that $j_p = \sigma(j_{p-1})$ for $p = 1, 2, \ldots, k - 1$.

Assume that $i$ is not contained in $Q$. Consider a new permutation $\sigma'$ such that $\sigma'(i) = \sigma(j)$, $\sigma'(j) = \sigma(i)$, and $\sigma'(k) = \sigma(k)$ for other $k \neq i, j$. From the fact that $\sigma$ is a connected permutation, it follows that $\sigma'$ is also a connected permutation. By Proposition 1,

$$S(\sigma') \leq S(\sigma),$$

(7)

hence, $\sigma'$ is also a shortest connected permutation. Moreover, the number of crossing pairs of $\sigma'$ is less than one of $\sigma$. This is a contradiction.

Assume that $i$ is contained in $Q$ (see Fig. 5). Thus, $Q$ is $\langle \sigma(j) = j_0, j_1, \ldots, j_{h-1} = i, j_h = \sigma(i), j_{h+1}, \ldots, j_{k-1} = j \rangle$. Let $Q' = \langle \sigma(j) = j_0, j_1, \ldots, j_{h-1} = i \rangle$. Consider a new permutation $\sigma'$ such that $\sigma'(j) = i$, $\sigma'(\sigma(i)) = \sigma(j)$, $\sigma'(j_q) = j_{q-1}$ for $q = 1, 2, \ldots, h - 1$, and $\sigma'(q) = \sigma(q)$ for other $q \notin Q' \cup \{j\}$. From the fact that $\sigma$ is a connected permutation, it follows that $\sigma'$ is also a connected permutation. By Proposition 1, (7) is obtained,
hence, $\sigma'$ is also a shortest connected permutation. Moreover, the number of crossing pairs of $\sigma'$ is less than one of $\sigma$. This is a contradiction.

Next, we consider the case that the polygon is strictly convex. Let $\sigma$ be a shortest permutation. We assume that $\sigma$ includes at least one crossing pair. The equality of Proposition 1 does not hold. Thus by using the same discussion, $S(\sigma') < S(\sigma)$ is obtained in place of (7). It contradicts that $\sigma$ is a shortest permutation. Therefore $\sigma$ never includes a crossing pair. ⊓⊔

**Lemma 4.** If a permutation $\sigma$ includes no separated pair, then $\sigma$ is $\sigma_{[n/2]}$. If a connected permutation $\sigma$ includes no crossing pair, then $\sigma$ is $\sigma_{1}$.

**Proof.** First we try to prove the first half of the lemma. Suppose that $\sigma$ is not $\sigma_{[n/2]}$. That is, there exists $i \in \{0, 1, \ldots, n-1\}$ such that $|N[i, \sigma(i)]| < \lfloor n/2 \rfloor$ or $|N[i, \sigma(i)]| > \lceil n/2 \rceil$. First assume that $|N[i, \sigma(i)]| < \lfloor n/2 \rfloor$. Without loss of generality, we may assume that $\sigma(0) = k < \lfloor n/2 \rfloor$. From the fact that $\sigma$ includes no separated pair, it follows that $\sigma(j) \in N[0, k - 1]$ for all $j \in N[k + 1, n - 1]$. However such a mapping is infeasible because $|N[0, k-1]| = k < \lfloor n/2 \rfloor \leq n-k-1 = |N[k+1, n-1]|$. Thus, it is a contradiction. If we assume that $|N[i, \sigma(i)]| > \lceil n/2 \rceil$, a similar discussion can be obviously used, then it is omitted.

Next, we try to prove the latter half of the lemma. Suppose that a connected permutation $\sigma$ is not $\sigma_{1}$. That is, there exists $i \in \{0, 1, \ldots, n-1\}$ such that $1 < |N[i, \sigma(i)]| < n - 1$. From the fact that $\sigma$ is connected, it follows that there exists a $j \in \{0, 1, \ldots, n-1\}$ such that $j \in N[i, \sigma(i)]$ and $\sigma(j) \not\in N[i, \sigma(i)]$. From the property of permutations,

$$|\{k | k \in N[i, \sigma(i)], \sigma(k) \not\in N[i, \sigma(i)]\}| = |\{k | k \not\in N[i, \sigma(i)], \sigma(k) \in N[i, \sigma(i)]\}|.$$

Thus, there exist $j, j' \in \{0, 1, \ldots, n-1\}$ such that $j, \sigma(j') \in N[i, \sigma(i)]$ and $\sigma(j), j' \not\in N[i, \sigma(i)]$. $\sigma$ has no crossing pair, hence $j = \sigma(i)$ and $\sigma(j') = i$. Similarly, there must exist $h, h' \in \{0, 1, \ldots, n-1\}$ such that $h, \sigma(h') \in N[\sigma(i), i]$ and $\sigma(h), h' \not\in N[\sigma(i), i]$. Further, we must obtain that $h = \sigma(i)$ and $\sigma(h') = i$, hence $j = h$ and $j' = h'$. However, $j' \not\in N[i, \sigma(i)]$ and $h' \in N(i, \sigma(i)) \subset N[i, \sigma(i)]$. This is a contradiction. ⊓⊔

**Proof of Theorem 2.** It is clear from Lemmas 2, 3, and 4. ⊓⊔

The following property can be also obtained from Theorem 2.

**Corollary 1.** For any convex polygon, “$\sigma_{1}$ is a shortest tour” and “$\sigma_{[n/2]}$ is a longest tour if $n$ is odd”. If the polygon is strictly convex, “$\sigma_{1}$ is a unique shortest tour” and “$\sigma_{[n/2]}$ is a unique longest tour if $n$ is odd”.

### 4. Related problems

If $x_{0}, x_{1}, \ldots, x_{n-1}$ are arbitrary points in the plane, the problem for finding a shortest or longest tour or permutation can also be considered.
A problem of finding a shortest permutation can be reduced to Minimum-Cost Maximum-Matching Problem [3, pp. 252–258] as follows. Let the vertex set of a graph \( G \) be \( \{x^i_1, x^i_2 | i = 0, 1, \ldots, n - 1\} \). Let the edge set of \( G \) be \( \{(x^i_k, x^j_h) | i \neq j, i, j \in \{0, 1, \ldots, n - 1\}, k, h \in \{1, 2\}\} \). Let a weight of edge \((x^i_k, x^j_h)\) be \( w(x^i_k, x^j_h) := d(i, j) \). Find a minimum weight maximum matching \( M \) on \((G, w)\). Clearly, \(|M| = n\). By shrinking each \( \{x^i_1, x^i_2\} \) into a vertex \( x_i \), a new graph \( \tilde{G} \) of order \( n \) is obtained. An edge subset \( \tilde{M} \) in \( \tilde{G} \) corresponding to \( M \) defines a spanning subgraph in which every vertex has degree two. Thus, a permutation on \( \{0, 1, \ldots, n - 1\} \) can be constructed from \( \tilde{M} \). The permutation is obviously a shortest permutation. Thus, it can be solved in \( O(n^3) \) time. A longest permutation is obtained similarly.

Finding a shortest “tour” is NP-hard. It is a well-known Geometric Traveling Salesman Problem [2, pp. 212]. The problem for finding a longest “tour” was considered in Ref. [1]. The problem seems also NP-hard, however, it has not been proved yet. It is an open problem.

Theorems 1 and 2 are very basic properties of geometry, thus they seem important in theory. Moreover, from the viewpoint of applications, they are also valuable. As mentioned before, the problem treated in this paper can be regarded as a special case of Geometric Traveling Salesman Problem (GTSP) or Minimum-Cost Maximum-Matching Problem. Then for example a GTSP in which all cities are located around a (convex) gulf can be solved in \( O(1) \) time by using our results.

In this paper, Euclidean distance is used. However, for any distance (for example, \( L_k \) distance) in which the triangle inequality holds, similar results can be obtained. The difference is only that the equality of Proposition 1 may hold even if the four vertices construct a strictly convex quadrilateral. If such a measure of distance is used, we can say that

1. \( S_p \) is an increasing function in the wide sense for \( 1 \leq p \leq \lfloor n/2 \rfloor \) for any convex polygon, and
2. \( \sigma_1 \) is a shortest connected permutation and \( \sigma_{\lfloor n/2 \rfloor} \) is a longest permutation for any convex polygon.

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References
