1. Introduction

In this paper we study in depth a class of infinite dimensional linear time invariant systems. We get concrete analytic conditions characterizing exact controllability and exact observability in the sense of Helton. Also, we characterize the class of transfer functions having realizations that are exactly controllable and exactly observable.

There are currently two approaches to the problem of describing linear systems, using external and internal descriptions. The external description gives the input/output relations, whereas the internal description gives the dynamics of the system that produce the given input/output relations. The problem of realization is to find from a given external description all the possible internal descriptions and the relations among them. This is an impossible task unless some additional assumptions are made. One natural assumption is that in some sense a realization should be minimal. When this assumption is made precise, we get a complete theory for finite dimensional time invariant linear systems, whether discrete or continuous.

We will quote some of the highlights of the finite dimensional theory, that will serve as reference and motivation for the results of this paper. For a more complete exposition excellent accounts are in [3, 13].

Two linear spaces $U$ and $Y$ are given. $U$ is called the control space and $Y$ the output space. A (discrete) linear time invariant input/output map is a linear map sending sequences of elements of $U$ into sequences of elements $Y$ such that $y_n = \sum_{j=0}^{n-1} A_j u_{n-j-1}$, where $A_j \in L(U, Y)$. The sequence $(A_0, A_1, \ldots)$ is called the impulse response function, whereas $\sum A_j z^j$ (or $\sum A_j z^{j+1}$) is called the transfer function of the system. Usually we identify
The internal description of a discrete constant linear system is given by the dynamical equations

\[
x_{n+1} = Ax_n + Bu_n, \quad y_n = Cx_n,
\]

where \( x_n \) belong to a linear space \( X \) called the state space and \( A \in L(X, X) \), \( B \in L(U, X) \), and \( C \in L(X, Y) \). Starting with \( x_0 = 0 \), it is easy to check that \( y_n = C A^{n-1} B u_0 + \cdots + C B u_{n-1} \). Hence the system \( \{A, B, C\} \) is a realization of the given input/output relation if and only if \( A_i = C A^{i-1} B \) for all \( i \).

The minimality requirement mentioned earlier will be that the realization should be both controllable and observable, i.e., that \( \bigcap_i \ker B^* A^{*i} = \{0\} \) and \( \bigcap_i \ker C A^i = \{0\} \), respectively.

Given an infinite sequence of matrices \( A_i \in L(C^p, C^r) \), the corresponding Hankel matrix is the infinite dimensional block matrix whose \( i,j \)th element is \( A_i \). The basic result of finite dimensional system theory is the following complete characterization.

**Theorem 1.1 [13].** (a) An impulse response function \( \{A_0, A_1, \ldots\} \) has a finite dimensional realization if and only if the rank of the Hankel matrix is finite.

(b) Two controllable and observable realizations \( \{A, B, C\} \) and \( \{A_1, B_1, C_1\} \) realize the same impulse response function if and only if they are similar, i.e., there exists an invertible linear map \( R: X \to X \), which makes the following diagram commutative.

\[
\begin{array}{c}
A \\
\downarrow R \\
X \\
\downarrow R \\
C_1
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
B \\
\downarrow R \\
P \\
\downarrow R \\
P_1
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
C \\
\downarrow R \\
C_1 \\
\downarrow R \\
C_1
\end{array}
\end{array}
\]

Part (b) is usually referred to as the state space isomorphism theorem.

Our aim in this paper is to examine some of the notions involved in an infinite dimensional setting in the case that all spaces involved are Hilbert spaces. To simplify things we will assume single input/single output systems. Thus we have \( U = Y = C \). In this case \( Bx = ab \) for some \( b \in X \) and \( Cx = (x, c) \) for some \( c \in X \). We will use \( \{A, b, c\} \) as an alternate notation of the system and will assume \( A \in B(X) \) the Banach algebra of all bounded linear operators on \( X \). For us the controllability of the system means that \( b \)
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is a cyclic vector for $A$, whereas observability means that $c$ is a cyclic vector for $A^*$. Given an impulse response function $(a_0, a_1, \ldots)$ or the corresponding transfer function $a(z) = \sum a_n z^n$, the realization problem is easily resolved. For definitions and terminology we refer to Section 2.

THEOREM 1.2 [1, 9, 11]. Assume $a \in H^2$; then $a$ has a controllable and observable realization $\{T^*, a, k_0\}$ in the state space $K$, where $K \in H^2$ is the minimal left invariant subspace of $H^2$ including $a$ and $k_0$ is the projection of the constant function 1 onto $K$, $T^*$ is the left shift in $H^2$ restricted to $K$.

However, a straightforward generalization of the state space isomorphism theorem for infinite dimensional systems is not available [9]. Another interesting counter example is presented in Appendix 1. To get a state space isomorphism theorem additional assumptions on the realizations involved have to be made. With this in mind J. W. Helton introduced the notions of exact controllability and exact observability.

Let $D(n) = \{[\alpha_i]_{i=0}^{\infty} \mid \alpha_i \in L^2(0, \infty), \alpha_i = 0 \text{ for } i > n\}$ and let $d = \bigcup_{n = 0}^{\infty} D(n)$. $d$ is a dense subset of $L^2(0, \infty)$.

DEFINITION 1.3 [11]. The controllability operator of the system $\{A, b, c\}$ $\mathcal{C}: \Delta \rightarrow X$ is defined by $\mathcal{C}([\alpha_n]) = \sum_{n=0}^{\infty} \alpha_n A^nb$. We will say that the system is exactly controllable if $\mathcal{C}$ can be extended to a continuous map of $L^2(0, \infty)$ onto $X$. Similarly we define the observability operator $\mathcal{O}: \Delta \rightarrow X$ by $\mathcal{O}([\alpha_n]) = \sum_{n=0}^{\infty} \alpha_n A^*nc$ and define exact observability analogously.

Remark. Actually our definition differs slightly from Helton's, but in the rest of this paper the difference is irrelevant. With this stronger definition of controllability, a state space isomorphism is at hand.

THEOREM 1.4 [11]. Two exactly controllable and (not necessarily exactly) observable systems realize the same impulse response function if and only if they are similar.

Of course we may assume the two systems to be controllable and exactly observable and get the same result. Similarity is meant in the sense of Theorem 1.1(b) with $R$ a boundedly invertible operator.

In Section 2 we will assemble the mathematical machinery that will be applied to the study of systems. In Section 3 we will study, in detail shift systems, conditions guaranteeing their exact controllability and exact observability, and we will characterize the class of transfer functions realizable by exactly controllable and exactly observable systems. In Section 4 we will use the Cayley transform to get some of the results in the semigroup setting.
Appendix 1 contains a counterexample, and Appendix 2 contains an observation on the stability of exactly controllable systems.

Much of the work presented here has been motivated and facilitated by Helton's manuscript [11] and in particular by D. N. Clark's remark on Hankel operators. In particular I would like to thank Roger W. Brockett for many stimulating discussions during the course of this work.

2. MATHEMATICAL PRELIMINARIES

In the sequel we will be using some relatively recent results in operator theory in Hilbert spaces. To make this paper more accessible to system theorists we summarize in this section all these results with full reference to the papers where the proofs may be found. Most of the following material can be found in the monographs by Fillmore [6], Helson [10], Hoffman [12], and Sz.-Nagy and Foias [19].

The Hilbert spaces we use are $l^2(0, \infty)$, $l^2(-\infty, \infty)$, $L^2(\mathbb{T})$, where $\mathbb{T}$ is the unit circle $\{\lambda : |\lambda| = 1\}$. The Hardy space $H^2$ is the subspace of $L^2$ defined by $H^2 = \{f \in L^2, \int f(e^{it}) e^{-nt} dt = 0, n = 1, 2, \ldots\}$. We note [12] that $H^2$ functions have analytic extensions to the open unit disc from which they can be recaptured a.e. (Fatou's theorem) as radial limits. This possibility of analytic extension gives us the extra structure in terms of which we can solve explicitly some problems of operator theory.

The Fourier transform $\mathcal{F}: l^2(-\infty, \infty) \to L^2(\mathbb{T})$ is defined by $\mathcal{F}(f) = \sum a_n e^{int} = a(e^i)$. $\mathcal{F}$ is a unitary map, and $\mathcal{F}l^2(0, \infty) = H^2$. The bilateral shift $U_1$ in $l^2(-\infty, \infty)$ is defined by $U_1(\{a_n\}) = \{\beta_n\}$, with $\beta_n = a_{n+1}$. $U_1$ is unitary, and $l^2(0, \infty)$ is invariant. Let $U = \mathcal{F}U_1\mathcal{F}^{-1}$; then $(Ua)(e^{it}) = e^{it}a(e^{it})$ for all $a \in L^2(\mathbb{T})$. Let $S = U | H^2$. $S$ will be called the right (unilateral shift). In terms of the analytic representation of $H^2$ we have $(Sf)(z) = zf(z)$, $(S^*f)(z) = (f(z) - f(0))/z$.

**DEFINITION 2.1.** A subspace of $H^2$ invariant under $S(S^*)$ will be called right (left) invariant.

We note that the orthogonal complement of a right invariant subspace is left invariant and v.v.

**DEFINITION 2.2.** $\phi \in H^\infty$ is called inner if it is nonconstant and $|\phi(e^{it})| = 1$ a.e.

The structure of inner function is well known and we refer to [12] for full details.

The importance of inner functions arises from the following fundamental theorem of Beurling.
Theorem 2.3 [2]. Every proper right invariant subspace of $H^2$ is of the form $qH^2$ with $q$ inner. Moreover, $q$ is determined uniquely up to a constant factor of modulus one.

Given a proper left invariant subspace $K$ of $H^2$ let $P_K$ be the orthogonal projection on $K$. We define the restricted shift operator by

$$T f = P_K S f \quad f \in K$$

and then

$$T^* = S^* | K.$$  \hspace{1cm} (2.2)

The importance of the restricted shift operators goes back to a theorem of Rota [7] on the universality of this class of operators. This has been refined by de Branges and Rovnyak, Sz.-Nagy and Foias, and Lax and Phillips [15] in the continuous case (semigroup setting). The universality of the shifts is gained, however, generally at the expense of having to deal with shifts of infinite multiplicity. To get more concrete results we will have to assume finite multiplicity. To avoid the technicalities of vector and operator valued functions in this paper we will restrict ourselves to the case of multiplicity one and treat the vector valued case in a subsequent publication.

Now the restricted shift operator is completely determined by the corresponding left invariant subspace $K$ and hence by the inner function $q$ for which $K = \{ qH^2 \}$. Thus we must be able to extract from $q$ all the relevant spectral information about $T$. $\sigma(T)$, $\sigma_p(T)$ denote the spectrum and point spectrum of $R$, respectively.

Theorem 2.4 [16, 10].

$$\sigma(T) = \{ \lambda \mid | \lambda | < 1, \quad q(\lambda) = 0 \},$$

$$\cup \{ \lambda \mid | \lambda | = 1, \quad q \text{ has no analytic continuation at } \lambda \}.$$  \hspace{1cm} (2.3)

Moreover, we have [15]

$$\sigma_p(T) = \{ \lambda \mid | \lambda | < 1, q(\lambda) = 0 \} \quad \text{and} \quad \sigma_p(T^*) = \{ \lambda \mid | \lambda | < 1, q(\lambda) = 0 \}.$$  \hspace{1cm} (2.4)

A special case of a more general functional calculus developed by Sz.-Nagy and Foias [19] is given by the following definition.

Definition 2.5. Given $\phi \in H^\infty$ and a restricted shift $T$ in a left invariant subspace $K$, we define $\phi(T)$ by

$$\phi(T) f = P_K (\phi f) \quad \text{for all } f \in K.$$  \hspace{1cm} (2.5)

The following is a generalized spectral mapping theorem for this functional calculus.
Theorem 2.6 [7]. (a) $0 \in \sigma_{\rho}(\phi(T))$ if and only if $\phi$, $q$ have a common nontrivial inner factor.

(b) $0 \in \rho(\phi(T))$ if and only if there exists a $\delta > 0$ such that for all $z$, 
$$|z| < 1 \Rightarrow |\phi(z)| + |q(z)| \geq \delta.$$ 

An important theorem in the sequel will be Sarason’s commutant theorem characterizing all bounded operators commuting with restricted shifts.

Theorem 2.7 [18]. Any bounded operator $\Phi$ in $K$ commuting with the restricted shift $T$ has the form $\Phi = \phi(T)$ for some $\phi \in H^\infty$ satisfying $\|\Phi\| = \|\phi\|_\infty$.

We have a natural conjugation in $L^2(\mathbb{T})$ given by the map $f \rightarrow \hat{f}$, where $\hat{f}(e^{it}) = \bar{f}(e^{-it})$. Any inner function $q$ gives rise to a unitary transformation $\tau$ of $L^2(\mathbb{T})$, where

$$\tau f(\cdot e^{it}) = \tau f(\cdot e^{-it})$$

we have $\tau(K) = \hat{K}$, where $K = \{qH^2\}$ and $\hat{K} = \{\hat{q}H^2\}$. Let $\hat{T}$ be the restricted right shift in $K$; then the following diagram is commutative [7].

This simple result allows us to get results about left restricted shifts from corresponding ones on right restricted shifts.

Let $1$ be the constant function 1 in $H^2$. We denote, following Clark [4],

$k_0 = P_{K^1}$ and $K_0 = P_{K} e^{-it} q$, where $K = \{qH^2\}$. Thus $k_0(z) = 1 - \bar{q}(0) q(z)$ and $K_0(z) = (q(z) - q(0))/z$. There is a close relation between $k_0$ and $\hat{k}_0$.

If $k_0$, $\hat{k}_0$ are defined similarly in $\{qH^2\}$ and $\tau$ is the transformation defined in (2.4), then $\tau k_0 = \hat{k}_0$ and $\tau K_0 = \hat{K}_0$.

The function $\phi_\lambda$, $\phi_\lambda(z) = 1/(1 - \lambda z) |\lambda| < 1$ is a reproducing kernel for $H^2$, i.e., $(f, \phi_\lambda) = f(\lambda)$ for all $f$ in $H^2$. Hence $P_K \phi_\lambda = k_\lambda$ is a reproducing kernel for $K = \{qH^2\}$, and it is quite easy to check [4] that

$$k_\lambda(z) = \frac{1 - \bar{q}(\lambda) q(z)}{1 - \lambda z}$$

and

$$1 - \lambda T)^{-1} k_0 = k_\lambda.$$  

In $L^2(-\infty, \infty)$ the Fourier-Plancherel transform $\mathcal{F}$ is defined by

$$(\mathcal{F} f)(\omega) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x) e^{ix\omega} \, dx.$$  

(2.7)
A priori $\mathcal{F}$ is defined only on $L^1 \cap L^2$, which is dense in $L^2$. On this set $\mathcal{F}$ is isometric and has range that is dense in $L^2$. Thus we can extend by continuity to get a unitary map of $L^2(-\infty, \infty)$ onto itself.

Let $H^2(\mathbb{H}^+)$ be the Hilbert space of analytic functions in $\mathbb{H}^+$ normed by $\|f\|_2 = \sup_{y>0} \int_{-\infty}^{\infty} |f(x + iy)| \, dx$. $H^2(\mathbb{H}^+)$ functions have nontangential limits on the real axis a.e., and the limit function is in $L^2(-\infty, \infty)$. We can identify $H^2(\mathbb{H}^+)$ with the subspace of $L^2(-\infty, \infty)$ of limit functions. By the Paley–Wiener theorem [12] we have $\mathcal{F}(L^2(0, \infty)) = H^2(\mathbb{H}^+)$. 

Let $S(t)$ and $S_1(t)$ be the two strongly continuous semigroups in $L^2(0, \infty)$ and $H^2(\mathbb{H}^+)$, respectively, defined by

$$
(S(t)f)(x) = f(x-t), \quad x > t,
$$

and

$$(S_1(t)f)(x) = e^{itf}(x) = 0, \quad x < t; \quad (2.8)
$$

then it is easily checked that the semigroups are unitarily equivalent. In fact, $S_1(t) = \mathcal{F} S(t) \mathcal{F}^{-1}$.

Thus a subspace of $L^2(0, \infty)$ invariant under the translation semigroup $S(t)$ is mapped by $\mathcal{F}$ onto a subspace of $H^2(\mathbb{H}^+)$ invariant under the multiplication semigroup $S_1(t)$ and thus [12] under multiplication by all elements of $H^\infty(\mathbb{H}^+)$, i.e., bounded analytic functions in $\mathbb{H}^+$. Those subspaces are characterized by the Beurling–Lax theorem.

**Theorem 2.8 [14].** A subspace $M$ of $H^2(\mathbb{H}^+)$ is invariant under multiplication by all $H^\infty(\mathbb{H}^+)$ functions if and only if it has the form $M = QH^2(\mathbb{H}^+)$ for some inner function $Q$ in $\mathbb{H}^+$.

Let $\{T(t) \mid t \geq 0\}$ be a strongly continuous contraction semigroup, $A$ its infinitesimal generator. We define the infinitesimal cogenerator of the semigroup to be the contraction operator $T$ defined by $T = (A + I)(A - I)^{-1}$.

To relate the discrete and continuous case we will use the following isomorphism of $L^2(-\infty, \infty)$ and $L^2(\mathbb{T})$. Let $J: L^2(-\infty, \infty) \rightarrow L^2(\mathbb{T})$ be defined by

$$
(Jf)(e^{it}) = 2\pi^{1/2}(1 + e^{it})^{-1} f \left(i \frac{1 - e^{it}}{1 + e^{it}} \right). \quad (2.9)
$$

$J$ is a unitary map, and moreover $J(H^2(\mathbb{H}^+)) = H^2$. The multiplication semigroup $S_1(t)$ has multiplication by $i\omega$ as the infinitesimal generator and thus multiplication by $(1 + i\omega)/(1 - i\omega)$ as the infinitesimal cogenerator. It follows that for all $g \in H^2(\mathbb{H}^+)$, $J((1 + i\omega)/(1 - i\omega) g)(z) = z(Jg)(z)$. Hence the cogenerator of the right translation semigroup in $L^2(0, \infty)$ is unitarily equivalent to the "multiplication by $z$" operator in $H^2$ and hence to the right shift in $L^2(0, \infty)$. 

In this section we study in more detail discrete restricted shift systems. These are important in the light of Theorem 1.2 about realization and because the corresponding realization has a spectrum that coincides with the set of singularities of the transfer function. For further discussion of this we refer to [I].

**Definition 3.1.** A (single input/single output) restricted shift system is a triple \( \{T, b, c\} \) with \( T \) the bounded operator defined by (2.1) in a proper left invariant subspace \( K \) of \( H^2 \) and \( b, c \in K \).

By Beurling's theorem \( K = \{ qH^2 \}^\perp \) for some inner function \( q \).

**Theorem 3.2.**

(a) \( \{T, b\} \) is controllable if and only if \( b, q \) have no common nontrivial inner factor.

(b) \( \{T, b\} \) is exactly controllable if and only if \( b = \phi(T) k_0 \) and \( \phi(T) \) boundedly invertible.

(b') \( \{T, b\} \) is exactly controllable if and only if \( b = \phi(T) k_0 \) and \( \exists \delta > 0 \) such that \( |\phi(z)| + |q(z)| \geq \delta \) for all \( z, |z| < 1 \).

**Proof.** (a) \( \{T, b\} \) is not controllable if and only if \( \{T^n b | n \geq 0\} \) do not span \( K \), i.e., if and only if for some \( g \neq 0 \) in \( K \) (\( g, T^n b = 0 = (g, z^n b) \), i.e., \( g \) is orthogonal to the right invariant subspace by \( b \) which is \( \phi H^2 \), with \( \phi \) the inner factor of \( b \). But \( g \in K \) implies \( g \perp qH^2 \); hence \( g \perp qH^2 \cup \phi H^2 \) the right invariant subspace spanned by \( qH^2 \) and \( \phi H^2 \). This is given by \( \psi H^2 \), where \( \psi \) is the greatest common inner divisor of \( q \) and \( \phi \) and hence of \( q \) and \( b \). Since \( g \neq 0 \) if and only if \( \psi \) is trivial, the proof is completed.

(b) Assume \( \{T, b\} \) is exactly controllable, i.e., \( \mathcal{C}: l^2(0, \infty) \to K \) given by \( \mathcal{C}(\{\alpha_n\}) = \sum \alpha_n T^n b \) is onto. Obviously \( b = \mathcal{C}(1, 0, ...) \).

Using the Fourier transform we may as well assume that \( \mathcal{C} \) is a map of \( H^2 \) onto \( K \). From the definition we have

\[ \mathcal{C}S = TS. \]

Let \( M = \ker \mathcal{C} \). Clearly \( M \) is \( S \) invariant, and hence by Beurling's theorem \( M = q_1 H^2 \) for some inner function \( q_1 \). Let \( \mathcal{C} = \mathcal{C} | M^\perp \). Obviously \( \mathcal{C} \) is \( 1-1 \) as we factored out the kernel and onto \( K \) by the assumption of exact controllability.

Let

\[ T_1 = P_{M^\perp} S | M^\perp. \]
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From \( CS = T \mathcal{C} \) it follows that

\[
\mathcal{C} P_{M^\perp} S | M^\perp = T \mathcal{C} | M^\perp
\]

as \( \mathcal{C} T_1 = T \mathcal{C} \). Since \( \mathcal{C} \) is invertible, \( T \) and \( T_1 \) are similar completely non unitary contractions; hence, \( q_1 = q, K = M^\perp \), and \( T = T_1 \). By Theorem 2.7 \( \mathcal{C} = \phi(T), \phi \in H^\infty \), and \( \phi(T) \) is invertible. By Theorem 2.6(b) \( \phi(T) \) is invertible if and only if \( |\phi(z)| + |q(z)| \geq \delta \). Therefore,

\[
b = \mathcal{C} \mathcal{P} M^\perp 1 = \phi(T) k_0 = P_k \phi \cdot 1.
\]

Conversely, assume \( b = \phi(T) k_0 \) and \( \phi(T) \) invertible. The controllability operator is given by

\[
\mathcal{C}(a_0, a_1, \ldots) = \Sigma a_i T^i b = \Sigma a_i T^i \phi(T) k_0 = \phi(T) \Sigma a_i T^i k_0 = \phi(T) P_k \Sigma a_i z^i.
\]

As \( \phi(T) \) is invertible, \( \mathcal{C} \) is onto if and only if \( P_k H^2 = K \), which is obvious.

**Corollary 3.3.** If \( b \in K \cap H^\infty \), \( \{T, b\} \) is exactly controllable if and only if \( |b(z)| + |q(z)| \geq \delta > 0 \).

**Proof.** Let \( b \in H^\infty \cap K \).

Assume \( \{T, b\} \) exactly controllable. Then \( b = P_k b, \beta \in H^\infty \), and

\[
|\beta(z)| + |q(z)| \geq \delta > 0.
\]

Let \( \beta = (\beta - q^n) + q^n \) be the decomposition of \( \beta \) corresponding to \( H^2 = K \oplus K^\perp = K \oplus q H^2 \). Then \( b = \beta - q^n \). Since \( \beta, b \) are in \( H^\infty \) and \( q \) is inner, we get \( \gamma \in H^\infty \).

If \( |b(z)| + |q(z)| \) is not bounded away from zero, then there exists a sequence \( \{\zeta_n\} \) such that \( |\zeta_n| < 1 \) and \( \lim b(\zeta_n) = \lim q(\zeta_n) = 0 \). But then also \( \beta(\zeta_n) \to 0 \), contrary to the assumption \( |q(z)| + |\zeta(z)| \geq \delta \).

Conversely, assume \( |b(z)| + |q(z)| \geq \delta \); then trivially, \( b = P_k b \).

To reduce the problem of observability to that of controllability we will use the transformation \( \tau \) defined by (2.4).

Let \( \mathcal{O}: l^2(0, \infty) \to K \) be the observability operator.

From the commutative diagram

\[
\begin{array}{ccc}
l^2(0, \infty) & \xrightarrow{\mathcal{O}} & K \\
\downarrow & & \downarrow \tau \\
T & \xrightarrow{\tau} & \tilde{T} \\
\downarrow & & \downarrow \tau \\
K & \xrightarrow{\tau} & K
\end{array}
\]

we get

\[
\tau \mathcal{O}(\{a_n\}) = \tau \Sigma a_n T^n c = \Sigma a_n \tau T^n c = \Sigma a_n \tilde{T}^n(\tau c).
\]
Hence, \( \{T, \cdot, c\} \) is exactly observable if and only if \( \{\bar{T}, \tau c, \cdot\} \) is exactly controllable, i.e., if and only if

\[
\tau c = P_k \psi, \quad \psi \in H^\infty, \quad \text{and} \quad |\psi(z)| + |\bar{q}(z)| \geq \delta \quad \text{for all } z, \quad |z| < 1.
\]

Thus

\[
c = \tau^* P_k \psi = \tau^* \psi(\bar{T}) \bar{k}_0 = \psi(T^*) \tau^* \bar{k}_0 = \psi(T^*) K_0.
\]

In conclusion we have

**Theorem 3.4.** \( \{T, \cdot, c\} \) is exactly observable if and only if \( c = \psi(T^*) K_0 \) with \( \psi \in H^\infty \) and \( |\psi(z)| + |\bar{q}(z)| \geq \delta \).

It is possible now to characterize the class of all bounded transfer functions of exactly controllable and exactly observable linear systems.

**Theorem 3.5.** A function \( F \in H^\infty \) is the transfer function of an exactly controllable and exactly observable linear system if and only if it has a representation \( F = \phi g \) with an inner function \( \phi, G \in H_0^\infty \) for which there exists a \( \delta > 0 \) such that for all \( z, \quad |\phi(z)| + |\bar{q}(z)| \geq \delta \),

\[
\left| G(z) \right| + |\bar{q}(z)| \geq \delta,
\]

where

\[
G(e^{it}) = e^{-it} g(e^{-it}).
\]

**Proof.** Let \( F \in H^\infty \) be the transfer function of an exactly controllable and exactly observable linear system \( \{A, b, c\} \). Consider now the shift realization of \( F, \{T^*, F, k_0\} \) introduced in Theorem 1.2. The state space of the shift realization is the left invariant subspace generated by \( F \). Since the shift realization is always exactly observable and it is controllable by construction, we can apply Theorem 1.4 to obtain the similarity of the two systems. Thus, the shift realization is also exactly controllable. So without loss of generality we may restrict ourselves to the case of an exactly controllable and exactly observable shift system.

Now for an exactly controllable and exactly observable linear system the transfer function is necessarily noncyclic in the terminology of [5], i.e., the left invariant subspace \( K \) of \( H^2 \) spanned by it is proper. This follows from the proof of Theorem 3.2, as \( S^* \mid K \) is similar either to \( S \) or to \( T \). In case \( K = H^2 \) this is impossible as the left and right shifts in \( H^2 \) have different fine structures of the spectrum, e.g., \( \sigma_n(S^*) = \{\lambda \mid |\lambda| < 1\} \), whereas \( \sigma_n(S) = \phi \).

On the other hand, the similarity of \( S^* \) to a restricted shift \( T \) is excluded by virtue of Theorem 2.4. Let \( \{T^*, F, k_0\} \) be an exactly controllable and exactly observable system in \( \{qH^2\}^{-1} \). By applying the unitary map \( \tau \), it follows that
{T, F, K₀} is an exactly controllable and exactly observable system in \( qH^∞ \). From [10, Lemma 1] it follows that \( F = qg \) for some \( g \in H₀^∞ \). Since \( F \in H^∞ \) by assumption, \( g \) is actually in \( H₀^∞ \). Now \( τF = G \), and by Corollary 3.3 there exists a \( δ > 0 \) such that for all \( z, |z| < 1 \) (3.1) is satisfied.

Conversely, assume that \( F \in H^∞ \) has such a representation \( F = qg \) satisfying (3.1) for some \( δ > 0 \). The shift realization is exactly observable, and the exact controllability follows from the above argument.

Theorem 3.5 has a formulation in terms of Hankel operators. We can consider a Hankel operator to be defined on \( l^2(0, \infty) \) by a matrix \((a_{i+j})_{i,j=0}^∞\), and we assume the \( a_n \) to be the Taylor coefficients of an \( H^∞ \) function \( φ \). Equivalently we may consider a Hankel operator corresponding to \( φ \in H^∞ \) to be defined on \( H^2 \) by

\[
H_φ f = P_{H^φ} \phi(Wf),
\]

where \( W: L^2(0, 2\pi) \to L^2(0, 2\pi) \) is defined by \((Wf)(e^{it}) = f(e^{-it})\). Range \( H_φ \) is the left invariant subspace \( K \) of \( H^2 \), which is spanned by the left translates of \( φ \). It is clear that Range \( H_φ \) is closed if and only if the system \{\( T^*, φ, P_κ 1 \)\} is exactly controllable. Since that system is always exactly observable and its transfer function is \( φ \), we get the following theorem observed by Clark [11].

**Theorem 3.6.** Let \( φ \in H^∞ \); then the Hankel operator \( H_φ \) has closed range if and only if \( φ \) has a representation \( φ = qg \) with \( q \) an inner function and \( g \in H₀^∞ \), for which there exists a \( δ > 0 \) such that for all \( z, |z| < 1 \) \( |G(z)| + |q(z)| \geq δ \), where \( G(e^{it}) = e^{-it}G(e^{it}) \).

### 4. Continuous Systems

Let \( A \) be the infinitesimal generator of a strongly continuous semigroup \( \{T(t) | t \geq 0\} \) in a Hilbert space \( H \).

We consider the control system in \( H \) whose dynamics are given by \( \dot{x} = Ax + bu, x(0) = 0 \), where \( b \in H \). For us, controllability means that the linear span of the vectors of the form \( T(t) b, t \geq 0 \) span \( H \). Let \( L^2(0, \infty) \) be the linear manifold in \( L^2(0, \infty) \) of all functions of compact support. We define the controllability operator \( \mathcal{C}: L_c^2(0, \infty) \to H \) by

\[
\mathcal{C} u = \int_0^∞ T(t) bu(t) \, dt \quad \text{for all } u \in L_c^2(0, \infty).
\]

**Definition 4.1.** We will say that \( \{A, b\} \) is exactly controllable if \( \mathcal{C} \) has an extension to a continuous map of \( L^2(0, \infty) \) onto \( H \).

We will be interested in restricted translation systems described as follows.
Let $K \subset L^2(0, \infty)$ be a left translation invariant subspace, i.e., satisfying $S(t)^* K \subset K$ for $S(t)$ defined by (2.8). Its orthogonal complement $K^\perp$ is invariant under the right translation semigroup $S(t)$. We define the restricted translation semigroup $\{T(t) \mid t \geq 0\}$ by

$$T(t)f = P_K S(t)f \quad \text{for all } f \in K.$$  \hspace{1cm} (4.1)

Let $\mathcal{A}$ be the infinitesimal generator of this semigroup, and let $b \in K$. $\{\mathcal{A}, b\}$ will be called the restricted translation system. For this semigroup let $T$ be the infinitesimal cogenerator. To the continuous system $\{\mathcal{A}, b\}$ we relate the discrete system $\{T, b\}$. If $\mathcal{F}$ is the Fourier transform (2.7), then by Theorem 2.8 $\mathcal{F}K = (QH^2(I^+))^\perp$. Using the results of [8] and Theorem 3.2(a) we have

**Theorem 4.2.** The system $\{\mathcal{A}, b\}$ is controllable if and only if $\mathcal{F}b$ and $Q$ have no common nontrivial inner factor.

As in the discrete case the condition for exact controllability is stronger.

**Theorem 4.3.** Let $b \in K$ be such that $\mathcal{F}b \in H^\infty \cap H^2$; then the system $\{\mathcal{A}, b\}$ is exactly controllable if and only if for some $\delta > 0$

$$|\mathcal{F}b(\omega)| + |Q(\omega)| \geq \delta \quad \text{for all } \omega \in I^+.$$  \hspace{1cm} (4.1)

**Proof.** Assume $\{\mathcal{A}, b\}$ is exactly controllable. For the right translation semigroup in $L^2(0, \infty)$ we have

$$\mathcal{C} S(t) u = \int_0^\infty T(\tau) u(\tau - t) d\tau = \int_0^\infty T(t + \tau) b(\tau) d\tau = T(t) \int_0^\infty T(\tau) b(\tau) d\tau = T(t) \mathcal{C} u.$$  \hspace{1cm} (4.1)

Thus we have

$$\mathcal{C} S(t) = T(t) \mathcal{C}.$$  \hspace{1cm} (4.2)

Hence $\ker \mathcal{C}$ is a right translation invariant subspace of $L^2(0, \infty)$, and by Theorem 2.8 its Fourier transform has the form $\mathcal{F}(\ker \mathcal{C}) = Q_1 H^2(I^+)$ for some inner function $Q_1$. The possibility that $\ker \mathcal{C} = \{0\}$ is excluded by spectral considerations analogous to those used in the proof of Theorem 3.5.

Let us still keep the notation $\mathcal{C}$ for the restriction of the controllability operator to $M = \{\ker \mathcal{C}\}^\perp$. So now $\mathcal{C}$ is an invertible map from $M$ to $K$. Let us introduce in $M$ the semigroup $\{T_1(t) \mid t \geq 0\}$ defined by

$$T_1(t) = P_M S(t) \mid M;$$  \hspace{1cm} (4.3)
then

\[ cT_1(t) = T(t)c, \]  

(i.e., the semigroups are similar. It follows necessarily that \( Q_1 = Q \) (up to a constant factor of absolute value 1) and the infinitesimal generators and cogenerators coincide. Thus we get

\[ cT(t) = T(t)c \quad \text{and} \quad Tc = cT, \]

(i.e., \( c \) is in the commutant of \( T \), the semigroup cogenerator. Hence if we map \( L^q(0, \infty) \) into \( H^q \) by \( J\mathcal{F} \), then \( c \) is represented, by virtue of Theorem 2.7, as a multiplication operator by an \( H^\infty \) function. We proceed to compute directly that function. From

\[ c u = \int_0^{\infty} T(t)bu(t) dt = \int_0^{\infty} P_kS(t)bu(t) dt \]

\[ = P_k \int_0^{\infty} S(t)bu(t) dt, \]

it follows by applying the Fourier transform that

\[ \mathcal{F}c u = \mathcal{F}P_k \int_0^{\infty} S(t)bu(t) dt \]

\[ = \mathcal{F}P_k \int_0^{\infty} \mathcal{F}s(t) \mathcal{F}^{-1} \mathcal{F}bu(t) dt \]

\[ = \mathcal{F}P_k \int_0^{\infty} e^{iut} \mathcal{F}b u(t) dt \]

\[ = \mathcal{F}P_k(\mathcal{F}b) \int_0^{\infty} e^{iut}u(t) dt. \]

The integral of course has to be interpreted in the sense of the Fourier-Plancherel theorem.

Hence,

\[ (\mathcal{F}cF^{-1})\mathcal{F}u = P_{\mathcal{F}K}(\mathcal{F}b) \cdot (\mathcal{F}u) \]

and \( \mathcal{F}b \in H^\infty \) by assumption. On applying the unitary map \( J \) to (4.6) we get

\[ (J\mathcal{F}cF^{-1}J^{-1})(J\mathcal{F}u) = JP_{\mathcal{F}K}(\mathcal{F}b)(\mathcal{F}u) \]

\[ = P_{J\mathcal{F}K}(J(\mathcal{F}b)(\mathcal{F}u)) = P_{J\mathcal{F}K}(J\mathcal{F}u), \]

where \( \beta \in H^\infty \) of the unit disk is defined by \( \beta(z) = (\mathcal{F}b)(i(1 - z)/(1 + z)) \).

Now \( J \) preserves the invariant subspaces, and hence \( J\mathcal{F}K = \{qH^\infty\}^\perp \) with
\( q(x) = Q(i(1 - x)/(1 + x)) \). Hence the invertibility of \( C \) is equivalent to that of \( JF_a F^{-1} J^{-1} \), which in turn is equivalent to the existence of a \( \delta > 0 \) such that

\[
|\beta(x)| + |q(x)| \geq \delta \quad \text{for all } x, \quad |x| < 1. \tag{4.7}
\]

This is clearly equivalent to (4.1). The arguments are clearly reversible.

With the same notation as before we get the following result.

**Theorem 4.4.** The continuous system \( \{A, b\} \) is exactly controllable if and only if the discrete system \( \{T, (I + T)b\} \) is exactly controllable.

**Proof.** The exact controllability of the system \( \{A, b\} \) is equivalent to the existence of a \( \delta > 0 \) such that (4.1) holds, which is equivalent in turn to (4.7). But (4.7) is equivalent to the exact controllability of \( \{JFT, J^{-1}J^{-1}, PJF_k\beta\} \). So we want to compute \( J^{-1}P_JF_k\beta \). Now

\[
J^{-1}P_JF_k\beta = J^{-1}P_JF^{-1}T_fK^\beta.
\]

But

\[
(J^{-1}\beta)(w) = (J^{-1}\beta)(w)(J^{-1}1)(w) = (J\beta)(w) J^{-1}\beta(1 - iw).
\]

So

\[
J^{-1}P_JF_k\beta = J^{-1}P_JF_k\beta J^{-1}\beta(1 - iw) b,
\]

which implies

\[
J^{-1}P_JF_k\beta = J^{-1}\beta(1 - T)^{-1} b.
\]

Now

\[
T = (I + A)(I - A)^{-1} = 2(I - A)^{-1} - I,
\]

so \((I - A)^{-1} = \frac{1}{2}(I + T)\), and hence the result.

**APPENDIX A: A COUNTEREXAMPLE**

We will exhibit two discrete systems, \( \{T, g, h\} \) and \( \{T_1, g_1, h_1\} \), that are both controllable and observable and that realize the same transfer function and such that the first system is internally stable \((\|T^n\| \leq M)\), whereas the second is not.

Let \( \alpha \in H^2 \) be a nonrational function that is not cyclic (in \( H^2 \)) for the left shift. Consider \( \alpha_\rho \) defined by \( \alpha_\rho(z) = a(\rho z) \) for \( 0 < \rho < 1 \). Obviously \( \alpha_\rho \in H^2 \), and \( \alpha \) is analytic in \( |z| < 1/\rho \). Since \( a_\rho \) is not a rational function, it follows from [5, Theorem 2.2.4] that \( a_\rho \) is cyclic for the left shift in \( H^2 \). Consider now the two shift realization of \( a \) and \( a_\rho \), respectively, \( \{T^*, a, k_0\} \), and \( \{S^*, a_\rho, k_0\} \).
in the spaces $K \subseteq H^2$ and $H^2$. $K$ is the smallest left invariant subspace of $H^2$ containing $a$, $S^*$ the left shift in $H^2$, and $T^* = S^* \mid K$. Now the spectrum of $S^*$ is the closed unit disk, whereas from Theorem 2.4 the spectrum of $T^*$ is much smaller. Since $\{S^*, a_\rho, 1\}$ realizes $a_\rho$, the system $\{1/\rho, S^*, a_\rho, 1\}$ realizes $a_\rho$.

Now $T^*$ is a contraction, moreover an asymptotically stable one ($T^{*n} \to 0$ strongly). $(1/\rho) S^*$ on the other hand has spectral radius $1/\rho > 1$ and hence is not stable.

This example clearly shows that additional assumptions, beyond controllability and observability, are essential for any generalization of the state space isomorphism theorem to the infinite dimensional context.

**APPENDIX B: Exact Controllability and Stability**

A control system $\{T, b, c\}$ is stable if there exists an $M > 0$ such that for all $n \geq 0$, $\|T^n\| \leq M$.

**Theorem A.2.1.** Let $\{T, b, c\}$ be an exactly controllable system in a Hilbert space $H$; then the system is stable.

**Proof.** Let $\mathcal{C}: L^2(0, \infty) \to H$ be the controllability operator. Let $M = [\ker \mathcal{C}]^\perp$. We will consider $\mathcal{C}$ as defined from $M$ to $H$. Hence by the assumption of exact controllability, $\mathcal{C}$ is a boundedly invertible operator from $M$ to $H$. The same is true for $\mathcal{C}^*: H \to M$. It follows that there exists a $\rho > 0$ such that for all $x \in H \|\mathcal{C}^* x\| \geq \rho \|x\|$. Hence

$$\|\mathcal{C}^* T^n x\| \geq \rho \|T^n x\|.$$ 

It is simple to check that

$$\mathcal{C}^* x = \{(x, T^n b)\}_{n=0}^{\infty}.$$ 

So

$$\|\mathcal{C}^* x\|^2 = \sum_{m=0}^{\infty} \|(x, T^m b)\|^2.$$ 

Now

$$\|\mathcal{C}^* T^n x\|^2 = \sum_{m=0}^{\infty} \|(T^n x, T^m b)\|^2 = \sum_{m=n}^{\infty} \|(x, T^m b)\|^2 \leq \sum_{m=n}^{\infty} \|(x, T^m b)\|^2 = \|\mathcal{C}^* x\|^2.$$
Combining these results, we get for all $n \geq 0$.

$$\rho \| T^n x \| \leq \| C T^n \| \leq \| C^* x \| \leq \| C^* \| \| x \|.$$  

Stability follows by an application of the principle of uniform boundedness.

**Remarks.** (a) Exactly the same result holds in the case of exact observability.

(b) A system $(T, b)$ that is exactly controllable need not be asymptotically stable ($T^n x \to 0$ for all $x$).

**References**