JOURNAL OF DIFFERENTIAL EQUATIONS 10, 240–261 (1971)

Periodic Solutions of Nonlinear Functional Differential Equations

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Received June 11, 1970

1. INTRODUCTION

This paper is devoted to an existence study of periodic solutions in nonlinear functional differential equations [1–3]. Although some arguments of the present work can be applied to autonomous functional differential equations, our attention will be focused to the nonautonomous case. The method used here will be an extension of that proposed in [4] by the author for ordinary differential equations and reducing the problem to the existence of a fixed point for some operator in a suitable space of periodic functions. The corresponding operator for functional differential equations, which seems to be new, is introduced in Section 2 and a necessary and sufficient condition for the existence of periodic solutions is proved (Theorem 1). As an application of this result and of some properties of Leray–Schauder's degree [5–7], we obtain in Section 3 a sufficient condition for the existence of periodic solutions (Theorem 2) and then in Section 4 an extension to functional differential equations of a result first proved by G. Güssefeldt [8, 9] (see also [4]) for ordinary differential equations (Theorem 3). In Section 5 is proved Theorem 4 which covers cases where the previous result is not applicable and which generalizes a recent criterion of the author for equations without time lags [4, 10]. The remainder of the paper is devoted to applications of Theorem 4. First, in Section 6, Theorem 5 is an extension to functional differential equations of the basic theorem of the "method of guiding functions" introduced by M. A. Krasnosel'skii et al. [11–17] to study periodic solutions of ordinary and differential-difference equations. It is easy to verify that the new proof given here for the more general case of functional differential equations is much simpler and shorter than the original one given in [16] by Krasnosel'skii for ordinary differential equations and which was based upon the properties of the Poincaré-Andronov operator

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of translation along the solutions. Moreover, for differential-difference equations, Krasnosel'skii has still to use some homotopy argument to reduce these equations to ordinary differential equations [13–15], and it is not at all clear how to apply this process to general functional differential equations. As an application of Theorem 5 we extend a result of J. Cronin recently published in this Journal [18]. The last section is devoted to a direct application of Theorem 4 to systems of coupled Liénard functional differential equations (Theorem 6), the corresponding result in the case of ordinary Liénard equations having been given by the author in [19].

For other interesting papers concerning the periodic solutions of non-autonomous functional differential equations but not directly related to the method and the results introduced here, see [1, 3, 20–28].

2. A Necessary and Sufficient Condition for the Existence of Periodic Solutions of Functional Differential Equations

Let $\mathscr{C}$ denote the (Banach) space of mappings

$$\varphi : [-r, 0] \to R^n, \qquad \theta \mapsto \varphi(\theta)$$

($r \geqslant 0$ a fixed number) with the uniform norm

$$\| \varphi \|_{\mathscr{C}} = \sup_{\theta \in [-r,0]} \| \varphi(\theta) \|,$$

$\| \cdot \|$ being an arbitrary norm in R^n. If the mapping

$$x : R \to R^n, \qquad t \mapsto x(t)$$

is continuous, then, for every $t \in R$, we shall denote by x_t the element of $\mathscr{C}$ defined by

$$x_t(\theta) = x(t + \theta), \qquad \theta \in [-r, 0].$$

$\Omega \subset \mathscr{C}$ being an open set, let

$$q : R \times \Omega \to R^n, \qquad (t, \varphi) \mapsto q(t, \varphi) \tag{2.1}$$

be a continuous mapping such that

$$q(t + T, \varphi) = q(t, \varphi)$$

for every $(t, \varphi) \in R \times \Omega$, T (the period) a strictly positive fixed number.

Let us consider the functional differential equation

$$x'(t) = q(t, x_t), \tag{2.2}$$

where $x'(t)$ is the right-hand derivative with respect to t. It is well known that the class of Eqs. (2.2) contains ordinary differential equations (take $r = 0$), differential-difference equations and some integro-differential equations. A *T-periodic solution* of (2.2) (cf., for example, [3]) will be a solution $x(t)$ such that

$$x(t + T) = x(t) \tag{2.3}$$

for every $t \in R$.

We shall now briefly recall the definition and properties of the operators P and H introduced by Cesari [29, 30] in his studies of periodic solutions of nonlinear equations and used in [4]. Let C_T be the (Banach) space of mappings

$$x : R \to R^n, \qquad t \mapsto x(t)$$

continuous and T-periodic (i.e., satisfying (2.3)) with the norm

$$\| x \|_0 = \sup_{t \in R} \| x(t) \| = \sup_{t \in [0,T]} \| x(t) \|,$$

and let P be the operator

$$P : C_T \to R^n, \qquad x \mapsto \frac{1}{T} \int_0^T x(t)\, dt$$

(due to the isomorphism between R^n and the subspace of C_T of constant mappings, we denote in the following this last subspace also by R^n, the norm induced by $\| \cdot \|_0$ being $\| \cdot \|$). It is clear that

$$\| Px \|_0 = \| Px \| \leqslant \| x \|_0 .$$

For every $x \in C_T$, it is well known [10, 29, 30] that there exists an (unique) operator of primitivation H such that

$$H(I - P)\, x \in C_T$$

is of class C^1 (I the identity), and satisfies

$$PH(I - P)x = 0, \qquad \frac{d}{dt}\, [H(I - P)\, x(t)] = x(t) - Px$$

for every $t \in R$. Moreover [10, 29, 30],

$$\| H(I - P)x \|_0 \leqslant 3^{-1/2}(T/2)\, \| x \|_0 .$$

If $\omega \subset C_T$ is the set

$$\omega = \{x \in C_T : x_t \in \Omega \text{ for every } t \in R\},$$

and if A is an arbitrary $n \times n$ constant nonsingular matrix, then the operators

$$\tilde{q} : \omega \to C_T, \quad x(t) \mapsto q(t, x_t)$$

for every $t \in R$, and

$$F : \omega \to C_T, \quad x \mapsto Px + AP\tilde{q}x + H(I - P)\,\tilde{q}x \tag{2.4}$$

are well defined and, for every $x \in \omega$, $(Fx)(t)$ is of class C^1.

We can now prove the following

THEOREM 1. *The functional differential equation* (2.2) *will have a T-periodic solution if and only if there exists an $x \in \omega$ such that*

$$x = Fx. \tag{2.5}$$

Proof. If x is a fixed point of F, $x(t)$ is T-periodic and of class C^1, then differentiating both members of (2.5), we obtain

$$x'(t) = q(t, x_t) - P\tilde{q}x, \qquad t \in R.$$

If we now apply P to (2.5), we find

$$Px = Px + AP\tilde{q}x,$$

i.e.,

$$P\tilde{q}x = 0$$

and the condition is sufficient. Conversely, if $x(t)$ is a T-periodic solution of (2.2), then, applying P to both members of (2.2), we obtain

$$P\tilde{q}x = Px' = 0,$$

and hence

$$x - Px = H\tilde{q}x = H(I - P)\,\tilde{q}x = AP\tilde{q}x + H(I - P)\,\tilde{q}x,$$

i.e.,

$$x = Fx.$$

3. LERAY–SCHAUDER'S DEGREE AND THE EXISTENCE OF PERIODIC SOLUTIONS OF (2.2)

Using Theorem 1, the existence of a T-periodic solution for (2.2) is reduced to the study of the fixed points of the operator F in some set of the (Banach) space C_T and it is known that one of the most powerful tools in this domain is the theory of Leray–Schauder's degree for completely continuous perturbations of the identity operator. We have, therefore, to find conditions ensuring the complete continuity of the operator F. If E_1 and E_2 are Banach spaces, the (not necessarily linear) operator

$$W : S \subset E_1 \to E_2$$

will be said to satisfy the *B-property* if it transforms every bounded set contained in S into a bounded set of E_2. If W is linear, this is the usual boundedness property.

LEMMA 1. *If the mapping q defined in* (2.1) *is continuous and satisfies the B-property, then F is completely continuous in* ω.

Proof. The continuity of F is an immediate consequence of the continuity of P, H and q. Let us now show that F is compact. Let $\{x^k\}$ be a bounded sequence of elements of ω. There will exist a positive constant M such that

$$\| x^k \|_0 \leqslant M, \qquad k = 1, 2,...$$

and, if

$$y^k = Fx^k, \qquad k = 1, 2,...,$$

then, using B-property,

$$\begin{aligned} \| y^k \|_0 &\leqslant \| Px^k \|_0 + \| A \| \, \| P\tilde{q}x^k \|_0 + \| H(I - P)\, \tilde{q}x^k \|_0 \\ &\leqslant \| x^k \|_0 + \| A \| \, \| \tilde{q}x^k \|_0 + 3^{-1/2}(T/2) \, \| \tilde{q}x^k \|_0 \\ &\leqslant M + \| A \| \, N + 3^{-1/2}(T/2)N, \end{aligned}$$

where N is the positive constant such that

$$\| q(t, x_t) \|_0 \leqslant N$$

if $t \in R$, $x_t \in \Omega$ and

$$\| x_t \|_{\mathscr{C}} \leqslant \| x \|_0 \leqslant M.$$

Moreover, for every $k = 1, 2,..., y^k(t)$ is of class C^1 and

$$\| (y^k)' \|_0 = \| (I - P)\, \tilde{q}x^k \|_0 \leqslant 2 \, \| \tilde{q}x^k \|_0 \leqslant 2N, \qquad k = 1, 2,... .$$

Then $\{y^k\}$ is an equibounded and equicontinuous sequence of elements of C_T and the compactness of F follows immediately from Arzela–Ascoli's theorem.

It can be noted that if (2.2) is an ordinary differential equation

$$x'(t) = q[t, x(t)]$$

or a differential-difference equation

$$x'(t) = q[t, x(t), x(t - \tau_1), x(t - \tau_2), \dots, x(t - \tau_k)],$$

$\tau_j > 0, j = 1, 2, \dots, k$, the B-property for q is an immediate consequence of its continuity. The same is true if the constants τ_j are replaced by T-periodic continuous lags $\tau_j(t), j = 1, 2, \dots, k$.

Using Theorem 1, Lemma 1 and the basic properties of Leray–Schauder's degree, we obtain immediately the following

THEOREM 2. *If conditions of Lemma* 1 *are satisfied and if there exists an open bounded set* ω_0 *such that*

(i) $\bar{\omega}_0 \subset \omega$ ($\bar{\omega}_0$ *is the closure of* ω_0);

(ii) $x \neq Fx$ *for every* $x \in \partial\omega_0$ ($\partial\omega_0$ *is the boundary of* ω_0);

(iii) *the Leray–Schauder's degree* $d(x - Fx, \omega_0, 0)$ *is not zero, then Eq.* (2.2) *has at least one T-periodic solution contained in* ω_0.

To apply Theorem 2 it is often useful to introduce a family of functional differential equations depending continuously on a parameter, say

$$x'(t) = f(t, x_t, \lambda), \qquad \lambda \in [0, 1], \tag{3.1}$$

such that (3.1) reduces to (2.2) if $\lambda = 1$ and such that the Leray–Schauder's degree of Theorem 2 is easier to compute for the operator F corresponding to (3.1) with $\lambda = 0$. To apply the invariance property of Leray–Schauder's degree with respect to homotopy, we need the following

LEMMA 2. *With the notations of Section* 2, *if the mapping*

$$f : R \times \Omega \times [0, 1] \to R^n, \quad (t, \varphi, \lambda) \mapsto f(t, \varphi, \lambda)$$

is continuous, T-periodic with respect to t and satisfies the B-property, then the operator

$$\mathscr{F} : \omega \times [0, 1] \to C_T, \quad (x, \lambda) \mapsto Px + AP\tilde{f}(x, \lambda) + H(I - P)\tilde{f}(x, \lambda) \tag{3.2}$$

with

$$\tilde{f} : \omega \times [0, 1] \to C_T, \quad [x(t), \lambda] \mapsto f(t, x_t, \lambda), \quad t \in R,$$

is completely continuous.

Proof. It follows exactly the lines of the proof of Lemma 1. Instead of $\{x^k\}$, one has to use a bounded sequence $\{(x^k, \lambda^k)\}$ of elements of $\omega \times [0, 1]$.

4. An Extension of a Theorem of Güssefeldt

As a first application of Theorem 2 let us consider the functional differential equation

$$x'(t) = s(t, x_t) + p(t, x_t), \tag{4.1}$$

where the mappings

$$s : R \times \Sigma \to R^n, \quad (t, \varphi) \mapsto s(t, \varphi),$$
$$p : R \times \Sigma \to R^n, \quad (t, \varphi) \mapsto p(t, \varphi),$$

are continuous, T-periodic with respect to t, and satisfy the B-property with Σ an open set of $\mathscr{C}$ such that $-\varphi \in \Sigma$ if $\varphi \in \Sigma$. Moreover, let us suppose that

$$s(t, -\varphi) = -s(t, \varphi)$$

for every $(t, \varphi) \in R \times \Sigma$ and that there exists a mapping

$$g : R \times \Sigma \times [0, 1] \to R^n, \quad (t, \varphi, \lambda) \mapsto g(t, \varphi, \lambda)$$

continuous, T-periodic with respect to t, and satisfying the B-property and the relations

$$g(t, \varphi, 0) = 0, \qquad g(t, \varphi, 1) = p(t, \varphi)$$

for every $(t, \varphi) \in R \times \Sigma$.

If $\sigma = \{x \in C_T : x_t \in \Sigma \text{ for every } t \in R\}$, we can prove the following

Theorem 3. *Under the conditions quoted above, if there exists an open bounded set σ_0 such that $\bar{\sigma}_0 \subset \sigma$, $-x \in \bar{\sigma}_0$ if $x \in \bar{\sigma}_0$ and such that for every possible T-periodic solution of*

$$x'(t) = s(t, x_t) + g(t, x_t, \lambda), \qquad \lambda \in [0, 1],$$

we have

$$x \notin \partial\sigma_0, \tag{4.2}$$

then (4.1) *has at least one T-periodic solution situated in* σ_0.

Proof. Let us introduce the mappings

$$q : R \times \Sigma \to R^n, \quad (t, \varphi) \mapsto q(t, \varphi) = s(t, \varphi) + p(t, \varphi),$$
$$f : R \times \Sigma \times [0, 1] \to R^n, \quad (t, \varphi, \lambda) \mapsto f(t, \varphi, \lambda) = s(t, \varphi) + g(t, \varphi, \lambda).$$

If the corresponding operators F and $\mathscr{F}$ are, respectively, defined in (2.4) and (3.2) with σ instead of ω, and if we write

$$F_0 : \sigma \to C_T\,, \quad x \mapsto \mathscr{F}(x, 0) = Px + AP\tilde{s}x + H(I - P)\,\tilde{s}x$$

with

$$\tilde{s} : \sigma \to C_T\,, \quad x(t) \mapsto s(t, x_t), \quad t \in R,$$

and

$$F_1 : \sigma \to C_T\,, \quad x \mapsto \mathscr{F}(x, 1),$$

then it is clear that

$$F_0(-x) = -F_0 x, \qquad F_1 x = Fx \tag{4.3}$$

for every $x \in \sigma$. By Lemma 2, $\mathscr{F}$ is completely continuous on $\bar{\sigma}_0 \times [0, 1]$, and, using condition (4.2) and Theorem 1, we have

$$x \neq \mathscr{F}(x, \lambda)$$

for every $x \in \partial\sigma_0$ and $\lambda \in [0, 1]$. Therefore, by the property of invariance under homotopy of Leray–Schauder's degree,

$$d(x - Fx, \sigma_0\,, 0) = d(x - F_1 x, \sigma_0\,, 0) = d(x - F_0 x, \sigma_0\,, 0). \tag{4.4}$$

But, using (4.3), we obtain from a theorem of Krasnosel'skii [31] that $d(x - F_0 x, \sigma_0\,, 0)$ is an odd number. Theorem 3 is then an immediate consequence of (4.4) and of Theorem 2.

In the case of ordinary differential equations, Theorem 3 was first proved by G. Güssefeldt [8, 9] by writing (4.1) in the form

$$x'(t) = A(t)\,x(t) + s^{(1)}[t, x(t)] + p[t, x(t)], \tag{4.5}$$

with $A(t)$ a noncritical matrix [30], and using then the Green function of

$$x'(t) = A(t)\,x(t), \qquad x(0) = x(T)$$

to reduce the problem of periodic solutions of (4.5) into an integral equation of Hammerstein type. A direct proof analogous to that used here was given in [4] by the author.

5. A Sufficient Condition for the Existence of T-Periodic Solutions of (2.2)

It is clear from its proof that, contrarily to Theorem 2, Theorem 3 can only cover cases where the Leray–Schauder's degree of $x - Fx$ is an odd number but, of course, it is more suitable for applications. In this section we shall prove an existence theorem in which the Leray–Schauder's degree need not to be odd and which is also very suitable for applications as shown in Sections 5 and 6. This theorem is an extension to functional differential equations of a result first proved in [4, 10] by the author for ordinary differential equations.

To prove the existence of T-periodic solutions for (2.2), let us introduce the auxiliary equation

$$x'(t) = \lambda g(t, x_t, \lambda), \qquad \lambda \in [0, 1],$$

where the mapping

$$g : R \times \Omega \times [0, 1] \to R^n, \quad (t, \varphi, \lambda) \mapsto g(t, \varphi, \lambda)$$

is continuous, T-periodic with respect to t, satisfies the B-property and is such that

$$g(t, \varphi, 1) = q(t, \varphi)$$

for every $(t, \varphi) \in R \times \Omega$. Let $\mathscr{G}$ be the operator

$$\mathscr{G} : \omega \times [0, 1] \to C_T, \quad (x, \lambda) \mapsto \mathscr{G}(x, \lambda) = Px - P\tilde{g}(x, \lambda) + \lambda H(I - P)\tilde{g}(x, \lambda)$$

with

$$\tilde{g} : \omega \times [0, 1] \to C_T, \quad [x(t), \lambda] \mapsto g(t, x_t, \lambda)$$

for every $t \in R$.

We can now prove the following

THEOREM 4. *If the following conditions are satisfied*:

(i) *There exists an open ball* b_ρ *of center* 0 *and of radius* $\rho > 0$ *such that* $\bar{b}_\rho \subset \omega$ *and such that, for every possible* T*-periodic solution* x *of the functional differential equations*

$$x'(t) = \lambda g(t, x_t, \lambda), \qquad \lambda \in]0, 1], \tag{5.1}$$

we have

$$x \notin \partial b_\rho\,;$$

(ii) *Every possible solution* $a \in R^n$ *of the equation*

$$g_0(a) = \frac{1}{T} \int_0^T g(t, a, 0)\, dt = 0$$

is such that

$$a \notin \partial B$$

with

$$B_\rho = b_\rho \cap R^n = \{a \in R^n : \| a \| < \rho\};$$

(iii) *The topological (Brouwer) degree* $d[g_0(a), B_\rho, 0]$ *is not zero,*

then (2.2) *has at least one T-periodic solution situated in* b_ρ.

Remark. It is to be noted that, in $g(t, a, 0)$, a has to be considered as an element of the (isomorphic to R^n) subspace of $\mathscr{C}$ of constant mappings.

Proof of Theorem 4. From Theorem 1 it is clear that, for every $\lambda \in]0, 1]$, the fixed points of

$$x = \mathscr{G}(x, \lambda)$$

coïncide with the T-periodic solutions of (5.1), and hence, using condition (i), we have

$$x \neq \mathscr{G}(x,\lambda) \tag{5.2}$$

for every $x \in \partial b_\rho$ and $\lambda \in]0, 1]$. Moreover, if we write, for every $\lambda \in [0, 1]$,

$$G_\lambda : \omega \to C_T, \quad x \mapsto \mathscr{G}(x, \lambda),$$

then every fixed point of G_0 satisfies

$$x = Px - P\tilde{g}(x, 0), \tag{5.3}$$

and hence, applying P to both members of (5.3),

$$x = Px, \qquad P\tilde{g}(Px, 0) = 0.$$

As a consequence, the set of possible fixed points of G_0 is contained in R^n and coïncides with the set of zeroes of g_0. Therefore, using condition (ii),

$$x \neq G_0 x = \mathscr{G}(x, 0)$$

for every $x \in \partial b_\rho$ and then (5.2) holds for every $\lambda \in [0, 1]$. By Lemma 2, $\mathscr{G}$ is a completely continuous mapping and, using the invariance under homotopy and excision properties of Leray–Schauder's degree we have

$$\begin{aligned} d(x - Fx, b_\rho, 0) &= d(x - G_1 x, b_\rho, 0) = d(x - G_0 x, b_\rho, 0) \\ &= d(x - G_0 x \mid R^n, b_\rho \cap R^n, 0) = d(g_0(a), B_\rho, 0), \end{aligned}$$

because, on $b_\rho \cap R^n$,

$$x - G_0 x = a - a + P\tilde{g}(a, 0) = g_0(a).$$

Theorem 4 follows then from Theorem 2 and condition (iii).

6. An Extension of the Basic Theorem of the Method of Guiding Functions

Let us consider the functional differential equation

$$x'(t) = f[t, x(t), x_t], \tag{6.1}$$

where

$$f : R \times R^n \times \mathscr{C} \to R^n, \quad (t, x, \varphi) \mapsto f(t, x, \varphi)$$

is continuous, T-periodic with respect to t and satisfies the B-property. It is clear that (6.1) is only a particular (and convenient for what follows) manner to write (2.2) with $\Omega = \mathscr{C}$ because, if we introduce the (bounded) linear operator

$$L : \mathscr{C} \to R^n \times \mathscr{C}, \quad \varphi \to [\varphi(0), \varphi],$$

then we have

$$f[t, x(t), x_t] = f(t, Lx_t) \stackrel{\text{def}}{=} q(t, x_t),$$

and the mapping q satisfies the conditions given in Section 2 and the B-property. In this section and in Section 7, $\|\cdot\|$ will always denote the Euclidian norm and $\langle\ ,\ \rangle$ the scalar product in R^n.

As an easy consequence of Theorem 4 we shall prove first a generalization of the basic theorem of the "method of guiding functions" introduced and developed by Krasnosel'skii et al. [11–17]. We need then the following

Definition. A function

$$V : R^n \to R, \quad x \mapsto V(x)$$

will be a guiding function for Equation (6.1) it there exists a $\rho > 0$ such that

$$\langle \operatorname{grad} V(x), f(t, x, \varphi)\rangle > 0 \tag{6.2}$$

for every $t \in R$, $\varphi \in \mathscr{C}$ and $x \in R^n$ satisfying $\|x\| \geq \rho$.

When we shall have to consider some guiding function V_j with a subscript j we shall write ρ_j for the corresponding value of ρ. We can now prove the following

Theorem 5. *If there exist* $m + 1 (\geq 1)$ *guiding functions* $V_0, V_1, \ldots, V_m$ *for* (6.1) *such that*

$$\lim_{\|x\| \to \infty} [|V_0(x)| + |V_1(x)| + \cdots + |V_m(x)|] = \infty \tag{6.3}$$

and

$$d[\text{grad } V_0(x), B_{\rho_0}, 0] \neq 0$$

with

$$B_{\rho_0} = \{x \in R^n : \|x\| < \rho_0\},$$

then (6.1) *has at least one T-periodic solution.*

Proof. Let us introduce the auxiliary equation

$$x'(t) = \lambda f[t, x(t), x_t], \qquad \lambda \in]0, 1], \tag{6.4}$$

and let $x(t)$ be a possible T-periodic solution of (6.4). If we write

$$\mathscr{V}_j(t) = V_j[x(t)], \qquad j = 0, 1,..., m,$$

then

$$\mathscr{V}_j'(t) = \langle \text{grad } V_j[x(t)], x'(t)\rangle = \lambda\langle \text{grad } V_j[x(t)], f[t, x(t), x_t]\rangle$$
$$j = 0, 1,..., m; \quad t \in R.$$

For every τ_j, such that

$$|\mathscr{V}_j(\tau_j)| = \sup_{t\in R} |\mathscr{V}_j(t)| = \sup_{t\in[0,T]} |\mathscr{V}_j(t)|, \qquad j = 0, 1,..., m,$$

we have

$$\mathscr{V}_j'(\tau_j) = 0, \qquad j = 0, 1,..., m,$$

and then, using (6.2),

$$\|x(\tau_j)\| < \rho_j, \qquad j = 0, 1,..., m.$$

Hence, for every $t \in R$,

$$|V_j[x(t)]| \leqslant \sup_{\|x\|\leqslant\rho_j} |V_j(x)| = M_j, \qquad j = 0, 1,..., m,$$

for every possible T-periodic solution of (6.4). Therefore,

$$\sum_{j=0}^{m} |V_j[x(t)]| \leqslant \sum_{j=0}^{m} M_j = M, \qquad t \in R,$$

and, by condition (6.3), there will exist one $\rho_M > 0$ such that

$$\sup_{t\in[0,T]} \|x(t)\| < \rho_M$$

for every possible T-periodic solution of (6.4).

Using (6.2), we find now

$$\left\langle \operatorname{grad} V_0(a), \frac{1}{T} \int_0^T f(t, a, a)\, dt \right\rangle < 0$$

for every $a \in R^n$ such that $\| a \| \geqslant \rho_0$, and hence every possible solution of

$$g_0(a) = \frac{1}{T} \int_0^T f(t, a, a)\, dt = 0$$

is contained in B_{ρ_0}, and, by Poincaré–Bohl theorem [6],

$$d[g_0(a), B_\rho, 0] = d[\operatorname{grad} V_0(a), B_\rho, 0] \neq 0$$

for every $\rho \geqslant \rho_0$. With $\rho = \max(\rho_0, \rho_M)$, the conditions of Theorem 4 are satisfied and Theorem 5 is proved.

A direct application of this result is the following extension of a theorem proved for ordinary differential equations by Cronin [18] using the Brouwer fixed point theorem:

COROLLARY 1. *Let us consider the functional differential equation*

$$x'(t) = h[x(t)] + g[t, x(t), x_t], \tag{6.5}$$

where

$$h : R^n \to R^n, \quad x \mapsto h(x),$$
$$g : R \times R^n \times \mathscr{C} \to R^n, \quad (t, x, \varphi) \mapsto g(t, x, \varphi)$$

are continuous, h being homogeneous of degree $s > 1$ and g, T-periodic with respect to t, satisfying the B-property and being such that

$$\frac{\| g(t, x, \varphi) \|}{\| x \|^s} \to 0 \qquad \text{if} \quad \| x \| \to \infty \tag{6.6}$$

uniformly in (t, φ). If 0 is an asymptotically stable critical point of

$$x'(t) = h[x(t)], \tag{6.7}$$

then (6.5) has at least one T-periodic solution.

Proof. 0 being an asymptotically stable critical point of (6.7), there will exist, by virtue of a theorem due to Zubov [32, 33], two functions V and W:

$$V : R^n \to R^+ \cup \{0\}, \quad x \mapsto V(x),$$
$$W : R^n \to R^+ \cup \{0\}, \quad x \mapsto W(x),$$

that are of class C^1, homogeneous of respective degrees 2 and $s + 1$, and such that

$$V(x) = W(x) = 0 \qquad \text{if and only if} \quad x = 0$$

and

$$\langle \text{grad } V(x), p(x) \rangle = -W(x).$$

Therefore, we can find two strictly positive constants v and w such that

$$\| \text{grad } V(x) \| \leqslant v \| x \|, \qquad W(x) \geqslant w \| x \|^{s+1}$$

for every $x \in R^n$. If we write

$$V_0(x) = -V(x),$$

then

$$\begin{aligned} \langle \text{grad } V_0(x), h(x) + g(t, x, \varphi) \rangle &= W(x) - \langle \text{grad } V(x), g(t, x, \varphi) \rangle \\ &\geqslant w \| x \|^{s+1} - \| \text{grad } V(x) \| \, \| g(t, x, \varphi) \| \\ &\geqslant w \| x \|^{s+1} - v \| x \| \, \| g(t, x, \varphi) \|. \end{aligned}$$

But, for every $\epsilon > 0$ it follows from condition (6.6) that we can find one $\rho(\epsilon) > 0$ such that

$$\| g(t, x, \varphi) \| \leqslant \epsilon \| x \|^s,$$

if $t \in R$, $\varphi \in \mathscr{C}$, and $\| x \| \geqslant \rho(\epsilon)$, and hence

$$\langle \text{grad } V_0(x), h(x) + g(t, x, \varphi) \rangle \geqslant w \| x \|^{s+1} - \epsilon v \| x \|^{s+1} \geqslant \frac{w}{2} (\rho_0)^{s+1}$$

for every $t \in R$, $\varphi \in \mathscr{C}$ and $x \in R^n$ such that $\| x \| \geqslant \rho_0 = \rho(w/2v)$.

Now, the negative definite quadratic form $V_0(x)$ can be written

$$V_0(x) = \tfrac{1}{2} \langle x, \mathscr{V}_0 x \rangle$$

with $\mathscr{V}_0$ an $n \times n$ negative symmetric matrix such that

$$\text{grad } V_0(x) = \mathscr{V}_0 x$$

and hence [10]

$$d[\text{grad } V_0(x), B_\rho, 0] = d[\mathscr{V}_0 x, B_\rho, 0] = (-1)^n$$

for every $\rho > 0$. Then, $V_0(x)$ satisfies the conditions of Theorem 5 and Corollary 1 is proved.

7. Periodic Solutions of a System of Liénard Functional Differential Equations

The applications of Theorem 4 given up to now are all of theoretical nature. In this section we shall make a direct use of this theorem to prove the existence of T-periodic solutions for a class of systems of coupled Liénard functional differential equations. The criterion thus obtained is an extension of results, proved by the author in [19], for a system of ordinary Liénard differential equations. It is to be noted that a number of other existence theorems given in [10, 34, 35] for ordinary differential equations can also be extended in some way to corresponding functional differential equations.

Theorem 6. *Let us consider the functional differential equation*

$$y''(t) + \frac{d}{dt}\{\operatorname{grad}\Phi[y(t)]\} + \int_{-r}^{0} d\eta(\theta)\, y(t+\theta) = e(t) \tag{7.1}$$

where $y \in R^n$,

$$\Phi : R^n \to R, \quad y \mapsto \Phi(y) = \Phi_1(y) + \Phi_2(y)$$

is an application of class C^2 with Φ_1 homogeneous of degree $2p \geqslant 2$ and of definite sign and with Φ_2 such that

$$\| \operatorname{grad}\Phi_2(y)\| \leqslant C_2 \| y \|^{2p-2} + C_3, \qquad C_2, C_3 \geqslant 0. \tag{7.2}$$

Moreover, let us suppose that

$$e : R \to R^n, \qquad t \mapsto e(t)$$

is continuous, T-periodic and satisfies

$$\frac{1}{T}\int_0^T e(t)\, dt = 0,$$

and that $\eta(\theta)$, $-r \leqslant \theta \leqslant 0$, is an $n \times n$ matrix whose elements have bounded variation and are such that the matrix

$$M = \int_{-r}^{0} d\eta(\theta)$$

is nonsingular. Then (7.1) *has at least one T-periodic solution.*

To prove Theorem 6 we shall need the following

LEMMA 3. *With the notations of Theorem* 6, *if* $y \in C_T$ *and* $z \in C_T$, *there exists one constant* N *depending only upon* η, r *and such that*

$$\left| \frac{1}{T} \int_0^T \int_{-r}^0 \langle d\eta(\theta)\, y(t+\theta), z(t)\rangle \, dt \right| \leqslant N \left[\frac{1}{T} \int_0^T \| y(t)\|^2 \, dt \right]^{1/2} \left[\frac{1}{T} \int_0^T \| z(t)\|^2 \, dt \right]^{1/2}. \tag{7.4}$$

Proof. Every component of the matrix $\eta(\theta)$ having bounded variation, we have [36]

$$\eta_{ij}(\theta) = \eta_{ij}^{(1)}(\theta) - \eta_{ij}^{(2)}(\theta), \qquad i, j = 1, 2, \ldots, n,$$

where the $\eta_{ij}^{(k)}(\theta) (k = 1, 2; i, j = 1, 2, \ldots, n)$ are increasing functions. Then, using Fubini–Lebesgue theorem [36] and Schwarz inequality, we obtain

$$\begin{aligned}
&\left| \frac{1}{T} \int_0^T \int_{-r}^0 \langle d\eta(\theta)\, y(t+\theta), z(t)\rangle \, dt \right| \\
&\quad \leqslant \sum_{k=1}^{2} \sum_{i,j=1}^{n} \left| \frac{1}{T} \int_0^T \int_{-r}^0 y_i(t+\theta)\, z_j(t)\, d\eta_{ij}^{(k)}(\theta)\, dt \right| \\
&\quad \leqslant \sum_{k=1}^{2} \sum_{i,j=1}^{n} \int_{-r}^0 \left| \frac{1}{T} \int_0^T y_i(t+\theta)\, z_j(t)\, dt \right| d\eta_{ij}^{(k)}(\theta) \\
&\quad \leqslant \sum_{k=1}^{2} \sum_{i,j=1}^{n} \int_{-r}^0 \left[\frac{1}{T} \int_0^T y_i^2(t+\theta)\, dt \right]^{1/2} \left[\frac{1}{T} \int_0 z_j^2(t)\, dt \right]^{1/2} d\eta_{ij}^{(k)}(\theta) \\
&\quad \leqslant \sum_{i,j=1}^{n} \sum_{k=1}^{2} [\eta_{ij}^{(k)}(0) - \eta_{ij}^{(k)}(-r)] \left[\frac{1}{T} \int_0^T y_i^2(t)\, dt \right]^{1/2} \left[\frac{1}{T} \int_0^T z_j^2(t)\, dt \right]^{1/2}
\end{aligned}$$

which gives (7.4) if we take

$$N = \max_{i,j=1,2,\ldots,n} \left\{ \sum_{k=1}^{2} [\eta_{ij}^{(k)}(0) - \eta_{ij}^{(k)}(-r)] \right\}.$$

Proof of Theorem 6. Let us write (7.1) in the normal form

$$\begin{aligned}
y'(t) &= w(t) \\
w'(t) &= -\frac{d}{dt}\{\operatorname{grad} \Phi[y(t)]\} - \int_{-r}^0 d\eta(\theta)\, y(t+\theta) + e(t),
\end{aligned}$$

and introduce the auxiliary equation

$$\begin{aligned} y'(t) &= \lambda w(t) \\ w'(t) &= -\lambda \frac{d}{dt}\{\operatorname{grad} \Phi[y(t)]\} - \lambda \int_{-r}^{0} d\eta(\theta)\, y(t+\theta) + \lambda e(t), \\ &\qquad\qquad \lambda \in]0, 1], \end{aligned} \tag{7.5}$$

or, equivalently,

$$y''(t) + \lambda^2 \frac{d}{dt}\{\operatorname{grad} \Phi[y(t)]\} + \lambda^2 \int_{-r}^{0} d\eta(\theta)\, y(t+\theta) = \lambda^2 e(t), \qquad \lambda \in]0, 1]. \tag{7.6}$$

Let $y(t)$ be a possible T-periodic solution of (7.6). Then we have

$$\int_0^T \int_{-r}^0 d\eta(\theta)\, y(t+\theta)\, dt = 0$$

and hence, using Fubini–Lebesgue theorem [36],

$$M(Py) = 0,$$

i.e.,

$$Py = 0.$$

Therefore, there will exist an unique T-periodic application $z = Hy$ such that

$$Pz = 0, \qquad z'(t) = y(t), \qquad t \in R,$$

and, using Parseval formula, it is easy to see that if $\omega = 2\pi/T$,

$$\frac{\omega^2}{T}\int_0^T \| z(t)\|^2\, dt \leqslant \frac{1}{T}\int_0^T \| y(t)\|^2\, dt.$$

We deduce then from (7.6) the relation

$$\begin{aligned} &\frac{1}{T}\int_0^T \langle y''(t), z(t)\rangle\, dt + \frac{\lambda^2}{T}\int_0^T \left\langle \frac{d}{dt}\{\operatorname{grad} \Phi[y(t)]\}, z(t)\right\rangle dt \\ &\quad + \frac{\lambda^2}{T}\int_0^T \int_{-r}^0 \langle d\eta(\theta)\, y(t+\theta), z(t)\rangle\, dt = \frac{\lambda^2}{T}\int_0^T \langle e(t), z(t)\rangle\, dt, \end{aligned}$$

or, integrating by parts and using the properties of Φ,

$$\frac{2p}{T}\int_0^T \Phi_1[y(t)]\,dt = -\frac{1}{T}\int_0^T \langle \operatorname{grad}\Phi_2[y(t)], y(t)\rangle\,dt + \frac{1}{T}\int_0^T\int_{-r}^0 \langle d\eta(\theta)\,y(t+\theta), z(t)\rangle\,dt - \frac{1}{T}\int_0^T \langle e(t), z(t)\rangle\,dt.$$

Φ_1 being an homogeneous function of degree $2p \geqslant 2$ and of definite sign, there exists a $C_1 > 0$ such that

$$|\Phi_1(y)| \geqslant C_1 \|y\|^{2p} \tag{7.3}$$

for every $y \in R^n$ and then, by virtue of (7.2), (7.3), Hölder inequality and Lemma 3, we obtain

$$\frac{2pC_1}{T}\int_0^T \|y(t)\|^{2p}\,dt \leqslant C_2\left[\frac{1}{T}\int_0^T \|y(t)\|^{2p}\,dt\right]^{(2p-1)/2p} + \omega^{-1}N\left[\frac{1}{T}\int_0^T \|y(t)\|^{2p}\,dt\right]^{2/2p} + (C_3+E)\left[\frac{1}{T}\int_0^T \|y(t)\|^{2p}\,dt\right]^{1/2p},$$

with

$$\omega E = \left[\frac{1}{T}\int_0^T \|e(t)\|^2\,dt\right]^{1/2},$$

and hence

$$\left[\frac{1}{T}\int_0^T \|y(t)\|^2\,dt\right]^{1/2} \leqslant \left[\frac{1}{T}\int_0^T \|y(t)\|^{2p}\,dt\right]^{1/2p} \leqslant \rho_1, \tag{7.7}$$

with ρ_1 the (unique) positive root of the equation

$$2pC_1\rho^{2p-1} - C_2\rho^{2p-2} - \omega^{-1}N\rho - (C_3+E) = 0.$$

From (7.6) we can also obtain after integration by parts

$$-\frac{1}{T}\int_0^T \|y'(t)\|^2\,dt + \frac{\lambda^2}{T}\int_0^T\int_{-r}^0 \langle d\eta(\theta)\,y(t+\theta), y(t)\rangle\,dt = \frac{\lambda^2}{T}\int_0^T \langle e(t), y(t)\rangle\,dt,$$

and hence

$$\frac{1}{T}\int_0^T \|y'(t)\|^2\,dt \leqslant \lambda^2 N\left[\frac{1}{T}\int_0^T \|y(t)\|^2\,dt\right] + \lambda^2 E\left[\frac{1}{T}\int_0^T \|y(t)\|^2\,dt\right]^{1/2}$$
$$\leqslant \lambda^2\rho_1(N\rho_1 + E) = \lambda^2\rho_2^{\,2}, \tag{7.8}$$

or, equivalently,

$$\left[\frac{1}{T}\int_0^T \|w(t)\|^2\,dt\right]^{1/2} \leqslant \rho_2\,. \tag{7.9}$$

It follows from (7.7) and (7.8) [10, Chap. 1] that

$$\sup_{t\in[0,T]} \|y(t)\| \leqslant \rho_1 + 3^{-1/2}\left(\frac{T}{2}\right)\rho_2 = \rho_3 \tag{7.10}$$

for every possible T-periodic solution of (7.6). Lastly, we deduce still from (7.6)

$$\frac{1}{T}\int_0^T \|y''(t)\|^2\,dt + \frac{\lambda^2}{T}\int_0^T \left\langle \frac{d}{dt}\operatorname{grad}\Phi[y(t)], y''(t)\right\rangle dt$$
$$+ \frac{\lambda^2}{T}\int_0^T\int_{-r}^0 \langle d\eta(\theta)\, y(t+\theta), y''(t)\rangle\,dt = \frac{\lambda^2}{T}\int_0^T \langle e(t), y''(t)\rangle\,dt,$$

and, using (7.4), (7.8), and (7.10), it is easy to see that

$$\frac{1}{T}\int_0^T \|y''(t)\|^2\,dt \leqslant \lambda^2\rho_4\left[\frac{1}{T}\int_0^T \|y''(t)\|^2\,dt\right]^{1/2}$$

with ρ_4 a positive constant, and hence

$$\left[\frac{1}{T}\int_0^T \|w'(t)\|^2\,dt\right]^{1/2} \leqslant \rho_4\,.$$

Therefore,

$$\sup_{t\in[0,T]} \|w(t)\| \leqslant \rho_2 + 3^{-1/2}\left(\frac{T}{2}\right)\rho_4 = \rho_5$$

for every T-periodic solution $[y(t), w(t)]$ of (7.5).

If we write now $x = (y, w)$ and

$$\|x\|^2_{R^{2n}} = \|y\|^2 + \|w\|^2,$$

and if we adopt the notations of Theorem 4 (with $2n$ instead of n), it is easy to see that, if $a = (b, c)$, $b, c \in R^n$,

$$g_0(a) \equiv \begin{pmatrix} 0 & I \\ -M & 0 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix}$$

has the unique solution $a = 0$ and that

$$d[g_0(a), B_\rho, 0] = \operatorname{sgn} \det M = \pm 1$$

for every $\rho > 0$. If we take $\rho > (\rho_3^2 + \rho_5^2)^{1/2}$, the conditions of Theorem 4 are satisfied and Theorem 6 is proved.

As an example, it is easy to check that the forced van der Pol equation with retardation

$$y''(t) + k[x^2(t) - 1]\, x'(t) + \nu^2 x(t - \tau) = e(t)$$

with $k \neq 0$, $\nu^2 > 0$, $\tau \geqslant 0$, and $e(t)$ an arbitrary continuous T-periodic function of mean value zero verifies the conditions of Theorem 6. The same remains true if $k(x^2 - 1)$ is replaced by any polynomial of even order.

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