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$H - C^1$ Maps and elliptic SPDEs with polynomial and exponential perturbations of Nelson's Euclidean free field

Sergio Albeverio[a,b,c] and Minoru W. Yoshida[d,*]

[a] *Inst. Angew. Mathematik, Universität Bonn, Wegelerstr. 6, D-53115, Bonn, Germany*
[b] *SFB 256; BiBoS; CERFIM, Locarno, Switzerland*
[c] *Acc. Arch., USI, Switzerland*
[d] *Department of Systems Engineering, The Univ. Electro Commun, 182-8585 Chofu-shi, Tokyo, Japan*

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Abstract

Elliptic stochastic partial differential equations (SPDE) with polynomial and exponential perturbation terms defined in terms of Nelson's Euclidean free field on $\boldsymbol{R}^d$ are studied using results by S. Kusuoka and A.S. Üstünel and M. Zakai concerning transformation of measures on abstract Wiener space. SPDEs of this type arise, in particular, in (Euclidean) quantum field theory with interactions of the polynomial or exponential type. The probability laws of the solutions of such SPDEs are given by *Girsanov probability measures*, that are non-linearly transformed measures of the probability law of Nelson's free field defined on subspaces of Schwartz space of tempered distributions.

Keywords: Elliptic stochastic partial differential equations; Analysis on abstract Wiener space; Non-linear transformations on Wiener space; Random fields; Free Euclidean field; Quantum fields

*Corresponding author. Dept. Systems Engineering, The Univ. Electro Commun, 182-8585, Tokyo, Japan.
E-mail addresses: albeverio@uni-bonn.de (S. Albeverio), yoshida@se.uec.ac.jp (M.W. Yoshida).

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Introduction

In this paper we study elliptic stochastic partial (pseudo)-differential equations (SPDE) heuristically written as follows:

$$(-\Delta + 1)\psi(x) + V(\psi)(x) = (-\Delta + 1)^{\frac{1}{2}}\dot{W}(x), \quad x \in \boldsymbol{R}^d, \tag{1}$$

where Δ is the d-dimensional Laplace operator, V is a (renormalized) polynomial or exponential function, and W is an *isonormal Gaussian process* on $\boldsymbol{R}^d$ (cf. [44], and for precise definition of $(-\Delta + 1)^{\frac{1}{2}}\dot{W}$ see (2)). $\dot{W}$ is often referred to as the *Gaussian white noise* on $\boldsymbol{R}^d$ (cf. [30]).

The existence problem for the solution ψ of (1), as a tempered distribution valued random variable, and the problem of deriving probabilistic properties for the solution, such as characterizing a class of functionals of the solution possessing the so-called reflection positivity, will be solved by reducing these problems to the existence problem of the associated Girsanov probability measure and the absolutely continuity of the measure with respect to a reference measure (cf. (11)–(13)).

The investigation of such SPDEs is of importance in stochastic analysis as well as in *Euclidean* quantum field theory.

(i) *Stochastic analytic interest*: In order to give a solution for an ordinary stochastic differential equation (SDE), defining a stochastic process on $\boldsymbol{R}^1$ taking values in $\boldsymbol{R}^d$, the method of *change of variables* is a most powerful tool. By this method one can show the existence of a solution by showing the existence of a probability measure, called the associated *Girsanov measure*, that is the probability law of the solution. In particular, if the problem is formulated in terms of processes adapted to some filtration, then the existence of the associated Girsanov measure is equivalent to the existence of a solution of a corresponding *martingale problem*.

Similarly, if an SPDE is formulated on an abstract Wiener space and the solution is assumed to be a random field on $\boldsymbol{R}^d$, then the existence problem of the solution can be reduced to the existence problem for the associated *Girsanov measure* (cf. [40], [55]). The existence of the Girsanov measures is investigated by considering a *change of variable formula*. In [23] an existence problem for an SPDE defined in a bounded domain $D \subset \boldsymbol{R}^d$, $(d = 1, 2, 3)$, with Dirichlet boundary conditions and with a dynamics characterized by (1) with the RHS replaced by $\dot{W}$ is considered and solved by showing the existence of an associated Girsanov measure. These authors also study the Markov field property. In this work the solution is given as a random variable taking values in the space of continuous functions on D. For related work see also, e.g., [34], [7].

The equation given by (1) is an SPDE which belongs to a class of equations that arises in physical, engineering or economical problems. In order to give naturally a Euclidean quantum field theoretic interpretation, we have to set the RHS of (1) as $(-\Delta+1)^{\frac{1}{2}}\dot{W}$ (cf. Section 4). Also, in order to make the equation meaningful, we have to take V in (1) to be non-linear functions on $\mathscr{S}'(\boldsymbol{R}^d)$. As a consequence, this asks for new *mathematical developments*: With such a noise term the solutions of (1) become random variables which take values in $\mathscr{S}'(\boldsymbol{R}^d)$, the Schwartz space of tempered distributions. For the non-linear perturbation terms V one introduces the notions of Wick power and Wick exponential function of random variables (cf. [8,10,11]), which have to be interpreted as measurable functions from $\mathscr{S}'(\boldsymbol{R}^d)$ to $\mathscr{S}'(\boldsymbol{R}^d)$. We make these notions adaptable to a change of variable formula on an abstract Wiener space, which is a probability space defined by the probability law of $(-\Delta+1)^{-\frac{1}{2}}\dot{W}$, i.e. of Nelson's Euclidean free field measure. The formulation adopted here is an extension of that used in [11,48].

(ii) *Euclidean quantum field theoretic interest*: Nelson's Euclidean free field is a Gaussian random variable ϕ_ω taking values in $\mathscr{S}'(\boldsymbol{R}^d)$ defined on a probability space $(\Omega, \mathscr{F}, P)$ such that

$$E[\langle \varphi_1, \phi.\rangle \langle \varphi_2, \phi.\rangle] = \int_{\boldsymbol{R}^d} ((-\Delta+1)^{-1}\varphi_1)(x)\varphi_2(x)\,dx$$
$$\text{for real } \varphi_1, \varphi_2 \in \mathscr{S}(\boldsymbol{R}^d).$$

By making use of an isonormal Gaussian process W_ω on $\boldsymbol{R}^d$, we can give $\langle \varphi, \phi_\omega\rangle_{\mathscr{S},\mathscr{S}'}$ a stochastic integral expression such that

$$\langle \varphi, \phi_\omega\rangle_{\mathscr{S},\mathscr{S}'} = \int_{\boldsymbol{R}^d} ((-\Delta+1)^{-\frac{1}{2}}\varphi)(x)\,dW_\omega(x). \tag{2}$$

We may write this by

$$\phi_\omega = (-\Delta+1)^{-\frac{1}{2}}\dot{W}_\omega,$$

or we can write this as a linear elliptic SP(pseudo)-DE such that

$$-\Delta\phi_\omega + \phi_\omega = (-\Delta+1)^{\frac{1}{2}}\dot{W}_\omega. \tag{3}$$

For the convenience of the explanation we now use the notations adopted in Theorem II.17 of Simon [52] and Section X.7 of Reed and Simon [47]. It is possible to define the "time-zero field" of Nelson's Euclidean free field ϕ_ω on $\boldsymbol{R}^d$ defined by (2) as follows:

$$\langle \phi_\omega, \delta_{\{0\}} \otimes f\rangle \quad \text{for } f \in \mathscr{S}(\boldsymbol{R}^{d-1}),$$

where $\delta_{\{t_0\}} = \delta_{\{t_0\}}(t)$ (the Dirac measure on $t = t_0$). Since of course $\delta_{\{0\}} \otimes f$ is not in $\mathscr{S}(\boldsymbol{R}^d)$, $\langle \phi_\omega, \delta_{\{0\}} \otimes f\rangle$ has to be understood, e.g., in the sense of an $L^2(P)$-limit of $\langle \phi_\omega, \delta^\varepsilon_{\{0\}} \otimes f\rangle$ with $\delta^\varepsilon_{\{0\}}$ an $\mathscr{S}(\boldsymbol{R})$ approximation of $\delta_{\{0\}}$. For the existence of this limit see, e.g., [8,52]. (Also in the framework of an abstract Wiener space, $\langle \phi_\omega, h\rangle$ is defined as an $L^2(P)$-random variable for $h \in H^{-1} = (-\Delta+1)^{-\frac{1}{2}} L^2(\boldsymbol{R}^d)$ (cf. Remark 1.4 and Definition A.1 in the appendix).

We denote symbolically the time-zero field of the Euclidean free field by $\phi_\omega(0,\vec{x})$, $\vec{x} \in \boldsymbol{R}^{d-1}$. Let $\mathscr{H}_0$ be the Hilbert space defined by

$$\mathscr{H}_0 \equiv \overline{\left\{ \prod_{i=1}^n \langle \phi_\omega, \delta_{\{0\}} \otimes f_i \rangle \,|\, f_i \in \mathscr{S}(\boldsymbol{R}^{d-1}),\ i=1,\dots,n,\ n \in \boldsymbol{N} \right\}}^{L^2(P)}.$$

By making use of the isonormal Gaussian process $\dot{W}$ on $\boldsymbol{R}^{d-1}$ defined on some probability space $(\Omega', \mathscr{F}', P')$ and the Laplace operator Δ_{d-1} on $\boldsymbol{R}^{d-1}$, if we set $\tilde{\phi} = (-\Delta_{d-1}+1)^{-\frac{1}{4}} \dot{W}$, then the Hilbert space $\overline{\{\prod_{i=1}^n \langle \tilde{\phi}, f_i \rangle \,|\, f_i \in \mathscr{S}(\boldsymbol{R}^{d-1}),\ i=1,\dots,n,\ n\in \boldsymbol{N}\}}^{L^2(P')}$ can be identified with $\mathscr{H}_0$. Let H_0 be the operator on $\mathscr{H}_0$ which is the second quantization of the operator $(-\Delta_{d-1}+1)^{\frac{1}{2}}$ (cf. [52, Section I.4]):

$$H_0 \equiv d\Gamma((-\Delta_{d-1}+1)^{\frac{1}{2}}).$$

For $\varphi \in \mathscr{S}(\boldsymbol{R}^d)(\varphi(t,\vec{x}),\ t \in \boldsymbol{R}^1, \vec{x} \in \boldsymbol{R}^{d-1})$ let

$$\Phi_0(\varphi) = \int_{\boldsymbol{R}^1} e^{iH_0 t} \langle \phi(0,\cdot), \varphi(t,\cdot) \rangle e^{-iH_0 t}\, dt,$$

then by Theorem II.17 of [52] the operator $\Phi_0(\varphi)$ defined e.g. on the domain of analytic vectors for H_0 acting in $\mathscr{H}_0$ is the "free Hermitian scalar field of mass 1" (the free Hermitian scalar field can also be constructed through the Segal quantization, cf. [47, Section X7; also 17, Problem 8, Chapter 7, p. 206]. The operator valued distribution denoted heuristically by $\Phi_0(t,\vec{x}) = e^{iH_0 t} \phi(0,\vec{x}) e^{-iH_0 t}$ satisfies the following functional equation (on a dense domain in $\mathscr{H}_0$) (cf. [47, Theorem X42]):

$$\left(\frac{\partial^2}{\partial t^2} - \Delta_{d-1} \right) \Phi_0 + \Phi_0 = 0. \tag{4}$$

We have to notice that $\Phi_0(\varphi)$ is an operator on $\mathscr{H}_0$. It commutes with $\Phi_0(\eta)$, $\eta \in \mathscr{S}(\boldsymbol{R}^d)$, only when the supports of φ and η are space like separated in the sense of the Minkowski space, i.e. $|t-s|^2 - |\vec{x}-\vec{y}|^2 < 0$, $\forall (t,\vec{x}) \in \mathrm{supp}[\varphi], \forall (s,\vec{y}) \in \mathrm{supp}[\eta]$, $\langle \phi_\omega, \delta_{\{t\}} \otimes f \rangle$ is a random variable in $L^2(\Omega, P)$ (i.e. it is a multiplication operator), we have thus to strictly distinguish between $\Phi_0(t,\vec{x})$ and $\langle \phi_\omega, \delta_{\{t\}} \otimes f \rangle$. In a sense which can be made precise $\Phi_0(t,\vec{x})$ is an analytic continuation of $\phi_\omega(t,\vec{x})$, which is a solution of (3), at the point $t=0$ taking the boundary value $\phi_\omega(0,\vec{x}) = \Phi_0(0,\vec{x})$ (cf. [41,42,52] for precise Euclidean and Markov field strategies in the constructive quantum field theory). Also we should notice that the differential operator on the left-hand side of (4) formally comes from the differential operator which appeared on the left-hand side of (3) by exchanging its time component t by $\sqrt{-1}t$.

Correspondingly (cf. [47, (X.89)]), a scalar quantum field ϕ_I with a self-interaction V is supposed to satisfy, in the sense of operator-valued distribution, on a dense domain of the relevant Hilbert space $\mathscr{H}_I$ (cf. $\mathscr{H}_0$ defined above),

$$\left(\frac{\partial^2}{\partial t^2} - \Delta_{d-1}\right)\Phi_I + \Phi_I + V(\Phi_I) = 0. \tag{5}$$

By (3)–(5) we may naturally have an interest to the consideration of the SPDE (1).

Baez et al. [18] give a definition for functional equations of operator-valued distributions in the algebraic framework and call them "non-linear quantized equations" (cf. [18, Section 8.8]). In this note we restrict our considerations mainly to the analysis of *Euclidean* random fields. The study of the *time-zero field* and the *non-linear quantized equations*, as well as the discussions of the relations between our present results and the ones in [18], will essentially be postponed to future work except for a remark in Section 4.

In the framework of Euclidean quantum field theory various SDEs have been considered. Albeverio et al. [3,4] define a Euclidean random field by a solution of an elliptic SPDE (without non-linear perturbation term) driven by *general white noise processes* (including the Poisson noises) on $\boldsymbol{R}^d$. In the same note it has been shown that the constructed Euclidean field corresponds to an indefinite metric quantum field with a *non-trivial* interaction (of "non-polynomial type") (cf. [5,6,15,27] and references therein). For other considerations about SDE taking values in the space of distributions related to quantum fields see e.g. [9,14,19,22,33,43,49,50] and references therein.

The organization of this paper is as follows:

In Section 1, we firstly define the Nelson's Euclidean free field ϕ_ω on $\boldsymbol{R}^d$, its Wick powers and Wick exponential functions by making use of multiple stochastic integrals with respect to the isonormal Gaussian process W_ω on $\boldsymbol{R}^d$ as follows (we use formal notations here: rigorous definition will be given in Section 1)

$$\phi_\omega(x) = \int_{\boldsymbol{R}^d} J^{\frac{1}{2}}(x-y)\, dW_\omega(y) = (J^{\frac{1}{2}}\dot{W}_\omega)(x),$$

$$:\phi^p_{\alpha,\omega}:(x) = \int_{(\boldsymbol{R}^d)^p} J^\alpha(x-y_1)\cdots J^\alpha(x-y_p)\, dW_\omega(y_1)\cdots dW_\omega(y_p),$$
$$p = 2,3,\ldots,$$

$$:e^{\varepsilon\phi_{\alpha,\omega}}:(x) = \sum_{p=0}^{\infty}\frac{\varepsilon^p}{p!}:\phi^p_{\alpha,\omega}:(x), \quad x\in\boldsymbol{R}^d, \quad \omega\in\Omega,$$

where J^{α} is the integral kernel of the pseudo-differential operator $(-\Delta+1)^{-\alpha}$ on $\boldsymbol{R}^d$, α and ε are some real numbers. Briefly, on a complete probability space $(\Omega, \mathscr{F}, P)$, W_ω is defined as a generalized random field on $\boldsymbol{R}^d$ such that $dW_\omega = \dot{W}_\omega$ is Euclidean-invariant Gaussian white noise random field (or random measure) in the sense of, e.g. [30].

In Theorem 1.1 it is shown that the above quantities are well defined as random variables which take values in some subspace of the space $\mathscr{S}'(\boldsymbol{R}^d)$ of tempered distributions for suitably chosen α and ε. More precisely, it is shown that the Nelson's free field satisfies

$$\phi_\omega \in \bigcap_{b>d} \bigcap_{a>\frac{d-2}{4}} B_d^{a,b} \quad \text{a.s.,} \tag{6}$$

and for example if $\alpha = \frac{d}{4}$ then

$$:\phi^p_{\alpha,\omega}: \in \bigcap_{b>d} \bigcap_{a>0} B_d^{a,b} \quad \text{a.s.,} \tag{7}$$

if in addition $|\varepsilon| < a_0(d)$, then

$$:e^{\varepsilon\phi_{\alpha,\omega}}: \in \bigcap_{b>d} \bigcap_{a>\frac{\varepsilon^2 d}{4(a_0(d))^2}} B_d^{a,b} \quad \text{a.s,} \tag{8}$$

where $B_d^{a,b} = \{(|x|^2+1)^{\frac{b}{4}} J^{-a} f : f \in L^2(\boldsymbol{R}^d)\}$, and $a_0(d) = \sqrt{\frac{d(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}{2}}$. We remark that for $d=2$ these Wick powers, resp. Wick exponential, e.g., coincide with those discussed in [1,8,52].

Then, letting μ be the probability law of Nelson's free field ϕ_ω, that is a Borel probability measure on the topological vector space $B_d^{a,b}$ (in fact a Hilbert space), we define measurable functions $\tau_{(\alpha,p)}$ and $\tau_{(\alpha,e^\varepsilon)}$ from $B_d^{a,b}$ to $B_d^{a',b}$, which are random variables on $(B_d^{a,b}, \mathscr{B}(B_d^{a,b}), \mu)$, satisfying

$$\tau_{(\alpha,p)}(\phi_\omega) =: \phi^p_{\alpha,\omega}:, \qquad \tau_{\alpha,e^\varepsilon}(\phi_\omega) =: e^{\varepsilon\phi_{\alpha,\omega}}:, \qquad \mu\text{-a.s.,}$$

where a and a' satisfy the conditions for a in (6) and(7) (or (8)), respectively.

Next, we note that the Gaussian probability measure μ possesses the structure of an abstract Wiener space, namely $(B_d^{a,b}, J^1\mathscr{H}, \mu)$ is an abstract Wiener space, where $\mathscr{H} = H^{-1} = \{h : h = J^{-\frac{1}{2}}f, f \in L^2(\boldsymbol{R}^d)\}$, which is the index set of Nelson's free field. $\mathscr{H}$ is identified, via Riesz theorem, with the Cameron–Martin space H^1 by $J^1\mathscr{H} = H^1$. (The definitions, notions and notations associated with the analysis on an abstract Wiener space, which are used here, are found in Definitions A.1–A.3 and Remark A.1 in the Appendix (also cf. Remark 1.4).)

As a consequence, on the abstract Wiener space $(B_d^{a,b}, J^1\mathscr{H}, \mu)$ for a non-negative "space-cut-off function" $\eta_M \in C_0^\infty(\boldsymbol{R}^d)$ such that $\eta_M(x) = 1$ for

$|x| \leqslant M$ and $\eta_M(x) = 0$ for $|x| \geqslant 2M$, the maps $u_p(\psi) = J^{\alpha-\frac{1}{2}}(\eta_M \tau_{(\alpha,p)}(\psi))$ and $u_e(\psi) = J^{\alpha-\frac{1}{2}}(\eta_M \tau_{(\alpha,e^{\varepsilon})}(\psi))$ are well defined as measurable maps. Theorem 1.3 gives a sufficient condition for α, p and ε under which u_p and u_e become $H - C^1$ maps on this abstract Wiener space (cf. Definition A.3 and Remark 1.4).

In Section 2, we shall state some of the main results. For η_M as above and $\lambda \geqslant 0$ on the abstract Wiener space $(B_d^{a,b}, J^1\mathscr{H}, \mu)$ we define the shift

$$T_3(\psi) = \psi + \lambda J^1(\eta_M \tau_{(\alpha,3)}(\psi)). \tag{9}$$

In Theorem 2.3 by making use of general results given by Kusuoka [37–39], Shigekawa [51] and Üstünel and Zakai [55] we show the existence of a probability measure ν on $(B_d^{a,b}, \mathscr{B}(B_d^{a,b}))$, called the Girsanov probability measure associated with μ and T_3, such that

$$E^{\nu}[f \circ T_3] = E^{\mu}[f], \quad \forall f \in C_b(B_d^{a,b}), \tag{10}$$

where E^{ν} and E^{μ} denote the expectation with respect to the probability measures ν and μ, respectively. Since μ is the probability law of Nelson's free field $\phi_{\omega} = J^{\frac{1}{2}}\dot{W}_{\omega}$, (9) and (10) say that there exists an isonormal Gaussian process W on $\boldsymbol{R}^d$ such that

$$\psi + \lambda J^1(\eta_M \tau_{(\alpha,3)}(\psi)) = J^{\frac{1}{2}}\dot{W},$$

where ψ is a random variable with probability law ν. This equation can be written, similarly to (1), as

$$(-\Delta + 1)\psi + \lambda \eta_M \tau_{(\alpha,3)}(\psi) = (-\Delta + 1)^{\frac{1}{2}}\dot{W}. \tag{11}$$

In the same theorem we derive the explicit form of $\frac{d\nu}{d\mu}$, the Radon–Nikodym density of the Girsanov measure ν with respect to Nelson's free field measure μ. In the case $d = 2$ it is possible to take $\alpha = \frac{1}{2}$, then the Radon–Nikodym density $\frac{d\nu}{d\mu}(\phi)$, which is a random variable (Wiener functional) on $(B_2^{a,b}, J^1\mathscr{H}, \mu)$, is given by

$$\frac{d\nu}{d\mu}(\phi) = q(T_3(\phi))\Lambda_1(\phi)\Lambda_2(\phi)\exp\left\{-\lambda \int_{\boldsymbol{R}^2} \eta_M(x) : \phi^4 : (x)\, dx\right\},$$

where $q(T_3(\phi))$, $\Lambda_1(\phi)$ and $\Lambda_2(\phi)$ are non-linear (also non-local) functionals of $\phi \in B_2^{a,b}$ such that $q(T_3(\phi)) = \dfrac{1}{\sharp\{T_3^{-1}(T_3(\phi))\}}$ is the reciprocal of the cardinality of the elements that are mapped to the common point $T_3(\phi)$ by the map T_3 (multiplicity), $\Lambda_1(\phi) = |\det_2(I_{H^{-1}} + 3\lambda\eta_M(x) : \phi^2(x) : \delta_{\{x\}}(y))|$ is the absolute value of Carleman Fredholm determinant of the Hilbert Schmidt operator appeared in the parentheses and $\Lambda_2(\phi) = \exp\{-\frac{\lambda^2}{2}\int_{\boldsymbol{R}^2}(J^{\frac{1}{2}}(\eta_M : \phi^3 :)(x)^2\, dx\}$ (more precisely see Theorem 2.3 and

Remark 2.3). On the other hand, the $(\phi^4)_2$ Euclidean field with the space-cut-off η_M is defined as the random field on $\boldsymbol{R}^2$ with the probability measure ν_{η_M} such that (cf., e.g., Definition in Section V.1 of [52, pp. 141], [28]):

$$d\nu_{\eta_M}(\phi) = \frac{1}{Z_M} \exp\left\{ -\lambda \int_{\boldsymbol{R}^2} \eta_M(x) : \phi^4 : (x)\, dx \right\} d\mu(\phi),$$

with the normalization contact $Z_M = E^\mu[\exp\{-\lambda \int_{\boldsymbol{R}^2} \eta_M(x) : \phi^4 : (x)\, dx\}]$. Then there is a similarity between ν and ν_{η_M} in the sense that their Radon–Nikodym densities $\frac{d\nu}{d\mu}$, resp. $\frac{d\nu_{\eta_M}}{d\mu}$, have the common term $\exp\{-\lambda \int \eta_M : \phi^4 : dx\}$. But because of the existence of the non-linear term $K(\phi)$ in $\frac{d\nu}{d\mu}$, we have to distinguish ν from ν_{η_M} (see Remark 2.3).

In Theorem 2.4, it is shown that the shift T_e defined by $T_e(\psi) = \psi + \lambda J^1(u_e(\psi))$ is a *strongly monotone shift* on the abstract Wiener space, a result which has an interest in its own, because T_e is a *non-linear* transformation on a space of *distributions*. In the same theorem by making use of the general results for monotone shifts developed by Üstünel and Zakai [55] in an abstract Wiener space setting and through the consideration of the associated Girsanov probability measure we prove the existence of a solution ψ such that

$$(-\Delta + 1)\psi + \lambda\eta_M \tau_{(\alpha, e^\varepsilon)}(\psi) = (-\Delta + 1)^{\frac{1}{2}} \dot{W}. \tag{12}$$

In Corollary 2.5 the space-cut-off function η_M (or "infrared cut-off", in physicist's terminology) will be removed.

In Section 3, we derive a partial result for SPDEs of form (11) with $\tau_{(\alpha,3)}$ replaced by $\tau_{(\alpha,p)}$, $p \neq 3$.

Section 4, contains two discussions concerning the problem of Euclidean quantum field theory. One of them is a consideration of the so-called *reflection positivity property*. In Theorem 4.1 for the probability measure ν, giving the distributions of the solutions of (11) or (12), and for the corresponding shifts $T = T_3$ or T_e, respectively, we see that

$$E^\nu\left[\left(\prod_{i=1}^n \langle \varphi_i, T(\theta\psi) \rangle \right) \left(\prod_{i=1}^n \langle \varphi_i, T(\psi) \rangle \right) \right] \geqslant 0 \tag{13}$$

holds for any $n \in \boldsymbol{N}$ and $\varphi_i \in \mathcal{S}(\boldsymbol{R}^d)$ $(i = 1, \ldots, n)$ such that $\mathrm{supp}[\varphi_i] \subset \{(t, \vec{x}) \in \boldsymbol{R}^d : t > 0\}$, where θ is the time reflection operator on $\boldsymbol{R}^d$: $\theta f(t, \vec{x}) = f(-t, \vec{x})$, $(t, \vec{x}) \in \boldsymbol{R} \times \boldsymbol{R}^{d-1}$. In order to conclude that the Euclidean random field ψ with the probability measure ν has the reflection positivity property introduced in axiomatic Euclidean quantum field theory (Hegerfeldt T-positivity, cf. [29]), we have to show that for the same

$\varphi_i \in \mathscr{S}(\boldsymbol{R}^d)$ as above the following holds:

$$E^{\nu}\left[\left(\prod_{i=1}^{n} \langle \varphi_i, \theta\psi \rangle\right)\left(\prod_{i=1}^{n} \langle \varphi_i, \psi \rangle\right)\right] \geqslant 0.$$

Hence, (13) characterizes a sub-space of random variables on $(B_d^{a,b}, \nu)$ consisting of elements which satisfy the reflection positivity property (cf. Remark 4.1(i)).

In the same section, the consideration of the time-zero field corresponding to $\psi(t, \vec{x})$ we investigate whether the random field $\psi(t, \vec{x})$ can be analytically continued to a solution of the non-linear *quantized* equation (5) (cf. Remark 4.2(i)) or not.

The appendix contains the explanations of some fundamental notions and notations associated with an abstract Wiener space and the proofs of theorems and lemmas which were omitted in the main text.

1. Construction of non-linear $H - C^1$ maps on Nelson's free field

We shall first recall the definition of a stochastic process on a parameter space $\mathscr{D}$ and its equivalent class.

(i) *Let* $\mathscr{D}$ *be a locally convex topological vector space* (*TVS*) *which is separable, and* $(\Omega, \mathscr{F}, P)$ *be a complete probability space. A family of complex-valued random variables* $\{\Psi(\varphi, \omega)\}_{\varphi \in \mathscr{D}}$ *on* $(\Omega, \mathscr{F}, P)$ *is called as a* stochastic process *with parameter space* $\mathscr{D}$.

(ii) *Two stochastic processes* $\{\Psi(\varphi, \omega)\}_{\varphi \in \mathscr{D}}$ *and* $\{\tilde{\Psi}(\varphi, \omega)\}_{\varphi \in \mathscr{D}}$ *on* $(\Omega, \mathscr{F}, P)$ *are said to be* equivalent *if*

$$\forall \varphi \in \mathscr{D}, \quad P(\{\omega | \Psi(\varphi, \omega) = \tilde{\Psi}(\varphi, \omega)\}) = 1.$$

(iii) *Two stochastic processes* $\{\Psi(\varphi, \omega)\}_{\varphi \in \mathscr{D}}$ *and* $\{\tilde{\Psi}(\varphi, \omega)\}_{\varphi \in \mathscr{D}}$ *on* $(\Omega, \mathscr{F}, P)$ *are said to be* strongly equivalent *if*

$$P(\{\omega | \forall \varphi \in \mathscr{D}, \quad \Psi(\varphi, \omega) = \tilde{\Psi}(\varphi, \omega)\}) = 1.$$

Let $\mathscr{S}(\boldsymbol{R}^d)$ be the Schwartz space of rapidly decreasing test functions equipped with usual topology. $\mathscr{S}(\boldsymbol{R}^d)$ is a nuclear space. Let $\mathscr{S}'(\boldsymbol{R}^d)$ be its topological dual.

Let Δ be the d-dimensional Laplacian, and set $J^{\alpha} = (-\Delta + m^2)^{-\alpha}$ for some fixed $m > 0$. Precisely J^{α} is the pseudo-differential operator with the symbol $(|\xi|^2 + m^2)^{-\alpha}$, $\xi \in \boldsymbol{R}^d$. We denote the kernel representation of J^{α} by $J^{\alpha}(x - y)$: $(J^{\alpha}\varphi)(x) = \int_{\boldsymbol{R}^d} J^{\alpha}(x - y)\varphi(y)\, dy$, for $\varphi \in \mathscr{S}$. This is defined by the Fourier inverse transform such that

$$J^{\alpha}(x) = (2\pi)^{-d} \int_{\boldsymbol{R}^d} e^{\sqrt{-1}x\cdot\xi}(|\xi|^2 + m^2)^{-\alpha}\, d\xi \in L^1(\boldsymbol{R}^d; \lambda^d).$$

An integral representation of this Green kernel by means of a modified Bessel function, which also puts into evidence its regularity, will be described in the appendix.

For each $a, b, d > 0$ let $B_d^{a,b}$ be the linear subspace of $\mathscr{S}'(\boldsymbol{R}^d)$ defined by

$$B_d^{a,b} = \{(|x|^2 + 1)^{\frac{b}{4}} J^{-a} f : f \in L^2(\boldsymbol{R}^d; \lambda^d)\}, \tag{14}$$

where λ denotes the Lebesgue measure on $\boldsymbol{R}^d$. $B_d^{a,b}$ is a separable Hilbert space with the scalar product

$$\langle u|v \rangle = \int_{\boldsymbol{R}^d} J^a((|x|^2 + 1)^{-\frac{b}{4}} u(x)) \, J^a((|x|^2 + 1)^{-\frac{b}{4}} v(x)) \, dx, \quad u, v \in B_d^{a,b}. \tag{15}$$

Note that if $a, b, d > 0$, then $C_0(\boldsymbol{R}^d) \subset B_d^{a,b}$. From the consideration of cylinder sets constructed from $C_0(\boldsymbol{R}^d)$ and $B_d^{a,b}$ it is easy to see that

$$\mathscr{B}(C_0(\boldsymbol{R}^d \to \boldsymbol{R})) = \left\{ A \cap C_0(\boldsymbol{R}^d \to \boldsymbol{R}) : A \in \mathscr{B}(B_d^{a,b}) \right\}, \tag{16}$$

where $\mathscr{B}(C_0(\boldsymbol{R}^d \to \boldsymbol{R}))$ and $\mathscr{B}(B_d^{a,b})$ are the Borel σ-fields of $C_0(\boldsymbol{R}^d)$ and $B_d^{a,b}$, respectively (this is obvious because the Borel σ field of a locally convex topological vector space which is separable is generated by its cylinder sets, cf. [54], [56]).

We use the same terminology and notations concerning multiple stochastic integrals, abstract Wiener spaces and transformations between abstract Wiener spaces which are used in [44,55].

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space and consider an isonormal Gaussian process $W = \{W(h), h \in L^2_{\text{real}}(\boldsymbol{R}^d; \lambda^d)\}$, where λ^d denotes the Lebesgue measure on $\boldsymbol{R}^d$ and L^2_{real} is the real L^2 space: W is a centered Gaussian family of random variables on $(\Omega, \mathscr{F}, P)$ such that

$$E[W(h)W(g)] = \int_{\boldsymbol{R}^d} h(x)\, g(x) \lambda^d(dx), \quad h, g \in L^2_{\text{real}}(\boldsymbol{R}^d; \lambda^d),$$

where E denotes the expectation with respect to the probability measure P. Ω can be taken to be the complete separable metric space $\boldsymbol{R}^\infty$ equipped with the metric

$$d(\boldsymbol{x}, \boldsymbol{y}) = \sum_{n=1}^{\infty} 2^{-n} \min\{|x_n - y_n|, 1\}, \;\; \boldsymbol{x} = (x_1, x_2, x_3, \ldots),$$
$$\boldsymbol{y} = (y_1, y_2, y_3, \ldots),$$
$$P = N_{0,1}^{\infty} \tag{17}$$

and $\mathscr{F}$ to be the completion of the Borel σ-field of Ω with respect to P.

For $A \in \mathscr{B}(\boldsymbol{R}^d)$ such that $\lambda^d(A) < \infty$ we set

$$W(A) = W(\chi_A),$$

where χ_A is the indicator function of the set A.

Then, for $h \in L^2_{\mathrm{real}}(\boldsymbol{R}^d; \lambda^d)$ the random variable $W(h)$ can be regarded as a stochastic integral, and is denoted by

$$W(h) = \int_{\boldsymbol{R}^d} h\, dW.$$

In the sequel we sometimes use the notation $W(\varphi) = \langle \varphi, \dot{W} \rangle_{\mathscr{S},\mathscr{S}'}$ for $\varphi \in \mathscr{S}$. The multiple stochastic integrals, such as (24), are defined in the usual way. Namely a multiple stochastic integral is the limit of a sequence of multiple sums of Gaussian random variables such that $\sum_{i_1,\dots,i_p} a_{i_1,\dots,i_p} W(A_{i_1}) \times \cdots \times W(A_{i_p})$, where $a_{i_1,\cdots,i_p} = 0$ if $i_j = i_k$ for some $j \neq k$ (i.e. by taking sums with elimination of all diagonal parts), for a precise definition of multiple stochastic integral, cf. [44, Section 1.1.2].

We denote the Fourier and Fourier inverse transform of a function φ, respectively, by $\mathscr{F}[\varphi]$ and $\mathscr{F}^{-1}[\varphi]$, which are defined by

$$\mathscr{F}[\varphi](\xi) = \int_{\boldsymbol{R}^d} e^{-\sqrt{-1}x\cdot\xi} \varphi(x)\, dx,$$

$$\mathscr{F}^{-1}[\varphi](\xi) = (2\pi)^{-d} \int_{\boldsymbol{R}^d} e^{\sqrt{-1}x\cdot\xi} \varphi(x)\, dx \quad \text{for } \varphi \in \mathscr{S}(\boldsymbol{R}^d).$$

We sometimes denote $\mathscr{F}[\varphi] = \hat{\varphi}$. Let $\eta_1 \in C_0^\infty(\boldsymbol{R}^d)$ be such that $\eta_1(x) = \eta_1(y)$ for $|x| = |y|$ and

$$0 \leqslant \eta_1(x) \leqslant 1, \quad \eta_1(x) = \begin{cases} 1 & |x| \leqslant 1, \\ 0 & |x| \geqslant 2, \end{cases} \tag{18}$$

and let $\eta_k(x) = \eta_1(\frac{x}{k}) \in C_0^\infty(\boldsymbol{R}^d)$, $k = 1, 2, 3, \dots$. Also define $\rho \in C_0^\infty(\boldsymbol{R}^d)$ as follows:

$$\rho(x) = \begin{cases} C \exp\left(-\dfrac{1}{1 - |x|^2}\right) & |x| < 1, \\ 0 & |x| \geqslant 1, \end{cases}$$

where the constant C is taken to satisfy

$$\int_{\boldsymbol{R}^d} \rho(x)\, dx = 1. \tag{19}$$

Let

$$\rho_k(x) = k^d \rho(kx), \quad k = 1, 2, 3, \dots .$$

For $\alpha > 0$ we define $J_k^\alpha \in \mathscr{S}(\boldsymbol{R}^d)$, $\quad k = 1, 2, 3, \dots$ by

$$J_k^\alpha(x) = \int_{\boldsymbol{R}^d} J^\alpha(y) \rho_k(x - y)\, dy. \tag{20}$$

Also

$$F_k^\alpha(x; y_1, \dots, y_p) = (\eta_k(x))^p J_k^\alpha(x - y_1) \cdots J_k^\alpha(x - y_p), \tag{21}$$

and

$$F^{\alpha}(x; y_1, \dots, y_p) = J^{\alpha}(x - y_1) \dots J^{\alpha}(x - y_p), \quad p = 1, 2, 3, \dots \quad (22)$$

Then we see that the function F_k^{α} and F^{α} are symmetric in the last p variables $(y_1, \dots, y_p)$ and

$$F_k^{\alpha} \in \mathcal{S}((\boldsymbol{R}^d)^{p+1}), \qquad F_k^{\alpha}(x; y_1, \dots, y_p) = 0 \quad \text{for } |x| \geqslant 2k. \quad (23)$$

For each $\alpha > 0, p \geqslant 1$ and $k \geqslant 1$ we define the random variable $:_k \phi^p_{\alpha,\omega}:$ as a multiple stochastic integral such that

$$:_k \phi^p_{\alpha,\omega} : (x) = \int_{(\boldsymbol{R}^d)^p} F_k^{\alpha}(x; y_1, \dots, y_p)\, dW_{\omega}(y_1) \cdots dW_{\omega}(y_p). \quad (24)$$

Remark 1. (i) Using the relationship between Hermite polynomials of Gaussian random variables and multiple stochastic integrals (cf. [44, Theorem 1.1.2]) we see that the following equality holds (in the sense of two equivalent processes on $\boldsymbol{R}^d$):

$$:_k \phi^p_{\alpha,\omega} : (x) = p!(\eta_k(x))^p \sum_{n=0}^{[\frac{p}{2}]} \frac{(-\frac{1}{2}c_{\alpha,k})^n}{n!(p-2n)!} (k\phi_{\alpha,\omega}(x))^{p-2n},$$

where

$$c_{\alpha,k} = \int_{\boldsymbol{R}^d} (J_k^{\alpha}(y))^2\, dy.$$

In Theorem 1.1 it is shown that $\lim_{k\to\infty} :_k \phi^p_{\alpha,\omega}:$ exists as a $B_d^{a,b}$-valued random variable (for suitable α, a, b and p). In particular, when $d = 2$, $\lim_{k\to\infty} :_k \phi^p_{\frac{1}{2},\omega} :=: \phi^p_{\frac{1}{2},\omega}:$ exists for all $p = 1, 2, 3, \dots$. Moreover by (27) we see that the following holds:

$$E[\langle \phi_{\frac{1}{2}}, h_1 \rangle \cdots \langle \phi_{\frac{1}{2}}, h_p \rangle \langle :\phi^p_{\frac{1}{2}}:, g \rangle]$$
$$= p! \int_{(\boldsymbol{R}^2)^p \times \boldsymbol{R}^2} \prod_{j=1}^{p} h_j(x_j) J^1(x_j - y) g(y)\, dx\, dy,$$

for $h_j, g \in \mathcal{S}(\boldsymbol{R}^2)$, $j = 1, 2, 3, \dots, p$, where we denote $:\phi^1_{\frac{1}{2}}:$ by $\phi_{\frac{1}{2}}$. Hence $:\phi^p_{\frac{1}{2},\omega}:$ satisfies the definition of (*Euclidean*) Wick power of Nelson's free field on $\boldsymbol{R}^2$ introduced, e.g., in the Definition in Section V.1 of Simon [52, p. 135].

We have to distinguish strictly the *Euclidean* Wick power $:\phi^p:$ defined here, which we considered throughout the paper, from $:\Phi_0^p:$, the Wick power (renormalized product) of the free Hermitian scalar field operator Φ_0 on the physical Hilbert space (Φ_0 is the operator-valued distribution satisfying (4), which has been reviewed in the Introduction, and is an analytic continuation of the Nelson's free field $\phi_{\frac{1}{2}}$ by taking its boundary value, cf., e.g., [17,35] and in the book [18], also Section 4 in this paper).

(ii) The reason we adopt the expression of $:\phi^p:$ by means of the multiple stochastic integral is that by this expression the random variables $:\phi^p:$ can be studied on a same probability space $(\Omega, \mathscr{F}, P)$ and also their support (path) properties are easily established (cf. Theorem 1.1).

Remark 1.2 (Continuous version of $:_k \phi^p_\alpha :$). For each fixed $k \in \boldsymbol{N}$ it is easy to see that $\{:_k \phi^p_{\alpha,\omega} : (x)\}_{x \in \boldsymbol{R}^d}$ satisfies the Kolmogorov's continuity criterion for processes on $\boldsymbol{R}^d$ (cf., e.g., [44, Section A.3]), and has an equivalent process $\{:_k \tilde{\phi}^p_{\alpha,\omega} : (x)\}_{x \in \boldsymbol{R}^d}$ which is a $C_0(\boldsymbol{R}^d \to \boldsymbol{R})$-valued random variable:

$$P(:_k \phi^p_{\alpha,\omega} : (x) = :_k \tilde{\phi}^p_{\alpha,\omega} : (x)) = 1, \quad \forall x \in \boldsymbol{R}^d,$$

$$P(:_k \tilde{\phi}^p_{\alpha,\omega} : \in C_0(\boldsymbol{R}^d \to \boldsymbol{R})) = 1.$$

We always take $\{:_k \phi^p_{\alpha,\omega} : (x)\}_{x \in \boldsymbol{R}^d}$ as its continuous modification $\{:_k \tilde{\phi}^p_{\alpha,\omega} : (x)\}_{x \in \boldsymbol{R}^d}$ and drop the tilde in the following. Then by (16) $\{:_k \phi^p_{\alpha,\omega} : (x)\}_{x \in \boldsymbol{R}^d}$ is understood as a $B^{a,b}_d$ $(a, b \geqslant 0)$ valued random variable on $(\Omega, \mathscr{F}, P)$.

In the next Theorem 1.1 we give multiple stochastic integral expressions to Wick power and the Albeverio Høegh–Krohn Wick exponential (cf. [8,10,11]) of Nelson's Euclidean free field, which are $\mathscr{S}'$-valued random variables on $(\Omega, \mathscr{F}, P)$. (For the consideration of the Albeverio Høegh–Krohn trigonometric functions see Remark 1.5 in this section and Theorem 3.1 in Section 3).

Theorem 1.1. (i) *Suppose that the positive integer p and the positive real numbers a, b and α satisfy*

$$\min\left(1, \frac{4a}{d}\right) + p \times \min\left(1, \frac{4\alpha}{d}\right) > p, \quad b > d. \tag{25}$$

Then $\{:_k \phi^p_{\alpha,\omega} :\}_{k \in \boldsymbol{N}}$ *is a Cauchy sequence in* $L^2(\Omega \to B^{a,b}_d; P)$ (*cf. Remark 1.2*) *and there exists a* $B^{a,b}_d$*-valued random variable* $:\phi^p_{\alpha,\omega} : \in L^2(\Omega \to B^{a,b}_d, P)$ *such that*

$$\lim_{k \to \infty} \int_\Omega \| :_k \phi^p_{\alpha,\omega} : - : \phi^p_{\alpha,\omega} : \|^2_{B^{a,b}_d} P(d\omega) = 0, \tag{26}$$

$$P(\langle : \phi^p_{\alpha,\omega} :, \varphi \rangle_{\mathscr{S}', \mathscr{S}} = l_{p,\omega}(\varphi)) = 1, \quad \forall \varphi \in \mathscr{S}(\boldsymbol{R}^d), \tag{27}$$

where

$$l_{p,\omega}(\varphi) = \int_{(\boldsymbol{R}^d)^p} \left(\int_{\boldsymbol{R}^d} \varphi(x)\, J^\alpha(x - y_1) \cdots J^\alpha(x - y_p)\, dx \right) dW_\omega(y_1) \cdots dW_\omega(y_p).$$

(ii) *Let* $\alpha \geq \frac{d}{4}$ *and* $|\varepsilon| < a_0(d)$, *where* $a_0(d) = \sqrt{\frac{d(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}{2}}$. *For* $F_{k,l}(x; y_1, \ldots, y_p) = (\eta_l(x))^p J_k^\alpha(x - y_1) \cdots J_k^\alpha(x - y_p)$, *set*

$$:_{k,l}\ \phi^p_{\alpha,\omega} : (x) = \int_{(\boldsymbol{R}^2)^p} F_{k,l}(x; y_1, \cdots, y_p)\, dW_\omega(y_1) \cdots dW_\omega(y_p),$$

and define $: e^{\varepsilon_{k,l}\phi_{\alpha,\omega}} := \sum_{p=0}^{\infty} \frac{\varepsilon^p}{p!} :_{k,l} \phi^p_{\alpha,\omega} :$. *Then* $\{: e^{\varepsilon_{k,l}\phi_{\alpha,\omega}} :\}_{k \in \boldsymbol{N}}$ *is a Cauchy sequence in* $L^2(\Omega \to B_d^{a,b}; P)$ *and there exists a* $B_d^{a,b}$*-valued random variable* $: e^{\varepsilon\phi_{\alpha,\omega}} :$ *such that*

$$\lim_{k \to \infty} \lim_{l \to \infty} \int_\Omega \| : e^{\varepsilon_{k,l}\phi_{\alpha,\omega}} : - : e^{\varepsilon\phi_{\alpha,\omega}} : \|^2_{B_d^{a,b}} P(d\omega) = 0, \tag{28}$$

where b is an arbitrary positive number such that $b > d$ *and a is any positive number such that*

$$\frac{\varepsilon^2 d}{4(a_0(d))^2} < a < \frac{d}{4}.$$

The proof of Theorem 1.1 will be given in the appendix. By Remark 1.2 and (16), since the $C_0(\boldsymbol{R}^d \to \boldsymbol{R})$-valued random variable $:_k \phi^p_{\alpha,\omega} :$ can be understood as a $B_d^{a,b}$ $(a, b > 0)$-valued random variable by making use of its multiple stochastic integral expression, it is easy to see that this random variable is in $L^2(\Omega \to B_d^{a,b}; P)$. Then by making use of a Fubini type theorem concerning the stochastic integral, resp. Lebesgue integral, on $\boldsymbol{R}^d$, the theorem follows.

In the sequel we shall denote $:_k \phi^1_{\alpha,\omega} :$ and $: \phi^1_{\alpha,\omega} :$ by ${}_k\phi_{\alpha,\omega}$ and $\phi_{\alpha,\omega}$, respectively. In particular when $\alpha = \frac{1}{2}$, then for each given d the $\mathscr{S}'(\boldsymbol{R}^d)$-valued random variable (cf. Theorem 1.1) $\phi_{\frac{1}{2},\omega}$ is a stochastic integral expression for Nelson' s free Euclidean field, we denote it simply by ϕ_ω and we write

$$\phi_\omega = J^{\frac{1}{2}}\, \dot{W}_\omega.$$

Now, by making use of the above results and notations let us study non-linear shifts on Nelson's free field in the context of abstract Wiener spaces. For given d, let μ be the probability law of $\phi_\omega = \phi_{\frac{1}{2},\omega}$. Since ϕ_ω is a $B_d^{a,b}$-valued random variable ($a > \frac{d-2}{4}, b > d$ by Theorem 1.1) on $(\Omega, \mathscr{F}, P)$, μ is a probability measure on $B_d^{a,b}$:

$$\mu(A) = P(\{\omega | \phi_\omega \in A\}), \quad A \in \mathscr{B}(B_d^{a,b}) \quad \left(a > \frac{d-2}{4}, b > d\right). \tag{29}$$

We remark that for the complete probability space $(\Omega, \mathscr{F}, P)$ defined by (17), the following holds (cf. for e.g. [31]): If we let

$$\mathscr{B}^{\mu} = \{A | \{\omega | \phi_{\omega} \in A\} \in \mathscr{F}\},$$

then the probability space $(B_d^{a,b}, \mathscr{B}^{\mu}, \mu)$ is a complete probability space, i.e.

$$\mathscr{B}^{\mu} = \overline{\mathscr{B}(B_d^{a,b})}^{\mu} = \text{the completion of } \mathscr{B}(B_d^{a,b}) \text{ with respect to } \mu. \quad (30)$$

Hence, the map τ_k defined by (33) below is a $B_d^{a',b}$-valued random variable on $(B_d^{a,b}, \mathscr{B}^{\mu}, \mu)$.

Theorem 1.2. (i) *Suppose that a, β, a', p and b satisfy*

$$\min\left(1, \frac{4a}{d}\right) + \min\left(1, \frac{2}{d}\right) > 1, \quad (31)$$

$$\min\left(1, \frac{4a'}{d}\right) + p \times \min\left(1, \frac{4\beta}{d}\right) > p, \quad b > d. \quad (32)$$

For each k let $\tau_k = \tau_{(\beta,p),k}$ be the measurable map from $B_d^{a,b}$ to $B_d^{a',b}$ defined by

$$\tau_k(\psi)(x) = p!(\eta_k(x))^p \sum_{n=0}^{[\frac{p}{2}]} \frac{(-\frac{1}{2}c_{\beta,k})^n}{n!(p-2n)!}$$
$$(\langle J_k^{\beta}(x-\cdot), (J^{-\frac{1}{2}}\psi)(\cdot)\rangle_{\mathscr{S},\mathscr{S}'})^{p-2n}, \text{for } \psi \in B_d^{a,b}, \quad (33)$$

where

$$c_{\beta,k} = \int_{\boldsymbol{R}^d} (J_k^{\beta}(y))^2 \, dy.$$

Then

$$P(\{\omega \mid \tau_k(\phi_{\omega})(x) = :_k \phi_{\beta,\omega}^p : (x) \quad \forall x \in \boldsymbol{R}^d\}) = 1, \quad (34)$$

the $B_d^{a',b}$-valued measurable functions $\{\tau_k(\psi)\}$ on $(B_d^{a,b}, \mathscr{B}^{\mu}, \mu)$ form a Cauchy sequence in the Banach space $L^2(B_d^{a,b} \to B_d^{a',b}; \mu)$, and there exists a $\mathscr{B}(B_d^{a',b})/\mathscr{B}^{\mu}$-measurable function $\tau = \tau_{(\beta,p)} \in L^2(B_d^{a,b} \to B_d^{a',b}; \mu)$ such that

$$\lim_{k \to \infty} \int_{B_d^{a,b}} \|\tau_k(\psi) - \tau(\psi)\|^2_{B_d^{a',b}} \mu(d\psi) = 0, \quad (35)$$

or equivalently

$$\lim_{k \to \infty} \int_{\Omega} \|\tau_k(\phi_{\omega}) - \tau(\phi_{\omega})\|^2_{B_d^{a',b}} P(d\omega) = 0. \quad (36)$$

Moreover one has

$$\tau(\phi_{\omega}) = : \phi_{\beta,\omega}^p :, \quad P\text{-a.s.}, \ \omega \in \Omega. \quad (37)$$

(ii) *For* $\beta \geqslant \frac{d}{4}$ *and* $|\varepsilon| < a_0(d)$ (*with* $a_0(d)$ *defined in Theorem 1.1*) *and for each* k, l *let*

$$\tau^{k,l}_{(\beta,e^{\varepsilon})}(\psi) = \sum_{p=0}^{\infty} \frac{\varepsilon^p}{p!} \tau_{(p),k,l}(\psi),$$

where $\tau_{(p),k,l}$ *is defined by* (33) *in which* η_k *is replaced by* η_l. *Then by Theorem 1.1 there exists a* $\mathscr{B}(B_d^{a',b})/\mathscr{B}^{\mu}$*-measurable function* $\tau_{(\beta,e^{\varepsilon})} \in L^2(B_d^{a,b} \to B_d^{a',b}; \mu)$ *such that*

$$\lim_{k\to\infty} \lim_{l\to\infty} \int_{B_d^{a,b}} \|\tau^{k,l}_{(\beta,e^{\varepsilon})}(\psi) - \tau_{(\beta,e^{\varepsilon})}(\psi)\|^2_{B_d^{a',b}} \mu(d\psi) = 0, \tag{38}$$

and the following holds:

$$\tau_{(\beta,e^{\varepsilon})}(\phi_{\omega}) =: e^{\varepsilon\phi_{\beta,\omega}}:, \quad P\text{-a.s.}, \ \omega \in \Omega, \tag{39}$$

where a satisfies (31) *and* a' *is any number satisfying* $\frac{\varepsilon^2 d}{4(a_0(d))^2} < a' < \frac{d}{4}$ *and* $b > d$.

By the definition of Wick power and multiple stochastic integral (34) can easily be proved. The existence of τ is proved by using Theorem 1.1 and (34), these proofs will be given in the appendix.

Remark 1.3. By Theorem 1.2 we have the following identifications: The random variable $\tau_{(\frac{1}{2},1)}(\psi)$ on the probability space $(B_d^{a,b}, \mathscr{B}^{\mu}, \mu)$ can be identified with the random variable ϕ_{ω} on the probability space $(\Omega, \mathscr{F}, P)$. μ is then Nelson's free field measure, $\tau_{(\frac{1}{2},1)}(\psi)$ is Nelson's free field (cf. [41,52], also cf. (40) and Remark 1.4). Similarly (for $d = 2$) the random variable: $\phi^p_{\frac{1}{2},\omega}$: on $(\Omega, \mathscr{F}, P)$ can be identified with Nelson's free field Wick power $\tau_{(\frac{1}{2},p)}(\psi)$ on the probability space $(B_d^{a,b}, \mathscr{B}^{\mu}, \mu)$.

Next, we shall see that Nelson's Euclidean free field possesses the structure of an abstract Wiener space, and then show that the maps $\tau_{(\beta,p)}$ and $\tau_{(\beta,e^{\varepsilon})}$ on the abstract Wiener space have sufficient regularities. Definitions of the notations and terminologies concerning an abstract Wiener space (e.g., definitions of an abstract Wiener space, Gross–Sobolev derivative ∇, divergence operator δ, the *Sobolev space* $\boldsymbol{D}_{p,k}$ and $H - C^1$ maps) can be found in Definitions A.1–A.3 and Remark A.1 in the appendix.

As usual let $H^{\gamma} = H^{\gamma}(\boldsymbol{R}^d)$ be the Sobolev space on $\boldsymbol{R}^d$ such that

$$H^{\gamma}(\boldsymbol{R}^d) = \left\{ \phi \in \mathscr{S}'(\boldsymbol{R}^d) \,\middle|\, \int_{\boldsymbol{R}^d} |\mathscr{F}\phi|^2(x)(1+|x|^2)^{\gamma}\, dx < \infty \right\}.$$

In order to make the notations simple, we equip $H^{\gamma}(\mathbf{R}^d)$ with the inner product

$$\langle u, v \rangle_{H^{\gamma}} = (2\pi)^{-d} \int_{\mathbf{R}^d} (\mathscr{F}u)(x)(\mathscr{F}v)(x)(m^2 + |x|^2)^{\gamma}\, dx$$

for a given constant $m > 0$ (interpreted as "mass parameter").

Then by Theorem 1.1 for $a > \frac{d}{4} - \frac{1}{2}$ we see that $(B_d^{a,b}, \mu)$ is an abstract Wiener space and one has, for $\varphi \in \mathscr{S}(\mathbf{R}^d)$:

$$\begin{aligned}
&\int_{B_d^{a,b}} e^{\sqrt{-1}\langle \psi, \varphi \rangle_{\mathscr{S}', \mathscr{S}}} \mu(d\psi) \\
&\quad = \int_{\Omega} \exp\left[\sqrt{-1} \int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^d} \varphi(x) J^{\frac{1}{2}}(x-y)\, dx \right) dW_{\omega}(y)\right] P(d\omega) \\
&\quad = \exp\left(-\frac{1}{2}\|\varphi\|^2_{H^{-1}}\right) = \exp\left(-\frac{1}{2}\|J^1\varphi\|^2_{H^1}\right). \qquad (40)
\end{aligned}$$

The inclusion map $i : H^{-1} \to B_d^{a,b}$ defined by

$$i(h) = J^1 h, \quad h \in H^{-1} \qquad (41)$$

is continuous and $i(H^{-1}) = H^1$ is dense in $B_d^{a,b}$. By this we can identify H^{-1} with H^1, and we have the following continuous injection (cf. Definition A.1 in the appendix):

$$(B_d^{a,b})^* \hookrightarrow H^{-1} \cong H^1 \hookrightarrow B_d^{a,b}.$$

Setting

$$\mathscr{H} = H^{-1}$$

we will consider the abstract Wiener space $(B_d^{a,b}, i(\mathscr{H}), \mu)$ with Cameron–Martin space

$$i(\mathscr{H}) = J^1 H^{-1} = H^1. \qquad (42)$$

We then apply the results given by [55] concerning the (non-linear) shifts on Wiener spaces to the maps τ defined above.

Remark 1.4. Nelson's Euclidean free field is defined originally as a Gaussian process indexed by $\mathscr{H} = H^{-1}$ (cf. [41]), i.e. Gaussian process with the index set H^{-1} of which characteristic function is

$$\exp\left(-\frac{1}{2}\|\varphi\|^2_{H^{-1}}\right), \quad \varphi \in H^{-1} \text{ (cf. (40))}.$$

By this, here we prefer to denote the Cameron–Martin space by $i(\mathscr{H})$, and denote the abstract Wiener space by $(B_d^{a,b}, i(\mathscr{H}), \mu)$. Then our calculus on the abstract Wiener space will be performed through $\mathscr{H}$.

By remarking that the Cameron–Martin space H (in the present case $H = H^1$) is identified with its dual space denoting H^*, and not explicitly denoting the identity map i, then the general statements concerning an abstract Wiener space will be simplified and clarified. For e.g., if a shift T on an abstract Wiener space is defined by $T(\psi) = \psi + v(\psi)$, where v is a Cameron–Martin space-valued random variable (i.e. in the present case it is an H^1-valued random variable), then the properties of T, the corresponding Girsanov measure (cf. Section 2), and the Radon–Nikodym densities can be expressed without complicated notations. We will formulate a shift T on the abstract Wiener space $(B_d^{a,b}, i(\mathscr{H}), \mu)$ as follows: $T(\psi) = \psi + i(u(\psi))$, where u is a random variable that takes values in the dual space of the Cameron–Martin space (i.e. it is an $\mathscr{H} = H^{-1}$-valued random variable, and then $i(u) = J^1 u$ is an H^1-valued random variable). By introducing explicitly the identity map i we can interpret the shift T as an inverse operator of a differential operator with a non-linear perturbation term: $T(\psi) = \psi + (-\Delta + m^2)^{-1} u(\psi)$. Accordingly, we can consider the SPDEs on Nelson's Euclidean free field through such shift T.

Since, the identity map i plays the crucial role in the present study on the SPDEs on the abstract Wiener space, we give the definitions and notions corresponding to the abstract Wiener space by denoting explicitly the identity map i in Definitions A.1–A.3 and Remark A.1 in the appendix.

We should also notice that the essential elements in the abstract Wiener space are the Banach space $B_d^{a,b}$, the Gaussian measure μ on it and the Cameron–Martin space H^1 (or its continuous dual Hilbert space $\mathscr{H} = H^{-1}$ with the continuous injection i). Then, the Hilbert space, on which the calculus are performed, can be taken rather flexibly, as far as the continuous injection i is specified (cf. [44, Section 4.1], where by giving a continuous injection i from $L^2(\boldsymbol{R}^d, \lambda^d)$ to the Cameron–Martin space of an abstract Wiener space, calculus on the abstract Wiener space is carried out on $L^2(\boldsymbol{R}^d, \lambda^d)$).

In order to make the subsequent discussions clear we shall fix nice representatives for the random variables $\tau_{(\beta,p)}$ and $\tau_{(\beta,e^\varepsilon)}$.

Definition 1.1 (Representatives for $\tau_{(\beta,e^\varepsilon)}$ and $\tau_{(\beta,p)}$). (i) By (38) there exists a $\mathscr{B}(B_d^{a',b})/\mathscr{B}^\mu$-measurable function $\tau_{(\beta,e^\varepsilon)} \in L^2(B_d^{a,b} \to B_d^{a',b}; \mu)$, a subsequence $\{\tau_{(\beta,e^\varepsilon)}^{k_j,l_i}\}$ of $\{\tau_{(\beta,e^\varepsilon)}^{k,l}\}$ and a set $B(\beta,e) \in \mathscr{B}^\mu$ satisfying $\mu(B(\beta,e)) = 1$ such that

$$\lim_{k_j \to \infty} \lim_{l_i \to \infty} \|\tau_{(\beta,e^\varepsilon)}^{k_j,l_i}(\psi) - \tau_{(\beta,e^\varepsilon)}(\psi)\|_{B_d^{a',b}}^2 = 0, \quad \forall \psi \in B(\beta,e).$$

Denote by $\bar{B}(\beta,e)$ the set consisting of all $\psi \in B_d^{a,b}$ such that the limit $\lim_{k_j \to \infty} \lim_{l_i \to \infty} \tau_{(\beta,e^\varepsilon)}^{k_j,l_i}(\psi)$ exists in $B_d^{a'',b}$ for some $a' \leqslant a''$. Then $\bar{B}(\beta,e)$ is $\mathscr{B}^\mu$-measurable. In the sequel we fix a representative $\bar{\tau}_{(\beta,e^\varepsilon)}$ of $\tau_{(\beta,e^\varepsilon)}$ defined as follows:

$$\bar{\tau}_{(\beta,e^{\varepsilon})} = \begin{cases} \lim_{k_j\to\infty} \ \lim_{l_i\to\infty} \ \tau^{k_j,l_i}_{(\beta,e^{\varepsilon})}(\psi), & \psi\in\bar{B}(\beta,e), \\ 0 & \text{elsewhere.} \end{cases}$$

$\bar{\tau}_{(\beta,e^{\varepsilon})}$ will be simply denoted by $\tau_{(\beta,e^{\varepsilon})}$

(ii) For each p by (35) we can take subsequences $\{\tau_{(\beta,1),k_j}\},\dots,\{\tau_{(\beta,p),k_j}\}$ and a set $B(\beta,p)\in\mathscr{B}^{\mu}$ satisfying $\mu(B(\beta,p))=1$ such that

$$\lim_{k_j\to\infty} \|\tau_{(\beta,q),k_j}(\psi)-\tau_{(\beta,q)}(\psi)\|^2_{B^{a,b}_d}=0, \quad \forall\psi\in B(\beta,p), \ \ q=1,\dots,p.$$

We denote by $\bar{B}(\beta,p)$ the set of all $\psi\in B^{a,b}_d$ such that the limits $\lim_{k_j\to\infty}\tau_{(\beta,q),k_j}(\psi)$ exist, $q=1,\dots,p$, in $B^{a'',b}_d$ for some $a\leqslant a''$. Then $\bar{B}(\beta,p)$ is $\mathscr{B}^{\mu}$-measurable. In the sequel we fix a representative $\bar{\tau}_{(\beta,p)}$ of $\tau_{(\beta,p)}$ defined as follows:

$$\bar{\tau}_{(\beta,p)} = \begin{cases} \lim_{k_j\to\infty} \ \tau_{(\beta,p),k_j}(\psi), & \psi\in\bar{B}(\beta,p), \\ 0 & \text{elsewhere.} \end{cases}$$

$\bar{\tau}_{(\beta,p)}$ will be simply denoted by $\tau_{(\beta,p)}$.

Theorem 1.3 (Polynomial and exponential $H-C^1$ maps). *Let $b>d$ and a be a number such that $a>\frac{d}{4}-\frac{1}{2}$. Let $(B^{a,b}_d, i(\mathscr{H}),\mu)$ be the abstract Wiener space defined above, and denote the "Gross–Sobolev derivative" and "divergence" operators on $(B^{a,b}_d, i(\mathscr{H}),\mu)$ by ∇ and δ, respectively (cf. Definition A.2). For $M\geqslant 0$ let η_M be the space-cut-off such that $\eta_M(x)=\eta_1(\frac{x}{M})$* (cf. (18)).

(i.1°) *Let the integer p and the real number $\beta>0$ satisfy*

$$\beta>\frac{d}{4}\frac{p+1}{p+2}. \tag{43}$$

Then the map $u_p(\psi)=J^{\beta-\frac{1}{2}}(\eta_M\tau_{(\beta,p)}(\psi))$ ($\mathscr{H}$-valued Wiener functional) is an element of $\boldsymbol{D}_{2,k}(\mathscr{H})$ ($\forall k\geqslant 1$) (cf. Definition A.2), and the following holds:

$$\begin{aligned}\nabla u_p(\psi)(x,y) &= p\langle\eta_M,\tau_{(\beta,p-1)}(\psi)(\cdot)J^{\beta-\frac{1}{2}}(\cdot-x)J^{\beta-\frac{1}{2}}(\cdot-y)\rangle_{\mathscr{S},\mathscr{S}'} \\ &\in L^2(\mathscr{H}\otimes\mathscr{H};\mu).\end{aligned}$$

Let $B(\beta,p)$ be as in Definition 1.1-(ii) *for these p and β, then $\mu(B(\beta,p))=1$ and $B(\beta,p)+H^1\subset\bar{B}(\beta,p)$.*

The divergence of u_p is given by

$$\delta u_p(\psi)=\langle\eta_M,\tau_{(\beta,p+1)}(\psi)\rangle_{\mathscr{S},\mathscr{S}'}, \quad \mu\text{-a.s.}, \ \ \psi\in B^{a,b}_d. \tag{44}$$

(i.2°) *If*

$$\beta > \frac{d}{4}\left(\frac{p-2}{p-1} + \frac{2}{3(p-1)}\right) \tag{45}$$

(*which is a particular case of i.* 1°)), *then*

$$\begin{aligned}\nabla u_p(\psi + i(h))(x,y) &= p \sum_{q=0}^{p-1} \binom{p-1}{q} \langle \eta_M, (J^{\beta-\frac{1}{2}}(i(h)))^q \tau_{(\beta,p-1-q)}(\psi)(\cdot) \\ &\quad \times J^{\beta-\frac{1}{2}}(\cdot - x) J^{\beta-\frac{1}{2}}(\cdot - y) \rangle_{\mathscr{S},\mathscr{S}'}, \quad \forall \psi \in B(\beta,p),\ \forall h \in \mathscr{H},\end{aligned} \tag{46}$$

u_p *is an* $H - C^1$ *map on* $(B_d^{a,b}, i(\mathscr{H}), \mu)$ (*cf. Definition A.3*):

$$\mathscr{H} \ni h \mapsto \nabla u_p(\psi + i(h)) \in \mathscr{H} \otimes \mathscr{H} \quad \text{is continuous for all } \psi \in B(\beta,p). \tag{47}$$

(ii) *Let* $\beta \geqslant \frac{d}{4}$ *and set* $u_e(\psi) = J^{\beta-\frac{1}{2}}(\eta_M \tau_{(\beta,e^{\varepsilon})}(\psi))$.

(ii.1°) *Suppose that* $|\varepsilon| < a_0(d)$, *then the map* u_e *is an element of* $\boldsymbol{D}_{2,1}(\mathscr{H})$ (*cf. Definition A.2*):

$$\begin{aligned}\nabla u_e(\psi) = \varepsilon \langle \eta_M(\cdot), \tau_{(\beta,e^{\varepsilon})}(\psi)(\cdot) J^{\beta-\frac{1}{2}}(\cdot - x) J^{\beta-\frac{1}{2}}(\cdot - y) \rangle_{\mathscr{S},\mathscr{S}'} \in L^2 \\ \times (\mathscr{H} \otimes \mathscr{H}; \mu).\end{aligned} \tag{48}$$

The divergence of u_e *is given by*

$$\delta u_e(\psi) = \left\langle \eta_M, \sum_{p=0}^{\infty} \frac{\varepsilon^p}{p!} \tau_{(\beta,p+1)}(\psi) \right\rangle_{\mathscr{S},\mathscr{S}'}, \quad \mu\text{-a.s.},\ \psi \in B_d^{a,b}. \tag{49}$$

(ii.2°) *Suppose that*

$$|\varepsilon| < \frac{a_0(d)}{\sqrt{2}}. \tag{50}$$

Let $B(\beta,e)$ *be as in Definition 1.1*(i). *Then* $B(\beta,e) + H^1 \subset \bar{B}(\beta,e)$ *and* u_e *is an* $H - C$ *map on* $(B_d^{a,b}, i(\mathscr{H}), \mu)$ (*cf. Definition A.3*):

$$\mathscr{H} \ni h \mapsto u_e(\psi + i(h)) \in \mathscr{H} \quad \text{is continuous for } \forall \psi \in B(\beta,e).$$

(ii.3°) *Suppose that*

$$|\varepsilon| < \frac{a_0(d)}{\sqrt{3}}, \tag{51}$$

then u_e *is an* $H - C^1$ *map on* $(B_d^{a,b}, i(\mathscr{H}), \mu)$ (*cf. Definition A.3*):

$$\mathscr{H} \ni h \mapsto \nabla u_e(\psi + i(h)) \in \mathscr{H} \otimes \mathscr{H} \quad \text{is continuous for all } \psi \in B(\beta,e),$$

$$\nabla u_e(\psi + i(h)) = \varepsilon \langle \eta_M(\cdot), e^{\varepsilon \bar{h}(\cdot)} \tau_{(\beta,e^{\varepsilon})}(\psi)(\cdot) J^{\beta-\frac{1}{2}}(\cdot - x) J^{\beta-\frac{1}{2}}(\cdot - y) \rangle_{\mathscr{S},\mathscr{S}'}$$
$$\forall \psi \in B(\beta,e),\ \forall \bar{h} = J^{\beta-\frac{1}{2}}(i(h)) \quad \text{with } h \in \mathscr{H}.$$

In the case $\beta = \frac{1}{2}$ *we take* $J^{\beta-\frac{1}{2}}(x) = \delta_{\{0\}}(x)$ (*with* $\delta_{\{0\}}$ *the Dirac point measure at* $\{0\}$).

This theorem can be proved by a simple application of Fourier transforms and Young's inequality, see the proof in the Appendix and Remark A.1.

Remark 1.5 (Measurable maps corresponding to the Wick trigonometric functions). Suppose that the numbers β and ε satisfy the assumptions of Theorem 1.3.(ii.1°) (i.e. $\beta \geqslant \frac{d}{4}$, $|\varepsilon| < a_0(d)$ with $a_0(d)$ defined in Theorem 1.1(ii)). Then similarly as in Theorem 1.2(ii) it is possible to define measurable maps $\tau_{(\beta,\sin)}$ and $\tau_{(\beta,\cos)}$ on $B_d^{a,b}$ such that

$$\tau_{(\beta,\sin)}(\psi) = \lim_{k\to\infty} \lim_{l\to\infty} \sum_{p=0}^{\infty} \frac{(-1)^p \varepsilon^{2p+1}}{(2p+1)!} \tau_{(2p+1),k,l}(\psi),$$

$$\tau_{(\beta,\cos)}(\psi) = \lim_{k\to\infty} \lim_{l\to\infty} \sum_{p=0}^{\infty} \frac{(-1)^p \varepsilon^{2p}}{(2p)!} \tau_{(2p),k,l}(\psi),$$

where for given $\beta \geqslant \frac{d}{4}$ the map $\tau_{(p),k,l}(\psi)$ appearing on the right-hand side is defined by (33) in which η_k is replaced by η_l.

These are, respectively, the expression of the Albeverio Høegh–Krohn sin and cos perturbations (cf. [10]) by means of random variables on the Wiener space $(B_d^{a,b}, i(\mathscr{H}), \mu)$. Moreover it is possible to show that (cf. the proof of Theorem 1.3) the maps $u_s(\psi) = J^{\beta-\frac{1}{2}}(\eta_M \tau_{(\beta,\sin)}(\psi))$ resp. $u_c(\psi) = J^{\beta-\frac{1}{2}}(\eta_M \tau_{(\beta,\cos)}(\psi))$ are elements of $\boldsymbol{D}_{2,1}(\mathscr{H})$, $H-C$, resp. $H-C^1$ continuous (cf. Definitions A.2 and A.3) under the same assumptions of $\tau_{(\beta,e^\varepsilon)}$ given in Theorem 1.3(ii).

Definition 1.2. For $u \in \boldsymbol{D}_{2,1}(\mathscr{H})$ and $\lambda \in \boldsymbol{R}$ we define

$$\Lambda_{\lambda u}(\psi) = \det{}_2(I_{\mathscr{H}} + \lambda \nabla u(\psi)) \exp\left(-\lambda \delta u(\psi) - \frac{\lambda^2}{2} |u(\psi)|_{\mathscr{H}}^2\right), \tag{52}$$

where $\det_2(I_{\mathscr{H}} + \lambda \nabla u(\psi))$ denotes the Carleman–Fredholm determinant of the Hilbert–Schmidt operator $\lambda \nabla u(\psi) \in \mathscr{H} \otimes \mathscr{H}$ (cf. Definition A.2) and $|\ |_{\mathscr{H}}$ denotes the norm of the Hilbert space $\mathscr{H}$.

2. Main results for SPDEs with cubic and exponential perturbations

In this section we shall consider elliptic SPDEs on $\boldsymbol{R}^d$ formally given by

$$(-\Delta + m^2)\psi(x) + \lambda \eta_M(x) : \psi^3(x) := (-\Delta + m^2)^{\frac{1}{2}} \dot{W}(x), \quad x \in \boldsymbol{R}^d, \tag{53}$$

resp.

$$(-\Delta + m^2)\psi(x) + \lambda\eta_M(x) : e^{\varepsilon\psi}(x) := (-\Delta + m^2)^{\frac{1}{2}}\dot{W}(x), \quad x \in \boldsymbol{R}^d, \tag{54}$$

where $\eta_M(x) = \eta_1(\frac{x}{M})$ is the "space-cut-off" defined by (18), and W is an isonormal Gaussian process on $\boldsymbol{R}^d$. Using the measurable maps defined by Theorem 1.2 and Definition 1.1, the above SPDEs can be written in the following form:

$$(-\Delta + m^2)\psi(x) + \lambda\eta_M(x)\tau_{(\frac{1}{2},3)}(\psi)(x) = (-\Delta + m^2)^{\frac{1}{2}}\dot{W}(x), \quad x \in \boldsymbol{R}^d, \tag{55}$$

resp.

$$(-\Delta + m^2)\psi(x) + \lambda\eta_M(x)\tau_{(\frac{1}{2},e^{\varepsilon})}(\psi)(x) = (-\Delta + m^2)^{\frac{1}{2}}\dot{W}(x), \quad x \in \boldsymbol{R}^d. \tag{56}$$

We reduce the existence problem of the solution of (55), resp. (56), to the existence of corresponding *Girsanov measures*. We shall adopt the notion of "Girsanov measure" given in Section 1.3 of [55] for our problem as follows. Let S be a topological space and $\mathscr{B}(S)$ be its Borel σ-field. Let μ be a complete probability measure on $(S, \overline{\mathscr{B}(S)}^{\mu})$, and let T be a measurable map such that $T : (S, \overline{\mathscr{B}(S)}^{\mu}) \mapsto (S, \mathscr{B}(S))$, where $\overline{\mathscr{B}(S)}^{\mu}$ ="the completion of $\mathscr{B}(S)$ with respect to μ". A signed measure ν on $(S, \overline{\mathscr{B}(S)}^{\mu})$ will be called as a "*Girsanov measure on* $(S, \overline{\mathscr{B}(S)}^{\mu})$ *associated with* μ *and* T" if and only if it satisfies

$$\int_S f(T\phi)\, d\nu(\phi) = \int_S f(\phi)\, d\mu(\phi) \quad \text{for any bounded measurable} \quad f : (S, \mathscr{B}(S)) \mapsto (\boldsymbol{R}, \mathscr{B}(\boldsymbol{R})). \tag{57}$$

In particular if such a signed measure ν is a probability measure on $(S, \overline{\mathscr{B}(S)}^{\mu})$, then this will be called the "*Girsanov probability measure on* $(S, \overline{\mathscr{B}(S)}^{\mu})$ *associated with* μ *and* T".

Remark 2.1. (i) If a "*Girsanov probability measure* ν *on* $(S, \overline{\mathscr{B}(S)}^{\mu})$ *associated with* μ *and* T" exists, then by (57) the probability law of $T\phi$ under ν is μ. In other words, for a random variable ϕ taking values in S with probability law ν there exists a random variable ψ with probability law μ, and the following holds:

$$T\phi = \psi.$$

In case ν is not a probability measure but a signed *Girsanov measure on* $(S, \overline{\mathscr{B}(S)}^{\mu})$ *associated with* μ *and* T, if we set $\mathscr{B}_T \equiv \{T^{-1}A | A \in \mathscr{B}(S)\}$, and restrict ν to $\mathscr{B}_T$, then $\nu|_{\mathscr{B}_T}$ is a probability measure on $(S, \mathscr{B}_T)$ and the probability law of $T\phi$ under ν is μ. Such signed measures may be important to be considered in relation with the indefinite metric quantum field theory (cf. [3,4] and Remark 4.1 in Section 4).

Let μ be the probability law of Nelson's free field ϕ on $\boldsymbol{R}^d$, then μ is a complete probability measure on $(B_d^{a,b}, \mathscr{B}^\mu)$ (cf. (30)). Let T be the map defined on $B_d^{a,b}$ such that

$$T(\psi) = \psi + J^1(\lambda \eta_M \tau_{(\frac{1}{2},3)}(\psi)), \quad \psi \in B_d^{a,b}.$$

We may set $S = B_d^{a,b}$ and $\mathscr{B}(S) = \mathscr{B}(B_d^{a,b})$ in the above general discussion. If there exists v which is a "*Girsanov probability measure on* $(B_d^{a,b}, \mathscr{B}^\mu)$ *associated with* μ *and* T", then for a $B_d^{a,b}$-valued random variable ψ with probability law v there exists a Nelson's free field ϕ on $\boldsymbol{R}^d$ and the following holds:

$$\psi + J^1(\lambda \eta_M \tau_{(\frac{1}{2},3)}(\psi)) = \phi.$$

Since ϕ can be expressed by $\phi = J^{\frac{1}{2}} \dot{W}$ for some isonormal Gaussian process W on $\boldsymbol{R}^d$, in the sense of distribution-valued random variables this equation means that

$$(-\Delta + m^2)\psi(x) + \lambda \eta_M(x) \tau_{(\frac{1}{2},3)}(\psi)(x) = (-\Delta + m^2)^{\frac{1}{2}} \dot{W}(x), \quad x \in \boldsymbol{R}^d. \quad (58)$$

By this way we can reduce the existence problem of the solution of the SPDE (58) to the existence problem of the corresponding *Girsanov probability measure*.

In general we give the following definition

Definition 2.1 (Solution of SPDE). For given d let $(B_d^{a,b}, i(\mathscr{H}), \mu)$ be the abstract Wiener space, which is Nelson's Euclidean free field, defined in Section 1. For an $\mathscr{H}$ valued $\mathscr{B}^\mu$-measurable function $u : B_d^{a,b} \mapsto \mathscr{H}$ and for some $\lambda \in \boldsymbol{R}$ (note that by Theorem 1.3 $u(\psi) = \eta_M \tau_{(\beta,p)}(\psi)$ and $u(\psi) = \eta_M \tau_{(\beta,e^\varepsilon)}(\psi)$ satisfy this measurability condition) set

$$T(\psi) = \psi + \lambda J^1(u(\psi)), \quad \psi \in B_d^{a,b}.$$

We say that a probability measure v on $(B_d^{a,b}, \mathscr{B}^\mu)$ gives a solution of the SPDE

$$(-\Delta + m^2)\psi(x) + \lambda u(\psi)(x) = (-\Delta + m^2)^{\frac{1}{2}} \dot{W}(x), \quad x \in \boldsymbol{R}^d,$$

where W is an isonormal Gaussian process on $\boldsymbol{R}^d$, if and only if v is a *Girsanov probability measure on* $(B_d^{a,b}, \mathscr{B}^\mu)$ *associated with* μ *and* T.

Remark 2.2 (Inverse shift). From the projection theorem (cf. [21, Theorem III.23] [55, Theorem 4.2.1]) for a $\mathscr{B}^\mu$ measurable shift $T(\psi) = \psi + J^1(\lambda u(\psi))$ with a $\mathscr{B}^\mu$ measurable $H^1 = J^1(\mathscr{H})$-valued function $J^1 u$, we see that $T(A)$ is in the universally completed σ algebra of $\mathscr{B}^\mu$ for all $A \in \mathscr{B}^\mu$. But, by (30) since

$(B_d^{a,b}, \mathscr{B}^\mu, \mu)$ is already a complete probability space, we have $T(A) \in \mathscr{B}^\mu$ for all $A \in \mathscr{B}^\mu$. Hence, if there exists a measurable map S such that

$$S : (B_d^{a,b}, \sigma_T) \mapsto (B_d^{a,b}, \mathscr{B}^\mu), \qquad S \circ T(\psi) = \psi, \quad \mu\text{-a.s.}, \ \psi \in B_d^{a,b},$$

where $\sigma_T \equiv \{T(A) : A \in \mathscr{B}^\mu\}$, then S is a $\mathscr{B}^\mu / \mathscr{B}^\mu$-measurable left inverse of T. And the probability measure ν on $(B_d^{a,b}, \mathscr{B}^\mu)$ defined by $\nu(A) = \mu(S^{-1}(A))$, $A \in \mathscr{B}^\mu$ is a Girsanov probability measure on $(B_d^{a,b}, \mathscr{B}^\mu)$ associated with μ and T.

Lemma 2.1 (Key lemma for the cubic power perturbation). *Let $d \geqslant 2$ be given, and suppose that the assumptions of Theorem 1.3*(i.1°) *hold for $p = 3$. Also take the numbers $\lambda > 0$ and $\varepsilon > 0$ to satisfy $\lambda(1+\varepsilon) < \frac{2}{9L}$, where $L = \int_{\mathbf{R}^d} (J^{2\beta}(x))^2\, dx$. Then for*

$$u(\psi) = u_3(\psi) = J^{\beta - \frac{1}{2}}(\eta_M \tau_{(\beta,3)}(\psi))$$

defined by Theorem 1.3(i.1°), *the following holds*:

$$\exp\left\{ -\lambda \delta u + \frac{1+\varepsilon}{2} \lambda^2 \|\nabla u\|_2^2 \right\} \in \bigcap_{q<\infty} L^q(\mu), \tag{59}$$

where $\| \ \|_2$ denotes the Hilbert–Schmidt norm $\| \ \|_{\mathscr{H} \otimes \mathscr{H}}$.

By making use of the fact that δu and ∇u are the 4th and 2nd Wick power of ψ, respectively, this lemma can be proved by applying Nelson's exponential bounds. The proof will be given in the appendix.

Let $\Lambda_{\lambda u}(\psi)$ be the random variable given in Definition 1.2. Then from Theorem 1.3(i.1°), for u as in Lemma 2.1 the following holds:

$$\begin{aligned} &\Lambda_{\lambda u}(\psi) \\ &\quad = \det{}_2(I_{H^{-1}} + 3\lambda \langle \eta_M(\cdot), \tau_{(\beta,2)}(\psi)(\cdot) J^{\beta-\frac{1}{2}}(\cdot - x) J^{\beta-\frac{1}{2}}(\cdot - y) \rangle_{\mathscr{S},\mathscr{S}'}) \\ &\qquad \times \exp\left\{ -\lambda \langle \eta_M, \tau_{(\beta,4)}(\psi) \rangle_{\mathscr{S},\mathscr{S}'} - \frac{\lambda^2}{2} \left| J^{\beta-\frac{1}{2}}(\eta_M \tau_{(\beta,3)}(\psi)) \right|_{H^{-1}}^2 \right\}. \end{aligned} \tag{60}$$

Lemma 2.2. *Let $a > \frac{d}{4} - \frac{1}{2}$ and $b > d$. Under the assumptions of Theorem 1.3*(i.2°) *the following holds*:

$$\Lambda_{\lambda u} \in \bigcap_{q<\infty} L^q(\mu), \qquad E^\mu[\Lambda_{\lambda u}] = 1. \tag{61}$$

Let

$$D = \{ y \in B_d^{a,b} \mid \det{}_2(I_{\mathscr{H}} + \lambda \nabla u(y)) \neq 0 \},$$

and let $N(\psi, D)$ denote the cardinality of the set $T^{-1}\{\psi\} \cap D$ for $T(\psi) = \psi + i(\lambda u(\psi))$, then $N(\psi, D)$ is a measurable function and the *following holds*:

$$\mu(\{\psi \mid 1 \leqslant N(\psi, D) < \infty\}) = 1. \tag{62}$$

Proof. First of all we recall a crucial result for $H - C^1$ maps on abstract Wiener spaces derived by Kusuoka [37] (cf. also [55, Proposition 3.5.1]): For a map u that is $H - C^1$ let T be the shift defined by Definition 2.1, then there exists a sequence of measurable sets $G_n \subset B_d^{a,b}$, $n \in \boldsymbol{N}$, such that $\bigcup_n G_n = D$, and there exists a sequence of shifts T_n, $n \in \boldsymbol{N}$, such that $T_n = T$ a.s. on G_n, T_n is bijective and the inverse T_n^{-1} is measurable.

Under the assumptions of Theorem 1.3(i.2°) since u_3 is an $H - C^1$ map, by this fundamental observation we can consider the properties of such measurable functions $N(\psi, D)$ and $\sum_{y \in T^{-1}(\psi)} \operatorname{sign}(\Lambda_{\lambda u}(y))$. Namely, in Theorem 9.3.2 and Remark 9.3.3 of [55] it is shown that if u satisfies (59) then (61) holds. On the other hand, in Theorem 9.2.4 of [55] it is shown that (59) is also a sufficient condition for u under which the following holds:

$$E^\mu[\Lambda_{\lambda u}] = \sum_{y \in T^{-1}(\psi)} \operatorname{sign}(\Lambda_{\lambda u}(y)), \quad \mu\text{-a.s.}, \ \psi \in B_d^{a,b}. \tag{63}$$

Since $\Lambda_{\lambda u}(y) = 0$ and $\operatorname{sign}(\Lambda_{\lambda u}(y)) = 0$ for $y \notin D$, by (61) and (63) we see that

$$\sum_{y \in T^{-1}(\psi) \cap \mathscr{D}} \operatorname{sign}(\Lambda_{\lambda u}(y)) = \sum_{y \in T^{-1}(\psi)} \operatorname{sign}(\Lambda_{\lambda u}(y)) = 1 \quad \mu\text{-a.s.}, \psi \in B_d^{a,b}.$$

By this we have

$$1 \leqslant \sum_{y \in T^{-1}(\psi) \cap D} |\operatorname{sign}(\Lambda_{\lambda u}(y))| = \sum_{y \in T^{-1}(\psi) \cap D} 1 = N(\psi, D),$$
$$\mu\text{-a.s.}, \quad \psi \in B_d^{a,b}.$$

On the other hand, by (61) since $E^\mu[|\Lambda_{\lambda u}|] < \infty$, and by Theorem 3.5.2 of [55] since $E^\mu[|\Lambda_{\lambda u}|] = E^\mu[N(\cdot, D)]$ (cf. also Theorem 3.1 in the next section: in Theorem 3.1 if we set $f = g \equiv 1$, then this equality follows), we have

$$N(\psi, D) < \infty \qquad \mu\text{-a.s.}, \quad \psi \in B_d^{a,b}.$$

Combining these facts we have (62).

Theorem 2.3 (Solution for the space-cut-off cubic perturbation case). *For given d and $p = 3$ take the positive numbers a, a' and β to satisfy the assumptions of Theorem 1.3*(i.2°). *Also take number $\lambda \geqslant 0$ to satisfy $\lambda < \frac{2}{9L}$, where L is the number defined in Lemma* 2.1. *For some fixed positive number M let $\eta_M(x) = \eta_1(\frac{x}{M})$* (cf. (18)), *and define*

$$T_3(\psi) = \psi + i(\lambda u_3(\psi)), \qquad u_3(\psi) = J^{\beta - \frac{1}{2}}(\eta_M \tau_{(\beta,3)}(\psi)) \tag{64}$$

and

$$d\nu_3 = q\circ T_3|\Lambda_{\lambda u_3}|\,d\mu \quad \textit{for } q \textit{ such that}$$

$$q(\psi) = \begin{cases} \frac{1}{N(\psi,D)} & \text{if } N(\psi,D)\neq 0, \\ 0 & \textit{otherwise}, \end{cases}$$

where $\Lambda_{\lambda u_3}$ *is given by* (60), *and the measurable function* $N(\psi,D)$ *is defined in Lemma* 2.2. *Then* $\Lambda_{\lambda u_3}\mu$ *is a* (*signed*) *Girsanov measure and* ν_3 *is a Girsanov* probability *measure on* $(B_d^{a,b},\mathscr{B}^\mu)$ *associated with* μ *and* T_3:

(i)

$$E^\mu[f\circ T_3\Lambda_{\lambda u_3}] = E^\mu[f], \qquad E^\nu[f\circ T_3] = E^\mu[f]\ \forall f\in C_b(B_d^{a,b}). \tag{65}$$

(ii) ν_3 *gives a solution of* (66) *below in the following sense*: *if* ψ *is a* $B_d^{a,b}$-*valued random variable with probability law* ν_3, *then the following holds for some isonormal Gaussian process* W *on* $\boldsymbol{R}^d$:

$$(-\Delta+m^2)^{1+(\beta-\frac{1}{2})}\psi(x) + \lambda\eta_M(x)\tau_{(\beta,3)}(\psi(x)) = (-\Delta+m^2)^\beta\,\dot W(x). \tag{66}$$

Proof of Theorem 2.3. First of all we note that $q(T_3(\psi))|\Lambda_{\lambda u_3}(\psi)|$ can be taken as a $\mathscr{B}^\mu$-measurable function: For the $\mathscr{B}^\mu$-measurable shift $T_3(\psi)$ with the $H-C^1$ map u_3, since $T_3*(\mu|D)$ (the image measure of $T_3(\psi)$ restricted to D) is absolutely continuous with respect to μ (cf. [55, Theorem 3.5.2]), we can define the random variable $q(T_3(\psi))|\Lambda_{\lambda u_3}(\psi)|$ without ambiguity by using a Borel measurable $q(\psi)$ which is defined through any Borel measurable version $\tilde N(\psi)$ of $N(\psi)$ such that

$$N(\psi,D) = \tilde N(\psi,D), \quad \mu\text{-a.s.},\ \ \psi\in B_d^{a,b}$$

(cf. the proof of Lemma 2.2).

Noticing this, by (62) we can apply Corollary 3.5.3 of [55] to our shift T_3, which then yields the results. □

Remark 2.3 (Comparison with $(\phi^4)_2$ field). When $d=2$ we can take $\beta=\frac{1}{2}$ a case of special interest in Euclidean quantum field theory. In this case the above theorem tells us that the measure ν_3 gives a solution of (55) with space-cut-off:

$$(-\Delta+m^2)\psi(x) + \lambda\eta_M(x):\psi^3(x): = (-\Delta+m^2)^{\frac{1}{2}}\dot W(x), \quad x\in\boldsymbol{R}^2. \tag{67}$$

ν_3 can be written by

$$\begin{aligned}\nu_3(d\psi) = {} & q(T(\psi))|\det{}_2(I_{H^{-1}} + 3\lambda\eta_M(x):\psi^2(x):\delta_{\{x\}}(y))| \\ & \times\exp\Big\{-\lambda\int_{\boldsymbol{R}^2}\eta_M(x):\psi^4(x):dx - \frac{\lambda^2}{2}\int_{\boldsymbol{R}^2}(J^{\frac{1}{2}}(\eta_M:\psi^3:) \\ & (x))^2\,dx\Big\} \\ & \times\mu(d\psi),\end{aligned}$$

where we have used the fact that $J^{\beta-\frac{1}{2}}(x) = \delta_{\{0\}}(x)$ for $\beta = \frac{1}{2}$ (cf. Theorem 1.3).

On the other hand, the $(\phi^4)_2$ Euclidean field with space-cut-off η_M is a random field on $\boldsymbol{R}^2$ with the probability measure ν_{η_M} such that (cf., e.g., [52, Definition, Section 1, p. 141])

$$d\nu_{\eta_M}(\psi) = \frac{1}{Z_M}\exp\left\{-\lambda \int_{\boldsymbol{R}^2} \eta_M(x) : \psi^4(x) : dx\right\} d\mu,$$

with the normalization constant $Z_M = E^\mu[\exp\{-\lambda \int_{\boldsymbol{R}^2} \eta_M(x) : \psi^4(x) : dx\}]$. Then, there is a similarity between ν_3 and ν_{η_M} in the sense that their Radon–Nikodym densities $\frac{d\nu_3}{d\mu}$, resp. $\frac{d\nu_{\eta_M}}{d\mu}$, have the common term $\exp\{-\lambda \int_{\boldsymbol{R}^2} \eta_M(x) : \psi^4(x) : dx\}$. But, because of the existence of the other non-linear (also non-local) terms of ψ in $\frac{d\nu_3}{d\mu}$ such that $q(T(\psi)) = \frac{1}{N(T(\psi),D)} = \frac{1}{\sharp\{T^{-1}(T(\psi))\cap D\}}$, the reciprocal of the cardinality of the set $\{\psi' \in D \mid T(\psi') = T(\psi)\}$ (cf. Lemma 2.2 and Theorem 2.3), $\Lambda_1 = |\det_2(I_{H^{-1}} + 3\lambda\eta_M(x) : \psi^2(x) : \delta_{\{x\}}(y)|$ and $\Lambda_2 = \exp\{-\frac{\lambda^2}{2}\int_{\boldsymbol{R}^2} (J^{\frac{1}{2}}(\eta_M : \psi^3 :)(x))^2\, dx\}$, we have to distinguish ν_3 from ν_{η_M} (as far as $q(T(\psi))$, Λ_1 and Λ_2 do not cancel each other).

We also remark that $(J^{\frac{1}{2}}(\eta_M : \psi^3 :)(x))^2$, which is the integrand of Λ_2, is non-local in the sense that $(J^{\frac{1}{2}}(\eta_M : \psi^3 :)(x))^2 = (\int_{\boldsymbol{R}^2} J^{\frac{1}{2}}(x - y)\eta_M(y) : \psi^3(y) : dy)^2$ is not measurable with respect to the σ-field generated by the random variable $\langle \psi(\cdot), \delta^\varepsilon_{\{x\}} \rangle$ with δ^ε a $C_0^\infty(\boldsymbol{R}^2)$ approximation of the Dirac measure at the point x.

Moreover, since $\int_{\boldsymbol{R}^2}(J^{\frac{1}{2}}(\eta_M : \psi^3 :)(x))^2\, dx = \int_{\boldsymbol{R}^2\times\boldsymbol{R}^2} J^1(y - y')\eta_M(y)\eta_M(y')(: \psi^3(y) :)(: \psi^3(y') :)\, dy\, dy'$ and $J^1(y)$ on $\boldsymbol{R}^2$ diverges like "$-\log|y|$" (near 0) (cf. (A.5) in the appendix), it is possible to say that the exponent of Λ_2 contains a term of higher order than $: \psi^4 :$.

Theorem 2.4 (Solution for the exponential perturbation case). *For given $d \geqslant 2$ take β and ε to satisfy*

$$\beta \geqslant \frac{d}{4}, \quad 0 \leqslant \varepsilon < \frac{a_0(d)}{\sqrt{2}}, \tag{68}$$

let a and b be any numbers such that $a > \frac{d}{4} - \frac{1}{2}$, $b > d$. For some given $\lambda \geqslant 0$ and $M \geqslant 0$ define

$$T_e(\psi) = \psi + i(\lambda u_e(\psi)), \quad u_e(\psi) = J^{\beta-\frac{1}{2}}(\eta_M \tau_{(\beta,e^\varepsilon)}(\psi)). \tag{69}$$

Then the following holds:

(i) *The shift T_e on the abstract Wiener space $(B_d^{a,b}, i(\mathscr{H}), \mu)$ is strongly* monotone *in the sense of* [55] (*cf. Definition A.3*). *There* *exists an* $i(\mathscr{H})$*-invariant set* A *of* $B_d^{a,b}$ *with* $\mu(A)=1$ *and* T_e *is a bijection on* A.

(ii) *Let* S *be the inverse map of* T_e, *then* S *is a* $\mathscr{B}^\mu/\mathscr{B}^\mu$*-measurable function,* $S:(B_d^{a,b},\mathscr{B}^\mu)\mapsto(B_d^{a,b},\mathscr{B}^\mu)$. *A probability measure* ν_e *on* $(B_d^{a,b},\mathscr{B}^\mu)$ *defined by*

$$\nu_e(A)=\mu(S^{-1}A),\quad A\in\mathscr{B}^\mu$$

is a Girsanov probability measure on $(B_d^{a,b},\mathscr{B}^\mu)$ *associated with* μ *and* T_e:

$$E^{\nu_e}[f\circ T_e]=E^\mu[f],\quad \forall f\in C_b(B_d^{a,b}),\tag{70}$$

ν_e *solves the following SPDE in the sense of Definition 2.1*:

$$(-\Delta+m^2)^{1+(\beta-\frac{1}{2})}\psi(x)+\lambda\eta_M(x)\tau_{(\beta,e^\varepsilon)}(\psi(x))=(-\Delta+m^2)^\beta\dot W(x),$$

where ψ *is a* $B_d^{a,b}$*-valued random variable with probability law* ν_e *and* W *is an isonormal Gaussian process on* $\mathbf{R}^d$.

(iii) *In the case* $d=2$ *it is possible to take* $\beta=\frac{1}{2}$ *and* $0\leqslant\varepsilon<\sqrt{2\pi}$. *The probability measure* ν_e *gives then a solution of* (54) *with space-cut-off* η_M.

(iv) *If in particular* $0\leqslant\varepsilon<\frac{a_0(d)}{\sqrt{3}}$, *then* $\Lambda_{\lambda u_e}$ *defined by* (48), (49) *and* (52) *satisfies* $\Lambda_{\lambda u_e}\geqslant 0$, *a.s.,* $E^\mu[\Lambda_{\lambda u_e}]=1$ *and* ν_e *satisfies* $\nu_e=\Lambda_{\lambda u_e}\mu$.

Remark 2.4 (Why $0\leqslant\varepsilon<\frac{a_0(d)}{\sqrt{2}}$ in Theorem 2.4). In order to apply a change of variable formula concerning the monotone shifts to our exponential shift case, in Theorem 2.4 we had to assume condition (68) (i.e. $0\leqslant\varepsilon<\frac{a_0(d)}{\sqrt{2}}$) under which T_e becomes an $H-C$ continuous map (cf. the proof of Theorem 2.4). By this, in case $d=2$ if we take $\beta=\frac{1}{2}$, then ε should satisfy $0\leqslant\varepsilon<\sqrt{2\pi}$.

On the other hand, by Theorem 1.3(ii.1°) we have $u_e\in\mathbf{D}_{2,1}(\mathscr{H})$ when $0\leqslant\varepsilon<a_0(d)$, i.e. $0\leqslant\varepsilon<2\sqrt{\pi}$ in case $d=2$ and $\beta=\frac{1}{2}$.

However, generally speaking if a shift $T(\phi)=\phi+u(\phi)$ on an abstract Wiener space admits an application of some change of variable formula, then $u(\phi)$ should satisfy not only the differentiability condition such that $u\in\mathbf{D}_{p,k}(H)$ (H being the Cameron–Martin space) but also an H-regularity (e.g., $H-C$ and $H-C^1$). The property that $u\in\mathbf{D}_{p,k}(H)$ is characterized by means of the integrability of the Gross–Sobolev derivative ∇u, which is rather an element that connects to the algebraic structure of the Wiener space (cf. Definition A.2). On the other hand, H-regularities are the properties that come from the topological structure of the Wiener space (cf. Definition A.3 and the proof of Theorem 1.3). (Note that in our formulation we identify the Cameron–Martin space H^1 with $\mathscr{H}=H^{-1}$ by $H^1=i(\mathscr{H})$, cf. Remark 1.4 and Definitions A.1–A.3)

Proof of Theorem 2.4. For simplicity we write the detailed proof of Theorem 2.4 only for the case that $d=2$ and $\beta=\frac{1}{2}$, the other cases are similar. From Theorems 1.1 and 1.2 we see that $\tau_{(\frac{1}{2},e^{\varepsilon})}(\phi_\omega) =: e^{\varepsilon\phi_{\frac{1}{2},\omega}}:$ and $\lim_{k_j,l_i\to\infty} :e^{\varepsilon_{k_j,l_i}\phi_{\frac{1}{2},\omega}} :=: e^{\varepsilon\phi_{\frac{1}{2},\omega}}:$ P-a.s., $\omega\in\Omega$. For the Wick exponential: $e^{\varepsilon_{k,l}\phi_{\frac{1}{2},\omega}}$: the following equality in the sense of equivalent processes holds:

$$:e^{\varepsilon_{k,l}\phi_{\frac{1}{2},\omega}}:(x) = \exp\{\varepsilon({}_{k,l}\phi_{\frac{1}{2},\omega}(x))\}\exp\{-\varepsilon^2 c_{k,l}(x)\},\quad P\text{-a.s.},\quad \omega\in\Omega,$$

where ${}_{k,l}\phi_{\frac{1}{2},\omega}(x) = \eta_l(x)\int_{\boldsymbol{R}^2} J_k^{\frac{1}{2}}(x-y)\,dW_\omega(y)$ and $c_{k,l} = E[({}_{k,l}\phi_{\frac{1}{2},\omega}(x))^2] = (\eta_l(x))^2 \int_{\boldsymbol{R}^2}(J_k^{\frac{1}{2}}(x-y))^2\,dy$ (cf. [52, I.16]).

Noticing that the right-hand side of the above equality is non-negative, by Definition 1.1 and Theorem 1.3(ii.1°), for $\nabla u_e(\psi)\in L^2(\mathscr{H}\otimes\mathscr{H};\mu)$ it is easy to see that

$$\begin{aligned}
&((I_{H^{-1}} + \lambda\nabla u_e)h, h)_{H^{-1}}\\
&\quad = \|J^{\frac{1}{2}}h\|_{L^2}^2 + \lambda\varepsilon\int_{\boldsymbol{R}^2\times\boldsymbol{R}^2}(J^{\frac{1}{2}}h)(y_1)(J^{\frac{1}{2}}h)(y_2)\\
&\qquad \times\langle J^{\frac{1}{2}}(\cdot-y_1)J^{\frac{1}{2}}(\cdot-y_2)\eta_M(\cdot),\tau_{(\frac{1}{2},e^{\varepsilon})}(\psi)\rangle\,dy_1\,dy_2\\
&\quad \geqslant \|h\|_{H^{-1}}^2,\quad \mu\text{-a.s.},\quad \phi\in B_2^{a,b},\quad \forall h\in H^{-1}=\mathscr{H}.
\end{aligned}$$

This proves that the shift T_e defined by (69) satisfies the definition of strongly monotone shift (cf. [55, Lemma 6.2.1]).

In addition, by Theorem 1.3(ii.2°) since $u(\psi)$ is $H-C$, by applying [55, Theorem 6.4.1] we see that T_e is a bijection on some A such that $\mu(A)=1$. Then by Remark 2.2 for the measurable inverse S we can define a Girsanov probability measure, and assertion (ii) follows.

In particular if ε satisfies (iv), then from Theorem 1.3 (ii.3°) we see that u_e is $H-C^1$. Now by [55, Theorem 4.5.1] we have $E^\mu[|\Lambda_{\lambda u_e}|]=1$ and $v_e = |\Lambda_{\lambda u_e}|\mu$. But, for the monotone shift T_e the Carleman–Fredholm determinant obviously satisfies $\det_2(I_{\mathscr{H}} + \lambda\nabla u_e(\psi))\geqslant 0$, μ-a.s., and the non-negativity of $\Lambda_{\lambda u_e}$ follows. This proves (iv). □

Remark 2.5 (Crucial difference between v_e and the Albeverio Høegh-Krohn model). From Theorem 2.4(iv), if $0\leqslant\varepsilon<\frac{a_0(d)}{\sqrt{3}}$, then the Girsanov probability measure v_e associated with the monotone shift T_e has the expression $v_e = \Lambda_{\lambda u_e}\mu$. But if we compare $\Lambda_{\lambda u_e}$ with the multiplier considered in Albeverio and Høegh–Krohn [8,10] we find a crucial difference between them. Namely in [8,10] an Euclidean quantum field is defined through a probability measure such that

$$\frac{1}{Z_M}\exp\left\{-\lambda\int_{\boldsymbol{R}^2}\eta_M(x):e^{\alpha\psi}:(x)\,dx\right\},$$

where Z_M is a normalizing constant. But $\Lambda_{\lambda u_e}$ defined by (48), (49) and (52) has no term of the form $\int_{\boldsymbol{R}^2}\eta_M(x):e^{\alpha\psi}:(x)\,dx$.

Remark 2.6 (Regularities of $\tau_{(\beta,p)}$ and $\tau_{(\beta,e^{\varepsilon})}$ as Wiener functions). (i) In Theorem 2.3 we have assumed that (43) holds for given d and β (precisely the stronger condition (45) is assumed). This implies that if we take $\beta=\frac{1}{2}$, we have to restrict ourselves to $d=2$. Restriction (43) for $\beta=\frac{1}{2}$ and $p=3$ is just the assumption under which the "Sobolev divergence" of $u_3(\psi)=\eta_M\tau_{(\frac{1}{2},3)}(\psi)\in\mathscr{H}$ is given by $\delta u_3(\psi)=\langle\eta_M,\tau_{(\frac{1}{2},4)}(\psi)\rangle_{\mathscr{S},\mathscr{S}'}\in L^2(B_d^{a,b},\mu)$. In other words this is a condition under which δu_3 becomes a Skorohod integral (cf. Remark A.1). For $d\geqslant 3$ if we take $\beta=\frac{1}{2}$ and $p\geqslant 3$, then the divergence given by (44) cannot be defined any longer in the $L^2(\mu)$ sense.

However, also in this case, when $d=3$ there still exists a possibility that one can define $\delta u_3(\psi)=\langle\eta_M,\tau_{(\frac{1}{2},4)}(\psi)\rangle_{\mathscr{S},\mathscr{S}'}\in L^r(\mu)$ for some $1<r<2$. The study of this object and its relations to existing constructions, by different means, of a $(\phi^4)_3$ space-cut-off renormalized perturbation of Nelson's free field (see, e.g., [25]) will be pursued in forthcoming work.

(ii) In [38] the following statement is proven: Let $d=2$, and consider the exponential perturbation, that is equivalent (as a consequence) to the random variable $\tau_{(\frac{1}{2},e^{\varepsilon})}$ defined in Theorem 1.2 of the present paper. If $\varepsilon^2<4\pi$, then $\eta_M\tau_{(\frac{1}{2},e^{\varepsilon})}\in L^2(\mathscr{S}'(\boldsymbol{R}^2);\mu)$, and if $\varepsilon^2\in[4\pi,8\pi)$ then $\eta_M\tau_{(\frac{1}{2},e^{\varepsilon})}\in L^r(\mathscr{S}'(\boldsymbol{R}^2);\mu)$, $r\in(1,(\frac{8\pi}{\varepsilon^2})\wedge 2)$.

By the same reason mentioned in (i), for $d=2$ we have restricted our considerations to the case that $\varepsilon^2<4\pi$. But it would be interesting to reinterpret in the present framework the results of Albeverio and Høegh-Krohn [8,11], resp. Kusuoka [38], for the case $\varepsilon^2\in[4\pi,8\pi)$. Then for $\varepsilon^2\in[4\pi,8\pi)$ it may hold that $\eta_M\tau_{(\frac{1}{2},e^{\varepsilon})}\in\boldsymbol{D}_{r,1}(H^{-s})$ for some $1<r<2$ and $1<s$.

Remark 2.7 (Poincaré–Brascamp–Lieb and Log Sobolev inequalities). After finishing a preliminary version of this work, the authors were informed by Prof. A. S. Üstünel about the results of Feyel and Üstünel [24]. By applying the general results concerning the Poincaré–Brascamp–Lieb and Log Sobolev inequalities corresponding to (Gibbs type) measures defined through monotone shifts (concave function on Wiener spaces) given in Theorems 6.1 and 6.3 of [24], we immediately have the following: Suppose that the assumptions of Theorem 1.3(ii.1°) hold and that $\tau_{(\beta,e^{\varepsilon})}$ is the exponential perturbation defined there. Let ν_M be the probability measure on $(B_d^{a,b}, \mu)$ of the Albeverio Høegh-Krohn model with the space-cut-off η_M such that

$$\nu_M(d\phi) = \frac{1}{Z_M} \exp\{-\lambda \langle \eta_M, \tau_{(\beta,e^\varepsilon)}(\phi) \rangle\} \mu(d\phi),$$

where $\lambda \geqslant 0$ is some given constant and Z_M is a normalizing constant. Then for any smooth cylindrical Wiener functional G, we have

$$E^{\nu_M}[|G - E^{\nu_M}[G]|^2] \leqslant E^{\nu_M}[((I_{\mathscr{H}} + \lambda \nabla u_e)^{-1} \nabla G, \nabla G)_{\mathscr{H}}],$$

$$E^{\nu_M}[G^2\{\log G^2 - \log\|G\|^2_{L^2(\nu_M)}\}] \leqslant 2E^{\nu_M}[|\nabla G|^2_{\mathscr{H}}],$$

where $E^{\nu_M}[\cdot]$ denotes the expectation with respect to the probability measure ν_M and ∇u_e is given by (48).

When $d = 2$ the assumptions of Theorem 1.3(ii.$1^\circ, 2^\circ$ and 3°) are $|\varepsilon| < 2\sqrt{\pi}$, $|\varepsilon| < \sqrt{2\pi}$ and $|\varepsilon| < \sqrt{\frac{4\pi}{3}}$, respectively. In [8] there exists the considerations of the mass gaps corresponding to a class of Euclidean fields defined by the probability measures including the above ν_M as its special case. Definitely, in Theorem 7.1 of [8] (for the positivity of the corresponding Schwinger functions cf. Theorem A.1) the existence of the mass gap of the Hamiltonian corresponding to the model defined by ν_M has been proved under the assumption that $|\varepsilon| < \frac{4}{\sqrt{\pi}}$. It may be interesting to consider this Sobolev inequality in the framework of [8].

In [13] there are other considerations and applications of the log-concave property of the sharp time field measure of the Albeverio Høegh-Krohn model.

Next, let us try to remove the space-cut-off η_M from Theorems 2.3 and 2.4. The following Corollary 2.5 gives a first result in this direction, in which the probability measure is only constructed on a restricted σ-field.

Corollary 2.5 (Removing of the space-cut-off). *Let $\beta = \frac{2k+1}{2}$ for some $k = 0, 1, \ldots$.*

(i) *Suppose that the assumptions of Theorem 2.3 hold. In order to put in evidence the dependence on M we rewrite* (64) *as $T_3^M(\psi) = \psi + i(\lambda u_M(\psi))$ and $u_M(\psi) = J^{\beta-\frac{1}{2}}(\eta_M \tau_{(\beta,3)}(\psi))$. For each M we define a sub-σ-field $\mathscr{F}_M$ of $\mathscr{B}(B_d^{a,b})$ such that*

$$\begin{aligned} \mathscr{F}_M = \textit{sub-}\sigma-\textit{field of } \mathscr{B}(B_d^{a,b}) \textit{ generated by a family of} \\ \textit{random variables } \langle \varphi, T_3^M(\psi) \rangle, \ \varphi \in \Phi_{\beta,M}, \end{aligned} \tag{71}$$

where

$$\Phi_{\beta,M} = \{\varphi \in \mathscr{S} \mid \text{there exists } \varphi_0 \in C_0^\infty \text{ such that } \varphi = J^{-(\beta+\frac{1}{2})}\varphi_0,\ \operatorname{supp}[\varphi_0] \subset \{x \in \boldsymbol{R}^d \mid |x| < M\}\}.$$

Let $\mathscr{F}_\infty$ *be the smallest* σ*-field that includes all* $\mathscr{F}_M$, $M = 1, 2, \ldots$:

$$\mathscr{F}_\infty = \bigvee_{M \geqslant 1} \mathscr{F}_M.$$

Then there exists a probability measure v^∞ *on* $(B_d^{a,b}, \mathscr{F}_\infty)$ *such that*

$$v^\infty(B) = E^\mu[\chi_B q \circ T_3^M | \Lambda_{\lambda u_M}] \quad \text{for } B \in \mathscr{F}_M,\ M \in \boldsymbol{N}, \tag{72}$$

and the following holds:

$$E^{v^\infty}[F(\langle \varphi_{01}, J^{-(\beta+\frac{1}{2})}\psi + \lambda\tau_{(\beta,3)}(\psi)\rangle, \ldots, \langle \varphi_{0n}, J^{-(\beta+\frac{1}{2})}\psi + \lambda\tau_{(\beta,3)}(\psi)\rangle)] = E^P[F(\langle \varphi_{01}, J^{-\beta}\dot{W}_\omega\rangle, \ldots, \langle \varphi_{0n}, J^{-\beta}\dot{W}_\omega\rangle)] \tag{73}$$

for $\varphi_{0i} \in C_0^\infty$, $i = 1, \ldots, n$, *that satisfies* $J^{-(\beta+\frac{1}{2})}\varphi_{0i} = \varphi_i$ *for some* $\varphi_i \in \bigcup_M \Phi_{\beta,M}$ *and for* $F \in C_b(\boldsymbol{R}^n)$.

(ii) *Suppose that* (68) *holds. We rewrite* (69) *by* $T_e^M(\psi) = \psi + i(\lambda u_M(\psi))$ *and* $u_M(\psi) = J^{\beta-\frac{1}{2}}(\eta_M \tau_{(\beta,e^\varepsilon)}(\psi))$, *to underline the dependence on* M. *Let* S_M *be the inverse map of* T_e^M. *Replacing* T_3^M *by* T_e^M *in* (71) *we define the* σ*-fields* $\mathscr{F}_M$ *and* $\mathscr{F}_\infty$. *Then there exists a probability measure* v^∞ *on* $(B_d^{a,b}, \mathscr{F}_\infty)$ *such that*

$$v_\infty(B) = \mu(S_M^{-1}(B)) \quad \text{for } B \in \mathscr{F}_M,\ M \in \boldsymbol{N}.$$

The equation obtained by replacing $\tau_{(\beta,3)}$ *by* $\tau_{(\beta,e^\varepsilon)}$ *in* (73) *holds.*

Proof. For notational simplicity we denote T_3^M by T^M simply. By (18) and (64), if $M \leqslant M'$ then for $\varphi \in \Phi_{\beta,M}$ we see that

$$\langle \varphi, T^M(\psi)\rangle = \langle \varphi, T^{M'}(\psi)\rangle, \qquad \mu - \text{a.s.}$$

Hence by (65) for $\varphi_i \in \Phi_{\beta,M}$, $i = 1, \ldots, n$, $F \in C_b(\boldsymbol{R}^n)$, $n \in \boldsymbol{N}$ and $M \leqslant M'$,

$$\begin{aligned} &E^\mu[F(\langle \varphi_1, T^M(\psi)\rangle, \ldots, \langle \varphi_n, T^M(\psi)\rangle) q_{M'} \circ T^{M'} | \Lambda_{\lambda u_{M'}}] \\ &\quad = E^\mu[F(\langle \varphi_1, T^{M'}(\psi)\rangle, \ldots, \langle \varphi_n, T^{M'}(\psi)\rangle) q_{M'} \circ T^{M'} | \Lambda_{\lambda u_{M'}}] \\ &\quad = E^\mu[F(\langle \varphi_1, \psi\rangle, \ldots, \langle \varphi_n, \psi\rangle)] \\ &\quad = E^\mu[F(\langle \varphi_1, T^M(\psi)\rangle, \ldots, \langle \varphi_n, T^M(\psi)\rangle) q_M \circ T^M | \Lambda_{\lambda u_M}], \end{aligned}$$

(q_M is the q defined by Theorem 2.3). By this for each M if we define a probability measure v^M on $(B_d^{a,b}, \mathscr{F}_M)$ by

$$v^M(B) = E^\mu[\chi_B q_M \circ T^M | \Lambda_{\lambda u_M}] \quad \text{for } B \in \mathscr{F}_M,$$

then we remark that $\mathscr{F}_M \subset \mathscr{F}_{M'}$ $(M \leqslant M')$ we have

$$v^{M'}(B) = v^M(B) \quad \text{for } B \in \mathscr{F}_M, \quad M \leqslant M'. \tag{74}$$

Hence, $(B_d^{a,b}, \mathscr{F}_M, v^M)$, $M \in \boldsymbol{N}$, forms an inverse system of measures (cf. for e.g. [20, Section 9.4]), where for each M we assume that $B_d^{a,b}$ is a topological space equipped with the weakest topology by which T^M is a continuous map on it. Then, we have the existence of a probability measure v^∞ on $(B_d^{a,b}, \mathscr{F}_\infty)$ which satisfies (72).

By (65) and (72), for $\varphi_i \in \Phi_{\beta,M}$, $i = 1, \dots, n$ and $F \in C_b(\boldsymbol{R}^n)$, $(M \in \boldsymbol{N}, n \in \boldsymbol{N})$ we have

$$\begin{aligned}
&E^{v^\infty}[F(\langle \varphi_{01}, J^{-(\beta+\frac{1}{2})}\psi + \lambda\tau_{(\beta,3)}(\psi)\rangle, \dots, \langle \varphi_{0n}, J^{-(\beta+\frac{1}{2})}\psi + \lambda\tau_{(\beta,3)}(\psi)\rangle)] \\
&\quad = E^{v^\infty}[F(\langle \varphi_{01}, J^{-(\beta+\frac{1}{2})}\psi + \lambda\eta_M\tau_{(\beta,3)}(\psi)\rangle, \dots, \\
&\qquad \times \langle \varphi_{0n}, J^{-(\beta+\frac{1}{2})}\psi + \lambda\eta_M\tau_{(\beta,3)}(\psi)\rangle)] \\
&\quad = E^{v^M}[F(\langle J^{-(\beta+\frac{1}{2})}\varphi_{01}, T^M(\phi)\rangle, \dots, \langle J^{-(\beta+\frac{1}{2})}\varphi_{0n}, T^M(\phi)\rangle)] \\
&\quad = E^P[F(\langle J^{-(\beta+\frac{1}{2})}\varphi_{01}, J^{\frac{1}{2}}\dot{W}_\omega\rangle, \dots, \langle J^{-(\beta+\frac{1}{2})}\varphi_{0n}, J^{\frac{1}{2}}\dot{W}_\omega\rangle)],
\end{aligned}$$

and (73) is proved. This completes the proof of (i).

Moreover using $S_M^{-1} = T_e^M$ and (18), (69) and (70), for $\varphi^i \in \Phi_{\beta,M}$, $i = 1, \dots, n$, $F \in C_b(\boldsymbol{R}^n)$, $n \in \boldsymbol{N}$ and $M \leqslant M'$, we obtain

$$\begin{aligned}
&\mu(S_{M'}^{-1}\{\psi \in B_d^{a,b} : F(\langle \varphi_1, T_e^M(\psi)\rangle, \dots, \langle \varphi_n, T_e^M(\psi)\rangle) \in A\}) \\
&\quad = \mu(S_M^{-1}\{\psi \in B_d^{a,b} : F(\langle \varphi_1, T_e^M(\psi)\rangle, \dots, \\
&\qquad \times \langle \varphi_n, T_e^M(\psi)\rangle) \in A\}), \quad \forall A \in \mathscr{B}(\boldsymbol{R}).
\end{aligned}$$

The proof of assertion (ii) is then similar to the one of (i).

3. Case of trigonometric and general polynomial perturbations

For general $p \geqslant 2$, $p \neq 3$, we do not have Lemma 2.1, which is a key lemma for the case $p = 3$, nor for the shifts with the Albeverio Høegh-Krohn trigonometric perturbation term (cf. Remark 1.5) have we the monotonicity, which is satisfied by the exponential shift. Hence we cannot show in the above way the existence of a *Girsanov probability measure* associated with μ and $T(\psi) = \psi + J^1(\lambda J^{\beta-\frac{1}{2}}(\eta_M\tau_{(\beta,\xi)}(\psi)))$ $(\xi = p, s, c)$. The following Theorem 3.1 gives a partial substitute for these statements.

Let $C_b^+(B_d^{a,b})$ be the space of all non-negative-valued bounded continuous functions on $B_d^{a,b}$.

Theorem 3.1 (Trigonometric or $:\psi^p:$ perturbations with $p \neq 3$). (i) *Suppose that the assumptions of Theorem 1.3*(i.2°) *hold, and let* $u_p(\psi) =$

$J^{\beta-\frac{1}{2}}(\eta_M \tau_{(\beta,p)}(\psi))$ *be the* $H - C^1$ *map considered there. For* $\lambda \in \boldsymbol{R}$ *let*

$$T(\psi) = \psi + i(\lambda u_p(\psi)).$$

Then for any $f, g \in C_b^+(B_d^{a,b})$ *the following holds*:

$$E^\mu[f \circ T | \Lambda_u | g] = E^\mu \left[f \sum_{y \in T^{-1}\{\psi\} \cap D} g(y) \right],$$

where

$$D = \{\psi \in B_d^{a,b} \mid \det{}_2(I_{\mathscr{H}} + \nabla u_p(\psi)) \neq 0\}.$$

(ii) *Suppose that the assumptions of Theorem 1.3*(ii.3°) *hold, and let* $u_s(\psi) = J^{\beta-\frac{1}{2}}(\eta_M \tau_{(\beta,s)}(\psi))$ *and* $u_c(\psi) = J^{\beta-\frac{1}{2}}(\eta_M \tau_{(\beta,c)}(\psi))$ *be the* $H - C^1$ *trigonometric maps considered in Remark 1.5. Then the assertion obtained by replacing* u_p *by* u_s *or* u_c *in* (i) *holds.*

Proof. By Theorem 1.3(i.2°) and Remark 1.5 it has been proved that $u_p(\psi)$, u_s and u_c are $H - C^1$ maps on $(B_d^{a,b}, i(\mathscr{H}), \mu)$. Then Theorem 3.1 follows from Theorem 3.5.2 in [55]. □

4. A note on the reflection positivity and the time-zero field

This section contains two considerations in relation with *constructive quantum field theory*: We firstly discuss a restricted reflection positivity property of the random field defined by the solutions of the SPDEs considered in the previous sections, and secondly we establish a relation between the solutions of the Euclidean SPDEs and that of *quantized non-linear equations* (cf. [18,47]), by observing the corresponding *time-zero* fields.

Let $T : B_d^{a,b} \to B_d^{a,b}$ be the shift T_3, resp. T_e, considered in Theorems 2.3 and 2.4, and let $v = v_3$, resp. $v = v_e$. In this section we will only consider such T.

Let θ be the time reflection operator on $\boldsymbol{R}^d$:

$$\theta f(t, \vec{x}) = f(-t, \vec{x}), \tag{75}$$

for a complex-valued function f on $\boldsymbol{R}^d$, with $(t, \vec{x}) \in \boldsymbol{R} \times \boldsymbol{R}^{d-1}$. For $\psi \in \mathscr{S}'(\boldsymbol{R}^d)$, we define $\theta\psi$ as the tempered distribution $\theta\psi \in \mathscr{S}'(\boldsymbol{R}^d)$ that satisfies

$$\langle \varphi, \theta\psi \rangle_{\mathscr{S},\mathscr{S}'} = \langle \theta\varphi, \psi \rangle_{\mathscr{S},\mathscr{S}'}, \quad \forall \varphi \in \mathscr{S}(\boldsymbol{R}^d). \tag{76}$$

Note that for each $\varphi \in \mathscr{S}(\boldsymbol{R}^d)$

$$\langle \varphi, T(\cdot) \rangle_{\mathscr{S},\mathscr{S}'} : B_d^{a,b} \ni \psi \mapsto \langle \varphi, T(\psi) \rangle_{\mathscr{S},\mathscr{S}'} \in \boldsymbol{R}$$

defines a *non-linear* measurable functional on $(B_d^{a,b}, \mathscr{B}^{\mu})$. We regard this functional as a "non-linear test functional" on $\mathscr{S}'$.

We observe that for $T = T_3$ or $T = T_e$ satisfying $0 \leqslant \varepsilon < \frac{a_0(d)}{\sqrt{3}}$ the following holds:

$$\langle \varphi, \theta T(\psi) \rangle_{\mathscr{S},\mathscr{S}'} = \langle \varphi, T(\theta\psi) \rangle_{\mathscr{S},\mathscr{S}'}, \quad \nu\text{-a.s.}, \ \psi \in B_d^{a,b}. \tag{77}$$

This can be shown as follows: Since $(-\Delta + m^2)^{-\alpha}$ is translation invariant, in fact its kernel as $x, y \in \boldsymbol{R}^d$ is given by $J^{\alpha}(x-y)$, and by definition we have $\eta_k(t, \vec{x}) = \eta_k(-t, \vec{x})$, moreover the inclusion map i is defined by (41), from (20), (33), (35), (38), (64) and (69) we have

$$\langle \varphi, \theta T(\psi) \rangle_{\mathscr{S},\mathscr{S}'} = \langle \varphi, T(\theta\psi) \rangle_{\mathscr{S},\mathscr{S}'}, \quad \mu\text{-a.s.}, \ \psi \in B_d^{a,b}. \tag{78}$$

By Theorems 2.3 and 2.4(iv) since the probability measure ν is *absolutely continuous* with respect to μ, from (78) we obtain (77).

Theorem 4.1 (Subspace of reflection positive random variables). *For given* d *let* $T = T_3$, $\nu = \nu_3$ *or* $T = T_e$, $\nu = \nu_e$ *where* T_3 *and* ν_3, *resp.* T_e *and* ν_e *are the shifts and measures defined in Theorem* 2.3, *resp. Theorem* 2.4(iv). *Let* μ *be the probability law of Nelson's Euclidean free field on* $\boldsymbol{R}^d$ *defined in Section* 1. *Let* $\boldsymbol{F}_{T,+}$ *be a linear subspace of the random variables on* $(B_d^{a,b}, \mathscr{B}^{\mu}, \nu)$ *such that*

$$\boldsymbol{F}_{T,+} \equiv \overline{\{f \circ T(\psi) \,|\, f \in F_+\}}^{L^2(\nu)},$$

where

$$\begin{aligned} F_+ = \text{linear hull of } \ &\{f \mid \exists n \in \boldsymbol{N}, \exists f_i \in C_0^{\infty}(\boldsymbol{R}) \\ &\text{satisfying } f_i(0) = 0, \ \exists \varphi_i \in C_0^{\infty}(\boldsymbol{R}_+^d), \ i = 1, \dots, n, \\ &\text{such that } f(\cdot) = f_1(\langle \varphi_1, \cdot \rangle_{\mathscr{S},\mathscr{S}'}) \cdots f_n(\langle \varphi_n, \cdot \rangle_{\mathscr{S},\mathscr{S}'})\}, \end{aligned}$$

where

$$\boldsymbol{R}_+^d \equiv \{(t, \vec{x}) \in \boldsymbol{R}^d : t > 0\}.$$

Then, on $\boldsymbol{F}_{T,+}$ *the reflection positivity holds*

$$E^{\nu}[F(\theta\psi)F(\psi)] \geqslant 0 \quad \text{for } F \in \boldsymbol{F}_{T,+}. \tag{79}$$

Proof. By (65), (70) and (77) for $f \in F_+$ the following holds:

$$\begin{aligned} E^{\nu}[f \circ T(\theta\psi) f \circ T(\psi)] &= E^{\nu}[f(\theta T\psi) f(T\psi)] \\ &= E^{\mu}[f(\theta\psi) f(\psi)] \geqslant 0. \end{aligned}$$

The last inequality is the reflection positivity property of Nelson's free field (cf. for e.g. [3, Remark 5.2]), that comes from its Markov field property. This proves (79). □

Remark 4.1. (i) In order to conclude that the random fields ψ considered in Theorems 2.3 and 2.4(iv) possess the property of reflection positivity (Hegerfeldt T positivity given in [29]), we have to show that (79) holds for $F \in \boldsymbol{F}_+$, where

$$\boldsymbol{F}_+ \equiv \overline{\{f(\psi) \mid f \in F_+\}}^{L^2(\nu)}$$

Thus, Theorem 4.1 characterizes a sub-space of $L^2(\nu)$-random variables on which the reflection positivity holds. Nevertheless, $\boldsymbol{F}_{T,+}$ and $\boldsymbol{F}_+$ have non-empty intersection: e.g., in case $d = 2$, if we take $\beta = \frac{1}{2}$, then for $T = T_3$ and $\nu = \nu_3$ we have

$$\boldsymbol{F}_{T,+} \cap \boldsymbol{F}_+ \supset \overline{\{f \circ T(\psi) \mid f \in \tilde{F}_+\}}^{L^2(\nu)},$$

where $\tilde{F}_+$ is a subset of F_+ defined by restricting $\varphi_i \in C_0^\infty(\boldsymbol{R}_+^d)$ in the definition of F_+ to $\varphi_i \in C_0^\infty(\boldsymbol{R}_+^d)$ such that

$$\varphi_i = (-\Delta + m^2) g_i \quad \text{for some } g_i \in C_0^\infty(\boldsymbol{R}_+^d).$$

(ii) Let T_e, resp. ν_e, be the exponential shift, resp. the corresponding Girsanov probability measure, defined in Theorem 2.4 for $\frac{a_0(d)}{\sqrt{3}} \leqslant \varepsilon < \frac{a_0(d)}{\sqrt{2}}$, then we cannot conclude in general that ν_e is absolutely continuous with respect to μ, and (77) does not follow from (78). But from (70) we can still deduce the following:

$$E^\nu[f \circ \theta T(\psi) f \circ T(\psi)] \geqslant 0 \quad \text{for } f \in F_+.$$

Also for the *signed* measure $\Lambda_{\lambda u_3}\mu$ defined in (65) we have

$$\int_{B_2^{a,b}} f \circ \theta T(\psi) f \circ T(\psi) \Lambda_{\lambda u_3}\, d\mu(\psi) \geqslant 0 \quad \text{for } f \in F_+.$$

This can be seen as a weaker substitute for (79).

(iii) For discussions of the reflection-positivity property in axiomatic Euclidean quantum field theory see, e.g., [25,29,45,46,52] (and references therein). For the proof in models see, e.g., [2,10,12,16,25,32,52] (the latter references also contains a proof of the global Markov property of interacting (Euclidean) quantum fields).

Next, we give a short discussion on the time-zero fields of the Euclidean random fields which are defined by the solution of SPDEs considered in Section 2.

Let ψ be the solution of the (Euclidean) SPDE with cubic perturbation on $\boldsymbol{R}^2$, namely ψ is the Euclidean random field with the probability law ν_3 defined by Theorem 2.3 for $d = 2$ and $\beta = \frac{1}{2}$. Then under ν_3 the random variable $T_3(\psi)$ is a *Nelson's Euclidean free field* on $\boldsymbol{R}^2$, and by the definition of T_3 (cf. (64)) the following holds:

$$\left(-\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+m^2\right)\psi+\lambda\eta_M:\psi^3:=\left(-\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+m^2\right)T_3(\psi).$$

Since, at the point $t=0$ by taking the boundary-value Nelson's Euclidean free field it is analytically continued to Φ_0 which is a solution of (4) reviewed in the Introduction, we find that ψ satisfies

$$\begin{aligned}
&-\frac{d^2}{dt^2}E^{\nu_3}[\langle\psi(t,\cdot),f\rangle u]\Big|_{t=0}\\
&\quad+E^{\nu_3}\left[\left(\left\langle\psi(0,\cdot),\left(-\frac{d^2}{dx^2}+m^2\right)f\right\rangle+\lambda\langle:\psi^3(0,\cdot):,\eta_M(0,\cdot)f\rangle\right)u\right]\\
&\qquad=-\frac{d^2}{dt^2}E^{\nu_3}[\langle T_3(\psi)(t,\cdot),f\rangle u]|_{t=0}\\
&\qquad\quad+E^{\nu_3}\left[\left\langle T_3(\psi)(t,\cdot),\left(-\frac{d^2}{dx^2}+m^2\right)f\right\rangle u\right]\Big|_{t=0}=0 \qquad (80)
\end{aligned}$$

for real $f\in\mathcal{S}(\boldsymbol{R})$ and $u\in\tilde{\mathcal{U}}$, where

$$\tilde{\mathcal{U}}\equiv\text{linear hull of }\left\{\prod_{i=1}^{n}\langle T_3(\psi),\delta_{\{0\}}\otimes g_i\rangle|g_i\in\mathcal{S}(\boldsymbol{R}),\quad i=1,\ldots,n,\ n\in\boldsymbol{N}\right\}.$$

Thus, if $\tilde{\mathcal{U}}$ is dense in the (time-zero) Hilbert space $\mathcal{H}_I$ given by

$$\mathcal{H}_I\equiv\overline{\left\{\prod_{i=1}^{n}\langle\psi(0,\cdot),f_i(\cdot)\rangle|f_i\in\mathcal{S}(\boldsymbol{R}^1),\ i=1,\ldots,n,\ n\in\boldsymbol{N}\right\}}^{L^2(\nu_3)} \qquad (81)$$

then the Euclidean field ψ on $\boldsymbol{R}^2$ with the probability measure ν_3 may be analytically continued to a solution Φ_I (assuming its existence) of the non-linear quantized equation such that Φ_I is an operator-valued distribution on the physical Hilbert space $\mathcal{H}_I$ with a dense domain and satisfies

$$\begin{aligned}
&\frac{d^2}{dt^2}\Phi_I(\delta_{\{t\}}\otimes f)+\Phi_I\left(\delta_{\{t\}}\otimes\left(-\frac{d^2}{dx^2}+m^2\right)f\right)\\
&\quad+\lambda\langle:(\Phi_I(\cdot))^3:,\delta_{\{t\}}\otimes\eta_M(0,\cdot)f(\cdot)\rangle=0\quad\text{for } f\in\mathcal{S}(\boldsymbol{R}),\ t\in\boldsymbol{R}, \qquad (82)
\end{aligned}$$

where $:(\Phi_I(t,x))^3:$ is a renormalization of $(\Phi_I(t,x))^3$, and it is also a (linear) operator-valued distribution.

However, since $\tilde{\mathcal{U}}$ is a linear space spanned by the products of random variables such that $\langle T_3(\psi)(0,\cdot),g(\cdot)\rangle$ with

$$T_3(\psi)=\psi+\lambda J^1(\eta_M:\psi^3:) \qquad (83)$$

and J^1 is an integral operator, $\tilde{\mathscr{U}}$ may not be dense in $\mathscr{H}_I$ (cf. Remark 4.1(i)). Hence, we have the following Remark 4.2:

Remark 4.2 (*Time-zero* fields of the solutions). (i) The Euclidean random field ψ with the probability law ν_3 defined by Theorem 2.3 for the space-time dimension $d = 2$ and $\beta = \frac{1}{2}$ satisfies the functional differential equation (80) at least for $u \in \tilde{\mathscr{U}}$. In order that ψ can be analytically continued to some operator Φ_I, that is a solution of the functional equation (82), with a dense domain in the Hilbert space $\mathscr{H}_I$ defined by (81), the set $\bar{\tilde{\mathscr{U}}} \cap \mathscr{H}_I$ has to be dense in $\mathscr{H}_I$ ($\bar{\tilde{\mathscr{U}}} \supset \tilde{\mathscr{U}}$ is the set on which (80) holds).

We do not know whether $\bar{\tilde{\mathscr{U}}} \cap \mathscr{H}_I$ is sufficiently large. We should accordingly modify T_3 and the corresponding measure ν_3 adequately. Namely, in the definition of T_3 given by (83), $\eta_M(t,\vec{x}) \in C_0^\infty(\boldsymbol{R}^2)$ should be changed to $\eta(\vec{x}) \otimes \delta_{\{0\}}(t)$ for some $\eta \in C_0^\infty(\boldsymbol{R}^1)$ satisfying $\eta \geqslant 0$, and then define a map $\tilde{T}_3$ on $(B_2^{a,b}, i(\mathscr{H}), \mu)$ by

$$\tilde{T}_3(\psi)(t,\vec{x}) = \psi(t,\vec{x}) + \lambda \int_{\boldsymbol{R}^1} J^1((t,\vec{x}) - (0,\vec{x}'))\eta(\vec{x}') : \psi^3(0,\vec{x}') : d\vec{x}'.$$

Passing through similar arguments as in Theorem 1.3, it is not hard to show that

$$\int_{\boldsymbol{R}^1} J^1((t,\vec{x}) - (0,\vec{x}'))\eta(\vec{x}') : \psi^3(0,\vec{x}') : d\vec{x}' \in \boldsymbol{D}_{2,k}(i(\mathscr{H})) \quad (k \geqslant 0),$$

(i.e. $\eta(\vec{x})\delta_{\{0\}}(t) : \psi^3(t,\vec{x}) : \in \boldsymbol{D}_{2,k}(\mathscr{H})$). But the $H - C^1$ (or $H - C$) continuity of $\tilde{T}_3$ is not obvious (cf. Remark 2.4).

However, if everything is completed, and a corresponding Girsanov probability measure $\tilde{\nu}_3$ is defined, then for this modification $\tilde{\mathscr{U}}$ (using the same notation as above) we would have

$$\tilde{\mathscr{U}} \equiv \text{linear hull of } \left\{ \prod_{i=1}^{n} (\langle \psi(0,\cdot), f_i(\cdot) \rangle + \lambda \langle : \psi^3 : (0,\cdot), (J^{\frac{1}{2}} f_i)(\cdot)\eta(\cdot) \rangle) | f_i \in \mathscr{S}(\boldsymbol{R}),\ i = 1, \ldots, n,\ n \in \boldsymbol{N} \right\} \subset \tilde{\mathscr{H}}_I,$$

where $\tilde{\mathscr{H}}_I$ is the Hilbert space defined by (81) by replacing ν_3 by $\tilde{\nu}_3$. Also $\frac{d\tilde{\nu}_3}{d\mu}$, the Radon–Nikodym density of $\tilde{\nu}_3$ with respect to the Nelson's Euclidean free field measure μ on $\boldsymbol{R}^2$, has then a term such that

$$\exp\left\{ -\lambda \int_{\boldsymbol{R}} \eta(x) : \phi^4 : (0,x)\, dx \right\}.$$

For further discussions in this relation (cf. [18, Corollary 8.8.1], [17]) we refer to forthcoming papers.

(ii) By Theorem 2.4 analogous discussions as (i) can be performed for the case of Euclidean random fields that are solutions of the SPDE with an exponential perturbation.

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Appendix

In the first part of the appendix we shall quickly recall some fundamental notations and notions associated with the analysis on an abstract Wiener space. The following explanations on Definitions A.1–A.3 are mainly borrowed from Section 3.3 and Chapter B in [55], but some of the notations are changed in order that they are adaptable to the present paper (cf. Remark 1.4 and Remark A.1).

Definition A.1 (Abstract Wiener space). (i) Let H be a separable Hilbert space, denote by μ_0 the standard Gaussian cylindrical measure on H whose characteristic function is given by

$$\exp\{-\tfrac{1}{2}|h|_H^2\}, \quad h \in H.$$

In the infinite dimensional case, μ_0 is not a sigma-additive measure on H. By Gross [26] it has been proved that H can be completed under a weaker norm than the original norm of H to a Banach space W, and μ_0 can be extended to a probability measure μ on W.

The triple (W, H, μ) is called as an abstract Wiener space, H is called the Cameron–Martin space and μ is called the Wiener measure. The Cameron–Martin space H is identified with its continuous dual H^*, then

$$W^* \hookrightarrow H^* \cong H \hookrightarrow W.$$

Let i be the continuous injection such that

$$i : H^* \to W, \quad \text{i.e. } i(H^*) = H.$$

In the sequel we sometimes denote the triple (W, H, μ) by $(W, i(H^*), \mu)$. We notice that in the previous sections H^* has been denoted by $\mathscr{H}$.

(ii) Let $\alpha \in W^*$, then $w \mapsto \langle \alpha, w \rangle$ is a Gaussian random variable on the probability space (W, μ) whose characteristic function is

$$\exp\{-\tfrac{1}{2}|i(\alpha)|_H^2\} = \exp\{-\tfrac{1}{2}|\alpha|_{H^*}^2\}.$$

If $h^* \in H^*$, then there exists $(\alpha_n, n \in \boldsymbol{N})$ such that α_n converges to h^* in H^*, and $(\langle \alpha_n, w \rangle,\ n \in \boldsymbol{N})$ forms a Cauchy sequence in $L^p(\mu)$ for any $p \geqslant 0$.

We denote the limit by $\langle h^*, w \rangle$ $(h^* \in H^*)$.

Definition A.2 (Derivative and divergence operators). (i) Let $(W, i(H^*), \mu)$ be an abstract Wiener space. A measurable function $\varphi : W \to \boldsymbol{R}$ is called as a cylindrical Wiener functional if it is of the form

$$\varphi(w) = f(\langle h_1^*, w \rangle, \ldots, \langle h_n^*, w \rangle), \quad h_1^*, \ldots, h_n^* \in H^*, \ f \in \mathscr{S}(\boldsymbol{R}^n),$$

for some $n \in \boldsymbol{N}$. For the cylindrical Wiener functional φ and an element of the Cameron–Martin space $h \in H$ we define

$$\nabla_h \varphi(w) = \frac{d}{d\varepsilon} \varphi(w + \varepsilon h)\bigg|_{\varepsilon=0},$$

then

$$\nabla_h \varphi(w) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(\langle h_1^*, w \rangle, \ldots, \langle h_n^*, w \rangle) h_j^*(h),$$

where $h_j^*(h)$ denotes the dualization of $h_j^* \in H^*$ and $h \in H$. For each fixed w, the map $h \mapsto \nabla_h \varphi(w)$ is continuous and linear on H, it determines an element of the *dual space* of H. This element is called the Gross–Sobolev derivative of φ and is denoted by $\nabla \varphi$.

Since $h_j^*(h) = \langle i(h_j^*), h \rangle_H = \langle h_j^*, i^{-1}(h) \rangle_{H^*}$, where $\langle i(h_j^*), h \rangle_H$, resp. $\langle h_j^*, i^{-1}(h) \rangle_{H^*}$, are inner products with respect to the Hilbert spaces H, resp. H^*, depending on the adoption of the inner products the Gross–Sobolev derivative $\nabla \varphi$ can be understood as either H^* or H-valued random variable. In this paper we discuss $\nabla \varphi$ as an H^* (Hilbert space)-valued random variable.

(ii) In this paper we have denoted $H^* = \mathscr{H}$. The *Sobolev space* $\boldsymbol{D}_{p,1}$ is the set of equivalent class of the real-valued Wiener functionals defined as follows: $\varphi \in \boldsymbol{D}_{p,1}$ if and only if there exists a sequence of cylindrical random variables $(\varphi_n, n \in \boldsymbol{N})$ converging to φ in $L^p(\mu)$ such that $(\nabla \varphi_n, n \in \boldsymbol{N})$ is Cauchy in $L^p(\mu; \mathscr{H})$. In this case $\lim_{n \to \infty} \nabla \varphi_n$ is denoted by $\nabla \varphi$.

$\boldsymbol{D}_{p,1}$ is a Banach space with the norm defined by

$$\|\varphi\|_{p,1} = \|\varphi\|_{L^p(\mu)} + \|\nabla \varphi\|_{L^p(\mu, \mathscr{H})}.$$

This definition of $\boldsymbol{D}_{p,1}$ can be extended to the case of Wiener functionals with values in some separable Hilbert space $\mathcal{X}$. The corresponding Banach space will be denoted by $\boldsymbol{D}_{p,1}(\mathcal{X})$.

(iii) Let $p>1$, $k\geqslant 1$, and let $\mathcal{X}$ be a separable Hilbert space. The space $\boldsymbol{D}_{p,k}(\mathcal{X})$ is inductively defined by

(1) $F\in\boldsymbol{D}_{p,2}(\mathcal{X})$ if $\nabla F\in\boldsymbol{D}_{p,1}(\mathcal{X}\otimes\mathcal{H})$ and with $\nabla^2 F=\nabla(\nabla F)$.

(2) $F\in\boldsymbol{D}_{p,k}(\mathcal{X})$ if $\nabla^{k-1}F\in\boldsymbol{D}_{p,1}(\mathcal{X}\otimes\mathcal{H}^{\otimes(k-1)})$, where $\otimes$ denotes the completed Hilbert–Schmidt tensor product.

(iv) Let $p>1$, $\xi:W\to\mathcal{H}(=H^*)$ be a Wiener functional. $\xi\in\mathrm{Dom}_p(\delta)$ if and only if for any $\varphi\in\boldsymbol{D}_{q,k}$, $q^{-1}=1-p^{-1}$, the following holds:

$$|E^{\mu}[(\nabla\varphi,\xi)_{\mathcal{H}}]|\leqslant C\|\varphi\|_{L^q(\mu)},$$

where C is a constant depending only on ξ and p.

If $\xi\in\mathrm{Dom}_p(\delta)$, then there exists an element, denoted by $\delta\xi$, in $L^p(\mu)$ such that

$$E^{\mu}[(\nabla\varphi,\xi)_{\mathcal{H}}]=E^{\mu}[\varphi\cdot\delta\xi].$$

$\delta\xi$ is called the *divergence* of ξ.

(v) Let A be a Hilbert–Schmidt operator on the separable Hilbert space $\mathcal{H}$. The (modified) Carleman–Fredholm determinant of A, denoted by $\det_2(I_{\mathcal{H}}+A)$, is defined as

$$\det{}_2(I_{\mathcal{H}}+A)=\prod_{i=1}^{\infty}(1+\gamma_i)e^{-\gamma_i},$$

where $(\gamma_i, i\in\boldsymbol{N})$ are the eigenvalues of A counted with respect to their multiplicity.

Remark A.1. Let us examine the concrete actions of the operators ∇ and δ on the functionals on Nelson's Euclidean free field $(B_d^{a,b}, i(\mathcal{H}),\mu)$, which is an abstract Wiener space defined in Section 1. For this purpose it is convenient to use the identification of Nelson's free field $\tau_{(\frac{1}{2},1)}(\psi)$ with the stochastic integral $\int_{\boldsymbol{R}^d}J^{\frac{1}{2}}(x-x')\,dW_{\omega}(x')$ (this expression is "formal", but the following discussions can be carried out rigorously by making use of this expression (cf. Theorems 1.1 and 1.2 and Remark 1.3)).

Now, let $d=2$. Consider a cylindrical Wiener functional φ such that

$$\varphi=\int_{\boldsymbol{R}^{2(p+1)}}\left(\prod_{j=1}^{p+1}(J^{\frac{1}{2}}h^*)(x_j)\right)dW_{\omega}(x_1)\cdots dW_{\omega}(x_{p+1})\quad\text{for } h^*\in\mathcal{H}=H^{-1}.$$

This is identified with $:\langle h^*,\tau_{(\frac{1}{2},1)}(\psi)\rangle^{p+1}:$, which is the $(p+1)$th Wick product of the random variable $\langle h^*,\tau_{(\frac{1}{2},1)}(\psi)\rangle$ (cf., e.g., [52, Section I-1]).

Then by Definition A.1, it is easy to see that the derivative of: $\langle h^*, \tau_{(\frac{1}{2},1)}(\psi)\rangle^{p+1}$: is given by

$$\nabla : \langle h^*, \tau_{(\frac{1}{2},1)}(\psi)\rangle^{p+1} := (p+1)h^*(x) : \langle h^*, \tau_{(\frac{1}{2},1)}(\psi)\rangle^{p} : .$$

This can be identified with the following stochastic integral:

$$\nabla\varphi = (p+1)h^*(x)\int_{\boldsymbol{R}^{2p}}\left(\prod_{j=1}^{p}(J^{\frac{1}{2}}h^*)(x_j)\right)dW_\omega(x_1)\cdots dW_\omega(x_p)\in\mathscr{H}.$$

Also, $\tau_{(\frac{1}{2},p)}(\psi)$ is identified with

$$\int_{\boldsymbol{R}^{2p}}\left(\prod_{j=1}^{p}J^{\frac{1}{2}}(x-x_j)\right)dW_\omega(x_1)\cdots dW_\omega(x_p).$$

Hence, the map u_p, which is an $\mathscr{H}$-valued functional, defined by $u_p(\psi) = J^\beta(\eta_M\tau_{(\beta,p)}(\psi))$ is identified with (in case $\beta = \frac{1}{2}$)

$$u_p = \delta_{\{x'\}}(x)\eta_M(x')\int_{\boldsymbol{R}^{2p}}\left(\prod_{j=1}^{p}J^{\frac{1}{2}}(x'-x_j)\right)dW_\omega(x_1)\cdots dW_\omega(x_p).$$

If we set

$$\delta\Xi = \int_{\boldsymbol{R}^{2(p+1)}}\left(\int_{\boldsymbol{R}^2}\eta_M(x)\prod_{j=1}^{p+1}J^{\frac{1}{2}}(x-x_j)\,dx\right)dW_\omega(x_1)\cdots dW_\omega(x_{p+1}).$$

Then, for φ, $\nabla\varphi$, u_p and $\delta\Xi$ defined above by the properties of expectations with respect to (multiple) stochastic integrals (cf., e.g., [44, Section 1]), we obviously have

$$E[\langle\nabla\varphi, u_p\rangle_{\mathscr{H}}] = E[\varphi\cdot\delta\Xi].$$

Since $\delta\Xi$ is identified with $\langle\eta_M, \tau_{(\frac{1}{2},p+1)}\rangle$, by this we have relation (44):

$$\delta u_p = \langle\eta_M, \tau_{(\frac{1}{2},p+1)}\rangle.$$

Definition A.3 (H-regularity and monotone shifts). Let $(W, i(H^*), \mu)$ be an abstract Wiener space.

(i) Let $u(w)$ be a random variable taking values in a separable Hilbert space $\mathscr{X}$.

(1) $u(w)$ is said to be an $H-C$ map if, for almost all $w\in W$, $H^*\ni h^*\mapsto u(w+i(h^*))$ is a continuous function of $h^*\in H^*$.

(2) $u(w)$ is said to be an $H-C^1$ map if it is $H-C$, and for almost all $w\in W$, $H^*\ni h^*\mapsto u(w+i(h^*))$ is continuously Fréchet differentiable on H^* and this Fréchet derivative is an $H-C$ as a mapping from H^* into $\mathscr{X}\otimes H^*$ when the latter is equipped with the Hilbert–Schmidt topology.

(ii) Let $u(w)$ be an H^*-valued random variable. The shift $T : W \to W$ defined by $Tw = w + i(u(w))$ is called strongly monotone if there exists some $\alpha > 0$ such that

$$(T(w + i(h^*)) - T(w), i(h^*))_H \geqslant \alpha |i(h^*)|_H^2,$$

almost surely for all $h^* \in H^*$.

By Lemma 6.2.1 of [55], this condition is equivalent to (cf. Definition A.1(i))

$$((I_{H^*} + \nabla u)h^*, h^*)_{H^*} \geqslant \alpha |h^*|_{H^*}^2,$$

almost surely for all $h^* \in H^*$.

The following Proposition A.1 gives some evaluations for the functions that have been used in this paper, and they are well known or obvious.

Proposition A.1. (i) *Let $\rho \in C_0^\infty(\boldsymbol{R}^d)$ be such that*

$$\hat{\rho}_k(\xi) = \hat{\rho}\left(\frac{\xi}{k}\right), \quad |\hat{\rho}(\xi)| \leqslant 1, \quad \hat{\rho}(0) = 1 \tag{A.1}$$

(*with* $\hat{}$ *denoting Fourier transform*), *then* $\hat{\rho}_k(\xi)$ *converges to* 1 *uniformly on compact sets*: *For any* $M < \infty$ *and any* $\varepsilon > 0$ *there exists an* $N < \infty$ *such that*

$$0 \leqslant 1 - \hat{\rho}_n(\xi) < \varepsilon \quad \textit{for any } \xi \textit{ satisfying } |\xi| \leqslant M \textit{ and any } n \geqslant N. \tag{A.2}$$

(ii) *$J^\alpha(x)$ has the following integral representation* (*cf.* [53, V-3.1], *or* [48]):

$$J^\alpha(x) = \frac{1}{(4\pi)^{\frac{d}{2}} \Gamma(\alpha)} \int_0^\infty \exp\left\{ -\frac{|x|^2}{4s} - m^2 s \right\} s^{\frac{-d-2+2\alpha}{2}} \, ds, \quad x \in \boldsymbol{R}^d.$$

By this there exist some constants $C_1, C_2 > 0$ such that the following holds: *if* $0 < 2\alpha < d$, *then*

$$J^\alpha(x) \leqslant \begin{cases} C_1 |x|^{-d+2\alpha} & \textit{for } |x| < 1, \\ C_1 e^{-C_2|x|} & \textit{for } |x| \geqslant 1; \end{cases} \tag{A.3}$$

if $0 < d < 2\alpha$, *then*

$$J^\alpha(x) \leqslant C_1 e^{-C_2|x|} \quad \textit{for } x \in \boldsymbol{R}^d; \tag{A.4}$$

if $0 < 2\alpha = d$, *then*

$$J^\alpha(x) \leqslant \begin{cases} C_1 - \frac{2}{(4\pi)^{\frac{d}{2}} \Gamma(\frac{d}{2})} \log |x| & \textit{for } |x| < 1 \\ C_1 e^{-C_2|x|} & \textit{for } |x| \geqslant 1. \end{cases} \tag{A.5}$$

Proposition A.2 (Fubini type lemma for multiple stochastic integrals). *Let* $g \in L^2(\boldsymbol{R}^d)$ *and* $K \in L^2((\boldsymbol{R}^d)^{p+1})$. *Suppose that* K *satisfies the following*:

$K(x; y_1, \ldots, y_p)$ *is symmetric in the last* p *variables* $(y_1, \ldots, y_p) \in \boldsymbol{R}^{pd}$ (*for all* $x \in \boldsymbol{R}^d$);

there exists a compact set $D \subset \boldsymbol{R}^d$ *such that* $K(x; y_1, \ldots, y_p) = 0$ *for* $(x, y_1, \ldots, y_p) \in D^c \times (\boldsymbol{R}^d)^p$; *and the map* $\boldsymbol{R}^d \ni x \mapsto K(x; \cdot) \in L^2((\boldsymbol{R}^d)^p)$ *is continuous.*

Then, the stochastic process $I_p(K_x)(\omega) = \int_{(\boldsymbol{R}^d)^p} K(x; y_1, \ldots, y_p) \, dW_\omega(y_1) \cdots dW_\omega(y_p)$ *on* $\boldsymbol{R}^d$ *has an equivalent process which is measurable with respect to the two variables* (ω, x), *we simply denote this by* $I_p(K_x)(\omega)$. *The following Fubini type formula holds*:

$$\int_{\boldsymbol{R}^d} g(x) I_p(K_x)(\omega) \, dx = \int_{(\boldsymbol{R}^d)^p} \left(\int_{\boldsymbol{R}^d} g(x) \, K(x; y_1, \ldots, y_p) \, dx \right) dW_\omega(y_1) \cdots dW_\omega(y_p), \quad P\text{-a.s.}, \ \omega \in \Omega. \tag{A.6}$$

Proof. For each x we denote $K(x; y_1, \ldots, y_p)$ by $K_x(y_1, \ldots, y_p)$, and the multiple stochastic integrals with order p by I_p. Since for each $x \in \boldsymbol{R}^d$, $\|I_p(K_x)(\cdot)\|^2_{L^2(\Omega, P)} = p! \|K_x(\cdot)\|^2_{L^2((\boldsymbol{R}^d)^p; \lambda^{pd})}$ and since $\boldsymbol{R}^d \ni x \mapsto K(x; \cdot) \in L^2((\boldsymbol{R}^d)^p; \lambda^{pd})$ is assumed to be continuous, by Bochner Von Neumann measurability theorem $I_p(K_x)(\omega)$ has an equivalent process that is measurable with respect to (ω, x). Moreover from the assumption of K it is easy to see that $\int_{\boldsymbol{R}^d} I_p(K_x) \, dW(x)$ is well defined ($I_p(K_x)$ is a process that is Skorohod integrable). Since $E[(I_1(g))^2 \int_{\boldsymbol{R}^d} (I_p(K_x))^2 \, dx] < \infty$ for any $g \in L^2(\boldsymbol{R}^d)$, we can apply formula (1.49) in [44] (*the Skorohod integral of a process multiplied by a random variable*) to $I_1(g) I_p(K_x)$, then

$$\int_{\boldsymbol{R}^d} (I_1(g) I_p(K_x)) \, dW(x) = I_1(g) \int_{\boldsymbol{R}^d} I_p(K_x) \, dW(x) - \int_{\boldsymbol{R}^d} g(x) I_p(K_x) \, dx, \quad P\text{-a.s.}, \ \omega \in \Omega. \tag{A.7}$$

On the other hand, from the definition of multiple stochastic integrals the following equality holds in the sense of equivalent processes on $\boldsymbol{R}^d$:

$$I_1(g) I_p(K_x) = I_{p+1}(K_x(\cdot) g(\cdot)) + p I_{p-1} \left(\int_{\boldsymbol{R}^d} K_x(y_1, \ldots, y_p) g(y_1) \, dy_1 \right).$$

Moreover, if we let

$$R(x, y_1, \ldots, y_p) = \frac{1}{p+1} \{ K_x(y_1, \ldots, y_p) + K_{y_1}(x, y_2, \ldots, y_p) + \cdots + K_{y_p}(y_1, \ldots, y_{p-1}, x) \}$$

(the symmetrization of $K(x; y_1, \dots, y_p)$), then since $\int_{\boldsymbol{R}^d} I_p(K_x)\, dW(x) = I_{p+1}(R)$, we have

$$I_1(g) \int_{\boldsymbol{R}^d} I_p(K_x)\, dW(x) = I_{p+2}(Rg) + (p+1) I_p \left(\int_{\boldsymbol{R}^d} R(x, \cdot) g(x)\, dx \right),$$
$$P\text{-a.s.}, \quad \omega \in \Omega.$$

Substituting these two equalities for the integrand on the left-hand side, resp. for the first term on the right-hand side, of (A.7), we obtain (A.6). □

Lemma A.1. *Let* $p \in \boldsymbol{N}$, $a, b, \alpha > 0$. *For each fixed* k *let*

$$I(\omega, z) = \int_{(\boldsymbol{R}^d)^p} \left(\int_{\boldsymbol{R}^d} J^a(x-z)(1+|x|^2)^{-\frac{b}{4}} F_k^\alpha(x; y_1, \dots, y_p)\, dx \right) \times dW_\omega(y_1) \cdots dW_\omega(y_p),$$

then $I(\omega, z)$ *has an equivalent process which is measurable with respect to* (ω, z), *we simply denote this by* $I(\omega, z)$. *The following holds*:

$$P\left(I(\omega, z) = \int_{\boldsymbol{R}^d} J^a(x-z)(1+|x|^2)^{-\frac{b}{4}} :_k \phi^p_{\alpha,\omega} : (x)\, dx \right) = 1, \quad \forall z \in \boldsymbol{R}^d, \tag{A.8}$$

where $:_k \phi^p_{\alpha,\omega}:$ *is the* $C_0(\boldsymbol{R}^d \to \boldsymbol{R})$*-valued random variable given in Remark 1.2*.

Proof. Since the process $\{:_k \phi^p_{\alpha,\omega} : (x)\}_{x \in \boldsymbol{R}^d}$ defined by Remark 1.2 (cf. (24)) satisfies the assumption for $I_p(K_x)(\omega)$ in Proposition A.2, for each $z \in \boldsymbol{R}^d$ we have

$$\int_{\boldsymbol{R}^d} J^a_n(x-z)(1+|x|^2)^{-\frac{b}{4}} :_k \phi^p_{\alpha,\omega} : (x)\, dx$$
$$= \int_{(\boldsymbol{R}^d)^p} \left(\int_{\boldsymbol{R}^d} J^a_n(x-z)(1+|x|^2)^{-\frac{b}{4}} F_k^\alpha(x; y_1, \dots, y_p)\, dx \right) \times dW_\omega(y_1) \cdots dW_\omega(y_p), \quad \text{a.s.}, \quad \omega \in \Omega, \quad \forall n.$$

But for each fixed k we have $:_k \phi^p_{\alpha,\omega} : (\cdot) \in C_0(\boldsymbol{R}^d \to \boldsymbol{R})$, so if we let $n \to \infty$ in the above equation then by Lebesgue convergence theorem we obtain (A.8).

The fact that $I(\omega, z)$ has an equivalent process that is measurable with respect to (ω, z) follows from Bochner Von Neumann measurability theorem (cf. the proof of Proposition A.2). □

Proof of Theorem 1.1. Since the proof of Theorem 1.1(i) is similar and simpler than that of Theorem 1.1(ii), we only give the proof of (ii). Also since the proof of (28) for $d \geqslant 3$, $\alpha = \frac{d}{4}$, $|\varepsilon| < a_0(d)$, $\varepsilon^2 d \{4(a_0(d))^2\}^{-1} < a < \frac{d}{4}$

and $b > d$ can be done by exchanging the corresponding subscripts and superscripts in the proof of (28) for $d = 2$, $\alpha = \frac{1}{2}$, $|\varepsilon| < 2\sqrt{\pi}$, $\frac{\varepsilon^2}{8\pi} < a < \frac{1}{2}$ and $b > 2$, we give the proof only for the latter case. The other cases considered in Theorem 1.1(ii) (i.e. $\alpha > \frac{d}{4}$) are easy and omitted.

Let $d = 2$, $\alpha = \frac{1}{2}$ and $|\varepsilon| < 2\sqrt{\pi}$, and define

$$:_{k,l}\ \phi_\omega^p : (x) = \int_{(\boldsymbol{R}^2)^p} F_{k,l}(x; y_1, \dots, y_p)\, dW_\omega(y_1) \cdots dW_\omega(y_p),$$

where $F_{k,l}(x; y_1, \dots, y_p) = (\eta_l(x))^p J_k^{\frac{1}{2}}(x - y_1) \cdots J_k^{\frac{1}{2}}(x - y_p)$. Then by virtue of Lemma A.1 (with a slight modification) we see that for $a > 0$ and $b > 2$

$$\begin{aligned} &E[\| :_{k,l}\ \phi^p : - :_{k,l'}\ \phi^p : \|^2_{B_2^{a,b}}] \\ &= \int_\Omega \int_{\boldsymbol{R}^2} \left| \int_{\boldsymbol{R}^2} (1 + |x|^2)^{-\frac{b}{4}} J^a(x - z)(:_{k,l}\ \phi_\omega^p : (x) - :_{k,l'}\ \phi_\omega^p : (x))\, dx \right|^2 \\ &\quad \times dz P(d\omega) \\ &= (2\pi)^{-2(p+1)} p! \int_{\boldsymbol{R}^2} \int_{(\boldsymbol{R}^2)^p} \left| \hat{g}_{l,l'} \left(\sum_{j=1}^p \xi_j + \xi \right) \right|^2 (|\xi|^2 + 1)^{-2a} \\ &\quad \times \prod_{j=1}^p (|\xi_j|^2 + m^2)^{-1} (\hat{\rho}_k(\xi_j))^2\, d\xi_1 \cdots d\xi_p d\xi \\ &\leqslant (2\pi)^{-2(p+1)} p! (m^{-2})^p (\|\hat{\rho}_k\|_{L^1})^p \int_{|x| \geqslant \min(l,l')} (1 + |x|^2)^{-\frac{b}{2}}\, dx, \end{aligned} \tag{A.9}$$

where

$$g_{l,l'}(x) = (1 + |x|^2)^{-\frac{b}{4}} ((\eta_l(x))^p - (\eta_{l'}(x))^p) \in \bigcap_{r \geqslant 1} L^r,$$

and we have used Young's inequality repeatedly.

For

$$: \exp(\varepsilon_{k,l} \phi) := \sum_{p=0}^\infty \frac{\varepsilon^p}{p!} :_{k,l}\ \phi^p :$$

by making use of inequality (A.9) we can show that there exists: $\exp(\varepsilon_{k,\infty}\phi) : \in L^2(\Omega \to B_2^{a,b}; P)$ such that

$$\lim_{l \to \infty} E[\| : \exp(\varepsilon_{k,\infty}\phi) : - : \exp(\varepsilon_{k,l}\phi) : \|^2_{B_2^{a,b}}] = 0, \tag{A.10}$$

where a and b are any real numbers such that $a > 0$ and $b > 2$. Indeed by (A.9) we have

$$\begin{aligned} &E[\| : \exp(\varepsilon_{k,l}\phi) : - : \exp(\varepsilon_{k,l'}\phi) : \|^2_{B_2^{a,b}}] \\ &= \left(\exp\left(\frac{2\varepsilon^2 \|\hat{\rho}_k\|_{L^1}}{m^2 \pi^2} \right) - 1 \right) (2\pi)^{-2} \int_{|x| \geqslant \min(l,l')} (1 + |x|^2)^{-\frac{b}{2}}\, dx. \end{aligned}$$

By this $\{:\exp(\varepsilon_{k,l}\phi):\}_{l\in\boldsymbol{N}}$ forms a Cauchy sequence in $L^2(\Omega\to B_2^{a,b};P)$, for each fixed k, and (A.3) is proved.

Now, by (A.10) for any $\varepsilon>0, k$ and k' there exists $L(\varepsilon,k,k')$, and the following holds for all $l\geqslant L(\varepsilon,k,k')$:

$$(E[\|:\exp(\varepsilon_{k,\infty}\phi):-:\exp(\varepsilon_{k',\infty}\phi):\|^2_{B_2^{a,b}}])^{\frac{1}{2}} \leqslant (E[\|:\exp(\varepsilon_{k,l}\phi):-:\exp(\varepsilon_{k',l}\phi):\|^2_{B_2^{a,b}}])^{\frac{1}{2}}+\varepsilon. \tag{A.11}$$

Also, again from Lemma A.1

$$E[\|:\exp(\varepsilon_{k,l}\phi):-:\exp(\varepsilon_{k',l}\phi):\|^2_{B_2^{a,b}}] = \int_{\boldsymbol{R}^2\times\boldsymbol{R}^2} J^{2a}(x_1-x_2)(1+|x_1|^2)^{-\frac{b}{4}}(1+|x_2|^2)^{-\frac{b}{4}}(A(l;k,k;x_1,x_2) - 2A(l;k,k';x_1,x_2)+A(l;k',k';x_1,x_2))\,dx_1\,dx_2, \tag{A.12}$$

where

$$A(l;k,k';x_1,x_2)=\exp(\varepsilon^2\eta_l(x_1)\eta_l(x_2)J_{k,k'}(x_1-x_2))-1$$

with $J_{k,k'}(x_1-x_2)=\int_{\boldsymbol{R}^2\times\boldsymbol{R}^2}\rho_k(z)\rho_{k'}(z')J^1((x_1-x_2)-(z-z'))\,dz\,dz'$. Hence, for $a>0$ and $b>2$ if we let $l\uparrow\infty$ on the right-hand side of (A.11), then by (A.12) and the Lebesgue convergence theorem we have

$$E[\|:\exp(\varepsilon_{k,\infty}\phi):-:\exp(\varepsilon_{k',\infty}\phi):\|^2_{B_2^{a,b}}] \leqslant \int_{\boldsymbol{R}^2\times\boldsymbol{R}^2} J^{2a}(x_1-x_2)(1+|x_1|^2)^{-\frac{b}{4}}(1+|x_2|^2)^{-\frac{b}{4}}[(e^{\varepsilon^2 J_{k,k}(x_1-x_2))}-1) - 2(e^{\varepsilon^2 J_{k,k'}(x_1-x_2))}-1)+(e^{\varepsilon^2 J_{k',k'}(x_1-x_2))}-1)]\,dx_1\,dx_2. \tag{A.13}$$

Let us evaluate the right-hand side of (A.13), using an argument for Wick exponentials developed by [8] (a simple application of Fourier transform and Young's inequality would not give such a nice estimation). From Proposition A.1(ii) for $d=2$ since $J^1(x)=\frac{1}{2\pi}K_0(m|x|)$, for $|x|<1$ the function $J^1(x)+\frac{1}{2\pi}\log|x|$ is bounded, and for $|x|\geqslant 1$, $J^1(x)$ is a bounded function that decays exponentially to 0 as $|x|\to\infty$. Thus, if $|\varepsilon|<2\sqrt{\pi}$ and $a>\frac{\varepsilon^2}{8\pi}$, then $J^{2a}(x)(\exp(\varepsilon J^1(x))-1)\in L^1$. Let $C_{\varepsilon,a}\equiv\int_{\boldsymbol{R}^2}J^{2a}(x)(\exp(\varepsilon J^1(x)) -1)\,dx$, then by Young's inequality, for $b>2$ we have

$$\int_{\boldsymbol{R}^2\times\boldsymbol{R}^2} J^{2a}(x_1-x_2)(1+|x_1|^2)^{-\frac{b}{4}}(1+|x_2|^2)^{-\frac{b}{4}}(e^{\varepsilon^2 J^1(x_1-x_2))}-1)\,dx_1\,dx_2 \leqslant C_{\varepsilon,a}\|(|x|^2+1)^{-\frac{b}{2}}\|^2_{L^2}.$$

In addition since $\lim_{k,k'\to\infty}e^{\varepsilon^2 J_{k,k'}(x_1-x_2)}=e^{\varepsilon^2 J^1(x_1-x_2)}$ *a.e.* x_1,x_2, we can apply Fatou's lemma to each term on the right-hand side of (A.13), and then conclude that $\{:\exp(\varepsilon_{k,\infty}\phi):\}_{k=1,2,\dots}$ forms a Cauchy sequence in $L^2(\Omega\to B_2^{a,b})$. This proves Theorem 1.1(ii): □

Proof of Theorem 1.2. Since the map $\boldsymbol{R}^d \ni x \mapsto J_k^\beta(x-\cdot) \in \mathscr{S}$ is continuous, τ_k defined by (33) maps $\mathscr{S}'(\boldsymbol{R}^d)$ to $C_0(\boldsymbol{R}^d)$ and by (16) this defines a $\mathscr{B}(B_d^{a',b})/\mathscr{B}(B_d^{a,b})$ measurable map.

For each $x \in \boldsymbol{R}^d$ if we set $\varphi(\cdot) = J^{-\frac{1}{2}}(J_k^\beta(x-\cdot))$ then from (27) for $p=1$ we have

$$\begin{aligned}
&\eta_k(x)\langle J_k^\beta(x-\cdot), (J^{-\frac{1}{2}}\phi_\omega)(\cdot)\rangle_{\mathscr{S},\mathscr{S}'} \\
&\quad = \eta_k(x)\langle J^{-\frac{1}{2}}(J_k^\beta(x-\cdot)), \phi_\omega(\cdot)\rangle_{\mathscr{S},\mathscr{S}'} \\
&\quad = \eta_k(x)\int_{\boldsymbol{R}^d} J_k^\beta(x-y)\, dW_\omega(y) = {}_k\,\phi_{\beta,\omega}(x), \quad P\text{-a.s.}, \ \omega \in \Omega.
\end{aligned}$$

Hence, by (33) for each $x \in \boldsymbol{R}^d$

$$\begin{aligned}
\tau_k(\phi_\omega)(x) &= p\,!\sum_{n=0}^{[\frac{p}{2}]} \frac{(-\frac{1}{2}c_{\beta,k})^n}{n!(p-2n)!}(\eta_k(x))^{2n}\,({}_k\phi_{\beta,\omega}(x))^{p-2n} \\
&= :_k\,\phi_{\beta,\omega}^p : (x), \quad P\text{-a.s.}, \ \omega \in \Omega,
\end{aligned}$$

where the last equality follows from the well-known Wiener chaos expression of multiple stochastic integrals. But both sides of the above formula are continuous processes, hence we have (34).

By means of (16) Eq. (34) can be understood as an equality with respect to two $B_d^{a',b}$-valued random variables: $\tau_k(\phi_\omega) = :_k \phi_{\beta,\omega}^p:$ P-a.s., $\omega \in \Omega$. Hence from (26)

$$\begin{aligned}
&\int_{B_d^{a,b}} \|\tau_k(\psi) - \tau_m(\psi)\|^2_{B_d^{a',b}}\mu(d\psi) = \int_\Omega \|\tau_k(\phi_{\alpha,\omega}) - \tau_m(\phi_{\alpha,\omega})\|^2_{B_d^{a',b}} P(d\omega) \\
&\quad \to 0 \quad \text{as} \quad k,m \to \infty.
\end{aligned}$$

Thus, $\{\tau_k\}_{k\in \boldsymbol{N}}$ forms a Cauchy sequence in $L^2(B_d^{a,b} \to B_d^{a,b'};\mu)$, and (35) is proved.

The Proof of Theorem 1.2(ii) is similar and will therefore be omitted. □

Proof of Theorem 1.3. By the discussion made in Remark A.1, we can see that ∇u_p, δu_p, ∇u_e and δu_e have the expressions given in this theorem.

Before proving Theorem 1.3(i), we shall prove Theorem 1.3(ii). In order to simplify the notations we prove this theorem only for the case $d=2$, $\beta = \frac{1}{2}$. Since the proofs for the other cases are similar, we will only point out the differences between the former case and the other cases at the end of this proof.

The $\boldsymbol{D}_{2,1}(\mathscr{H})$ property of u_e has been essentially proven by Albeverio and Høegh-Krohn [8], namely, if β and ε satisfy the assumptions of

Theorem 1.3(ii-1°), by Proposition A.1(ii) (cf. also the proof of Theorem 1.1)

$$E^{\mu}[\|\nabla u_e\|^2_{\mathscr{H}\otimes\mathscr{H}}] = \varepsilon^2 \int_{\boldsymbol{R}^2\times\boldsymbol{R}^2} \eta_M(x)(J^1(x-y))^2 \exp\{\varepsilon^2 J^1(x-y)\} \times \eta_M(y)\,dx\,dy < \infty.$$

Moreover by Theorem 1.1(ii) and Theorem 1.2(ii) it has been proved that $u_e(\psi)\in\mathscr{H}$ (for μ-a.s., $\psi\in B^{a,b}_d$).

Next, we shall prove the $\mathscr{H}$-regularity of u_e. For this purpose we firstly note that for $h\in H^1(\boldsymbol{R}^2)$ and $\varepsilon\in\boldsymbol{R}$ the following holds:

$$e^{\varepsilon h}-1\in H^{1-\delta}, \quad \lim_{k\to\infty} \|(e^{\varepsilon h_k}-1)-(e^{\varepsilon h}-1)\|_{H^{1-\delta}} = 0, \;\; \forall\delta>0, \tag{A.14}$$

where

$$h_k(x) \equiv \langle J_k^{\frac{1}{2}}(x-\cdot), J^{-\frac{1}{2}}h\rangle_{\mathscr{S},\mathscr{S}'}.$$

Indeed, by a simple application of Fourier transform and Young's inequality it is easy to see that there exists a constant C and for any δ and δ' satisfying $0<\delta'<\delta<1$ the following hold:

$$\|(h)^p\|_{H^{1-\delta}} \leqslant Cp(\|h\|_{H^1})^p \|(1+|\xi|^2)^{-\frac{\delta}{2}}\|_{L^{\frac{2}{\delta'}}} \left(\frac{\pi(p-1-\delta')}{\delta'}\right)^{\frac{p-1-\delta'}{2}}, \quad \forall h\in H^1. \tag{A.15}$$

Using this and a simple calculation we obtain

$$\begin{aligned}\|e^{\varepsilon h}-1\|_{H^{1-\delta}} &\leqslant \sum_{p=1}^{\infty} \frac{\varepsilon^p}{p!}\|(h)^p\|_{H^{1-\delta}} \\ &\leqslant C\|(1+|\xi|^2)^{-\frac{\delta}{2}}\|_{L^{\frac{2}{\delta'}}} \sum_{p=1}^{\infty}\left(\varepsilon C'\sqrt{\frac{1}{\delta'}}\right)^p \frac{\|h\|^p_{H^1}}{\Gamma(\frac{p}{2}+1)}, \\ &\forall h\in H^1(\boldsymbol{R}^d) \quad \text{for any } 0<\varepsilon<\varepsilon',\end{aligned} \tag{A.16}$$

where C is a constant which does not depend on h, ε, δ, δ' and p.

From this, (A.14) follows immediately.

Next by Theorem 1.2(ii) (cf. definition of $\tau_{(p),k,l}$) we have

$$\tau^{k,l}_{(\frac{1}{2},e^{\varepsilon})}(\psi+h) = e^{\varepsilon h_k}\tau^{k,l}_{(\frac{1}{2},e^{\varepsilon})}(\psi), \quad \forall\psi\in B^{a,b}_2, \;\; \forall h\in H^1.$$

If we assume that ε satisfies the assumption of Theorem 1.3(ii2°):

$$|\varepsilon| < \frac{a_0(2)}{\sqrt{2}},$$

then by (38), (A.14) and Definition 1.1 for some a'' such that $a'<a''<\frac{1}{2}$ the following holds:

$$\lim_{k_j\to\infty}\lim_{l_i\to\infty}\|e^{\varepsilon h_k}\tau^{k_j,l_i}_{(\frac{1}{2},e^{\varepsilon})}(\psi)-e^{\varepsilon h}\tau_{(\frac{1}{2},e^{\varepsilon})}(\psi)\|^2_{B^{a'',b}_d}=0,$$
$$\forall\psi\in B(\tfrac{1}{2},e),\ \ \forall h\in H^1.$$

Again, from Definition 1.1 this equation tells us that

$$B(\tfrac{1}{2},e)+H^1\subset\bar{B}(\tfrac{1}{2},e)\quad\text{and}\quad\tau_{(\frac{1}{2},e^{\varepsilon})}(\psi+h)=e^{\varepsilon h}\tau_{(\frac{1}{2},e^{\varepsilon})}(\psi),$$
$$\forall\psi\in B(\tfrac{1}{2},e),\quad\forall h\in H^1. \tag{A.17}$$

Now, by (A.14), (A.16) and (A.17) by passing through a similar (and easier) discussion as in the proof of the $H-C^1$ property of u_e given below, it can be shown that for each $\psi\in B(\frac{1}{2},e)$ the map

$$\mathscr{H}=H^{-1}\ni h\mapsto\eta_M\tau_{(\frac{1}{2},e^{\varepsilon})}(\psi+i(h))\in H^{-1}$$

is continuous.

Next, we shall show that u_e is $H-C^1$ when ε satisfies the stronger assumption:

$$|\varepsilon|<\frac{a_0(d)}{\sqrt{3}}.$$

For this purpose and for later use we remark the following (A.19):

Let $d=2$ and $\beta=\frac{1}{2}$, and suppose that b and M are any non-negative numbers. Then for any non-negative a and δ satisfying

$$a+\frac{\delta}{6}<\frac{1}{6} \tag{A.18}$$

there exists $K=K(a,\delta;b,M)$ such that

$$\int_{\boldsymbol{R}^2\times\boldsymbol{R}^2}(\langle J^{\frac{1}{2}}(x-\cdot)J^{\frac{1}{2}}(y-\cdot),\eta_M(\cdot)h(\cdot)\varphi(\cdot)\rangle_{\mathscr{S}',\mathscr{S}})^2\,dx\,dy$$
$$\leqslant K\|h\|^2_{H^{1-\delta}}\|\varphi\|^2_{B^{a,b}_2},\quad\forall h,\ \ \forall\varphi\in\mathscr{S}(\boldsymbol{R}^2). \tag{A.19}$$

This inequality also follows easily from a standard argument by means of Fourier transforms and the theory of pseudo-differential operators (cf. for e.g. [36]). Actually this can be derived directly as follows: For h, $\varphi\in\mathscr{S}$, let h_0 and φ_0 be such that $h(x)=(J^{\frac{1-\delta}{2}}h_0)(x)$, $\varphi(x)=(|x|^2+1)^{\frac{b}{4}}(J^{-a}\varphi_0)(x)$. If we set $\eta'_M(x) = \eta_M(x)(|x|^2+1)^{\frac{b}{4}}$, then we have

$$\begin{aligned}
&\langle J^{\frac{1}{2}}(x-\cdot)J^{\frac{1}{2}}(y-\cdot), \eta_M(\cdot)h(\cdot)\varphi(\cdot)\rangle_{\mathscr{S}',\mathscr{S}} \\
&\quad = (2\pi)^2 \int_{(\boldsymbol{R}^2)^4} \hat{\eta}'_M(\xi_1)(|\xi_2|^2+m^2)^{-\frac{1-\delta}{2}}\hat{h}_0(\xi_2) \\
&\qquad \times (|\xi_1+\xi_2+\xi_4+\xi_5|^2+m^2)^a\hat{\varphi}_0(-\xi_1-\xi_2-\xi_4-\xi_5) \\
&\qquad \times (|\xi_4|^2+m^2)^{-\frac{1}{2}}(|\xi_5|^2+m^2)^{-\frac{1}{2}}e^{-\sqrt{-1}x\xi_4}e^{-\sqrt{-1}y\xi_5}\,\overline{d\xi_1}\,\overline{d\xi_2}\,\overline{d\xi_4}\,\overline{d\xi_5},
\end{aligned}$$

where $\overline{d\xi_j} = (2\pi)^{-2}\,d\xi_j$. Hence, by making use of the elementary inequality

$$\left|\sum_{j=1}^n x_j\right|^2 + L^2 \leqslant nL^{-2(n-1)}\prod_{j=1}^n(|x_j|^2+L^2),$$

and setting $\hat{\eta}''_M(\xi) = (|\xi|^2+m^2)^a\hat{\eta}'_M(\xi)$, we have

$$\begin{aligned}
&\int_{\boldsymbol{R}^2\times\boldsymbol{R}^2}(\langle J^{\frac{1}{2}}(x-\cdot)J^{\frac{1}{2}}(y-\cdot), \eta_M(\cdot)h(\cdot)\varphi(\cdot)\rangle_{\mathscr{S}',\mathscr{S}})^2\,dx\,dy \\
&\quad \leqslant (2\pi)^8(4m^{-6})^{2a}\int_{(\boldsymbol{R}^2)^6}\hat{\eta}''_M(\xi_1)\hat{\eta}''_M(\xi'_1)(|\xi_2|^2+m^2)^{-\frac{1-\delta}{2}+a} \\
&\qquad \times (|\xi'_2|^2+m^2)^{-\frac{1-\delta}{2}+a}\hat{h}_0(\xi_2)\hat{h}_0(\xi'_2) \\
&\qquad \times \hat{\varphi}_0(-\xi_1-\xi_2-\xi_4-\xi_5)\hat{\varphi}_0(-\xi'_1-\xi'_2-\xi_4-\xi_5) \\
&\qquad \times (|\xi_4|^2+m^2)^{-1+2a}(|\xi_5|^2+m^2)^{-1+2a}\,\overline{d\xi_1}\,\overline{d\xi_2}\,\overline{d\xi'}_1\,\overline{d\xi'}_2\,\overline{d\xi_4}\,\overline{d\xi_5} \\
&\quad \leqslant (2\pi)^4(4m^{-6})^{2a}\{\|\hat{\eta}''_M\|_{L^1}\|(|\xi|^2+m^2)^{-\frac{1-\delta}{2}+a}\|_{L^q} \\
&\qquad \times \|(|\xi|^2+m^2)^{-1+2a}\|_{L^s}\|\varphi_0\|_{L^2}\|h_0\|_{L^2}\}^2,
\end{aligned}$$

where the latter inequality holds under the condition that there exist q and s such that

$$(|\xi|^2+m^2)^{-\frac{1-\delta}{2}+a}\in L^q, \quad (|\xi|^2+m^2)^{-1+2a}\in L^s, \quad 1\leqslant q,\ s\leqslant\infty,\ \frac{1}{q}+\frac{1}{s}=1.$$

If a and δ satisfy (A.18), then there exist such q and s. Hence, under condition (A.18) we can take $K(a,\delta;b,M) = (2\pi)^4(4m^{-6})^{2a}(\|\hat{\eta}''_M\|_{L^1}\|(|\xi|^2+m^2)^{-\frac{1-\delta}{2}+a}\|_{L^q}\|(|\xi|^2+m^2)^{-1+2a}\|_{L^s})^2$, and since $\|h_0\|_{L^2} = \|h\|_{H^{1-\delta}}$ and $\|\varphi\|_{L^2} = \|\varphi\|_{B_2^{a,b}}$, for $h,\varphi\in\mathscr{S}$, we obtain (A.19).

Now, since $\mathscr{S}$ is dense in $H^{1-\delta}$ and $B_2^{a,b}$, from (A.14), (A.17) and (A.19) we see that

$$\begin{aligned}
&\|\nabla u_e(\psi+i(h)) - \nabla u_e(\psi)\|_{\mathscr{H}\otimes\mathscr{H}} \\
&\quad \leqslant K(a',\delta;b,M)\|\tau_{(\frac{1}{2},e^{\varepsilon})}(\psi)\|_{B_2^{a',b}}\|e^{\varepsilon i(h)}-1\|_{H^{1-\delta}}, \quad \forall\psi\in B, \quad \forall h\in H^{-1},
\end{aligned} \tag{A.20}$$

where from Theorem 1.1(ii) a' must satisfy

$$\varepsilon^2\{4(a_0(d))^2\}^{-1} < a'.$$

For $d = 2$ this means that

$$\frac{\varepsilon^2}{8\pi} < a'. \tag{A.21}$$

If $|\varepsilon| < 2\sqrt{\frac{\pi}{3}} = \frac{a_0(2)}{\sqrt{3}}$, then we can take a' satisfying both (A.18) (replacing a by a') and (A.21) (δ can be taken arbitrarily small cf. (A.14)). Hence, from (A.20) and(A.16) the $H - C^1$ property of u_e follows.

For general d if we take $\beta = \frac{d}{4}$, then the corresponding condition of (A.18) becomes

$$a' + \frac{\delta}{6} < \frac{d}{12}.$$

If we combine this with the condition that $\varepsilon^2 d\{4(a_0(d))^2\}^{-1} < a'$ (coming from Theorem 1.1(ii)), then we have a sufficient condition under which $J^{\beta-\frac{1}{2}}(\eta_M \tau_{(\beta, e^{\varepsilon})}(\phi))$ becomes an $H - C^1$ map. In other words $|\varepsilon| < \frac{a_0(d)}{\sqrt{3}}$ is a sufficient condition for the $H - C^1$ property of u_e.

The other parts of Theorem 1.3(ii) are obvious.

Next, we shall prove Theorem 1.3(i). Under the assumptions of Theorem 1.3(i-1°) from Theorem 1.1(i), Theorem 1.2(i) and Proposition A.1(ii) it is easy to see that the following holds:

$$u_p(\psi) \in \mathscr{H} \quad \text{for } \mu\text{-a.s., } B_d^{a,b} \text{ and}$$

$$E^{\mu}[\|\nabla^r u_p\|^2_{\mathscr{H}^{\otimes r}}] = p^2 \cdots (p - r + 1)^2 (p - r)! \int_{\boldsymbol{R}^d \times \boldsymbol{R}^d} \eta_M(x)(J^{2\beta}(x - y))^{p+1} \eta_M(y)\, dx\, dy < \infty, \quad 1 \leqslant r \leqslant p.$$

This proves the $\boldsymbol{D}_{2,r}(\mathscr{H})$ property of u_p.

Next, we write the detailed proof of the $H - C^1$ property of u_p only for $d = 2$, $\beta = \frac{1}{2}$ and $p = 3$ (proofs for the cases where $d \geqslant 3$, $\beta = \frac{d}{4}$ and $p = 3$ are essentially the same as in the former case and the other cases can be proved in a similar manner (cf. (A.15),(A.19) and (46))).

Let $B(\frac{1}{2}, 3)$ be the measurable set introduced in Definition 1.1(ii) for $d = 2$, $\beta = \frac{1}{2}$ and $p = 3$, then similar to (A.17), by noting (33) and (A.15) it is seen that

$$B(\tfrac{1}{2}, 3) + H^1 \subset \bar{B}(\tfrac{1}{2}, 3) \quad \text{and} \quad \tau_{(\frac{1}{2}, 2)}(\psi + h) = \tau_{(\frac{1}{2}, 2)}(\psi) + 2h\psi + h^2,$$
$$\forall \psi \in B(\tfrac{1}{2}, 3), \quad h \in H^1.$$

Hence, from (A.15) and the fact that $\tau_{(\frac{1}{2},p)}(\phi) \in \bigcap_{a>0} B_2^{a,b}$, $(p = 1, 2, \dots)$ by Theorem 1.2, the $H - C^1$ property of u_3 can be proved in the same manner as in the proof for this property of u_e given above.

The proofs of the other assertions in Theorem 1.4(i) are obvious. □

Proof of Lemma 2.1 (Key lemma for the map with cubic power). Here we will prove (59):

$$\exp\left\{-\lambda\delta u_3 + \frac{1+\varepsilon}{2}\lambda^2 \|\nabla u_3\|_2^2\right\} \in \bigcap_{q<\infty} L^q(\mu). \tag{A.22}$$

For simplicity we will give a proof only for the case $d = 2$ and $\beta = \frac{1}{2}$.

The proof will be performed by following a strategy given by Nelson [42]. Namely, let

$$V \equiv -\lambda\delta u_3 + \frac{1+\varepsilon}{2}\lambda^2 \|\nabla u_3\|_2^2 \quad \text{and} \quad V_k \equiv -\lambda\delta u_{3,k} + \frac{1+\varepsilon}{2}\lambda^2 \|\nabla u_{3,k}\|_2^2,$$

where

$$u_{3,k}(\psi) = J^{\beta-\frac{1}{2}}(\eta_M \tau_{(\beta,3),k}(\psi)).$$

Suppose that we can show that there exist κ_1, κ_2 and α that do not depend on k such that

$$V_k(\psi) \leqslant \kappa_1 (c_k)^2, \quad \forall k, \quad \mu\text{-a.s.}, \quad \psi \in B_2^{a,b}, \tag{A.23}$$

$$(E^\mu[|V_k - V|^q])^{\frac{1}{q}} \leqslant \kappa_2 (q-1)^2 k^{-\alpha}, \quad q \geqslant 2, \tag{A.24}$$

where $c_k = c_{\frac{1}{2},k} = \int_{\boldsymbol{R}^2} (J_k^{\frac{1}{2}}(y))^2\, dy$, defined in Theorem 1.2. Then through the same discussion as Lemma V.5 of [52], we see that there exist $\alpha' > 0$ and $\beta > 0$, independent of k, such that

$$\mu\{\psi | V \geqslant \beta(\log k)^2\} \leqslant e^{-k^{\alpha'}} \quad \text{for all large } k.$$

Eq. (A.22) easily follows from this inequality (cf. [52, Theorem V.7]).

Hence, it suffices to show that (A.23) and (A.24) hold for our exponent.

Eq. (A.23) can be shown as follows. For $\psi \in B_2^{a,b}$ let $\psi_k(z) \equiv \langle J_k^{\frac{1}{2}}(z - \cdot), (J^{-\frac{1}{2}}\psi)(\cdot)\rangle_{\mathcal{S},\mathcal{S}'}$, then by (33)

$$\tau_{(\frac{1}{2},2),k}(\psi)(z) = 2!(\eta_k(z))^2 \left\{\frac{1}{2!}(\psi_k(z))^2 - \tfrac{1}{2} c_k\right\},$$

by this we see that

$$\begin{aligned}
&\frac{1+\varepsilon}{2}(3\lambda)^2\|\langle \eta_M(\cdot), \tau_{(\frac{1}{2},2),k}(\psi)(\cdot)J^0(\cdot - x)J^0(\cdot - y)\rangle_{\mathcal{S},\mathcal{S}'}\|^2_{\mathcal{H}\otimes\mathcal{H}} \\
&\quad = \frac{(1+\varepsilon)(3\lambda)^2}{2}\int_{\boldsymbol{R}^2\times\boldsymbol{R}^2} \eta_M(z)(\eta_k(z))^2\{(\psi_k(z))^2 - c_k\}\eta_M(z')(\eta_k(z'))^2 \\
&\qquad \times \{(\psi_k(z'))^2 - c_k\}(J^1(z-z'))^2\,dz\,dz' \\
&\quad = -\frac{(1+\varepsilon)(3\lambda)^2}{4}\int_{\boldsymbol{R}^2\times\boldsymbol{R}^2}[\eta_M(z)(\eta_k(z))^2\{(\psi_k(z))^2 - c_k\} \\
&\qquad - \eta_M(z')(\eta_k(z'))^2\{(\psi_k(z'))^2 - c_k\}]^2(J^1(z-z'))^2\,dz\,dz' \\
&\qquad + \frac{(1+\varepsilon)(3\lambda)^2 L}{2}\int_{\boldsymbol{R}^2}(\eta_M(z))^2(\eta_k(z))^4\{(\psi_k(z))^2 - c_k\}^2\,dz, \\
&\qquad \forall\psi\in B_2^{a,b},
\end{aligned} \tag{A.25}$$

where $L = \int_{\boldsymbol{R}^2}(J^1(z))^2\,dz$.

On the other hand, from (33)

$$\begin{aligned}
-\lambda\delta u_3(\psi) &= -\lambda\langle \eta_M, \tau_{(\frac{1}{2},4),k}(\psi)\rangle \\
&= -\lambda\int_{\boldsymbol{R}^2}\eta_M(z)\left[4!(\eta_k(z))^4\left\{\frac{1}{4!}(\psi_k(z))^4 - \frac{\frac{1}{2}c_k}{2!}(\psi_k(z))^2 \right.\right. \\
&\qquad \left.\left. + \frac{(\frac{1}{2}c_k)^2}{2!}\right\}\right]dz, \quad \forall\psi\in B_2^{a,b}.
\end{aligned} \tag{A.26}$$

Since the first term of the right-hand side of (A.25) cannot be positive, from (A.25) and (A.26) we have the evaluation

$$\begin{aligned}
&-\lambda\delta u_k(\psi) + \frac{1+\varepsilon}{2}\lambda^2\|\nabla u_k(\psi)\|^2_{\mathcal{H}\otimes\mathcal{H}} \\
&\quad \leqslant \lambda\int_{\boldsymbol{R}^2}\eta_M(z)(\eta_k(z))^4\Big\{-(\psi_k(z))^4 + 6c_k(\psi_k(z))^2 - 3(c_k)^2 \\
&\qquad + \frac{3^2(1+\varepsilon)}{2}\lambda L\eta_M(z)((\psi_k(z))^4 - 2c_k(\psi_k(z))^2 + (c_k)^2)\Big\}dz, \quad \forall\psi\in B_2^{a,b}.
\end{aligned} \tag{A.27}$$

Since $0\leqslant\eta_M(z)\leqslant 1$, if ε and λ satisfy $\frac{3^2(1+\varepsilon)}{2}\lambda L < 1$, then the term in the bracket of the right-hand side of (A.27), the biquadratic formula of ψ_k, cannot be greater than $\kappa_1'(c_k)^2$, where κ_1' is a constant which is independent of z and k. Hence, we can take $\kappa_1 = \lambda\kappa_1'\int_{\boldsymbol{R}^2}\eta_M(z)\,dz$, and obtain (A.23).

Next, (A.24) can be proved as follows. By Hölder's inequality we see that

$$
\begin{aligned}
&(E^{\mu}[|\|\nabla u_{3,k}\|_2^2 - \|\nabla u_3\|_2^2|^q])^{\frac{1}{q}} \\
&\quad \leqslant (E^{\mu}[\|\nabla u_3\|_2^{2q}])^{\frac{1}{2q}}(E^{\mu}[\|\nabla u_{3,k} - \nabla u_3\|_2^{2q}])^{\frac{1}{2q}} \\
&\qquad + (E^{\mu}[\|\nabla u_{3,k}\|_2^{2q}])^{\frac{1}{2q}}(E^{\mu}[\|\nabla u_{3,k} - \nabla u_3\|_2^{2q}])^{\frac{1}{2q}}.
\end{aligned} \tag{A.28}
$$

But each term in the above expectation such as $\|\nabla u_3\|_2^2$, $\|\nabla u_{3,k} - \nabla u_3\|_2^2$ and $\|\nabla u_{3,k}\|_2^2$ has an expression by means of multiple stochastic integrals, for example

$$
\begin{aligned}
&\frac{\|\nabla u_3(\phi_\omega)\|_2^2}{3^2} \\
&\quad = \int_{\boldsymbol{R}^8}\left(\int_{\boldsymbol{R}^2\times\boldsymbol{R}^2}(J^1(z-z'))^2\eta_M(z)\eta_M(z')J^{\frac{1}{2}}(z-x_1)J^{\frac{1}{2}}(z-x_2)\right. \\
&\quad \left.\times J^{\frac{1}{2}}(z'-x_1')J^{\frac{1}{2}}(z'-x_2')\,dz\,dz'\right) \\
&\quad \times dW_\omega(x_1)\,dW_\omega(x_2)\,dW_\omega(x_1')\,dW_\omega(x_2') \\
&\quad + 4\int_{\boldsymbol{R}^4}\left(\int_{\boldsymbol{R}^2\times\boldsymbol{R}^2}(J^1(z-z'))^3\eta_M(z)\eta_M(z')J^{\frac{1}{2}}\right. \\
&\quad \left.\times (z-x_1)J^{\frac{1}{2}}(z'-x_1')\,dz\,dz'\right) \\
&\quad \times dW_\omega(x_1)\,dW_\omega(x_1') + 2\int_{\boldsymbol{R}^2\times\boldsymbol{R}^2}(J^1(z-z'))^4\eta_M(z)\eta_M(z')\,dz\,dz', \\
&\quad P\text{-a.s.},\quad \omega\in\Omega.
\end{aligned} \tag{A.29}
$$

Using (A.1) and (A.2) and passing to a standard argument concerning the calculation of the expectation of multiple stochastic integrals (cf. [52,56, Section V.1]) by (A.29) and the corresponding expressions through multiple stochastic integrals for the other terms, it is easy to see that there exists C_1 which depends only on M such that

$$(E^{\mu}[(\|\nabla u_3\|_2^2)^2])^{\frac{1}{2}} \leqslant C_1,$$

$$(E^{\mu}[(\|\nabla u_{3,k}\|_2^2)^2])^{\frac{1}{2}} \leqslant C_1,$$

also for each $\alpha > 0$ there exists C_2 which depends only on M such that

$$(E^{\mu}[(\|\nabla u_{3,k} - \nabla u_3\|_2^2)^2])^{\frac{1}{2}} \leqslant C_2 k^{-\alpha}.$$

Since for random variables having multiple stochastic integral representation we can apply Nelson's Hypercontractive bound (cf. [52, Theorem 1.22]), from the above inequalities we can deduce the following:

$$(E^{\mu}[(\|\nabla u_3\|_2^2)^q])^{\frac{1}{q}} \leqslant (q-1)^2 C_1, \tag{A.30}$$

$$(E^{\mu}[(\|\nabla u_{3,k}\|_2^2)^q])^{\frac{1}{q}} \leqslant (q-1)^2 C_1, \tag{A.31}$$

$$(E^{\mu}[(\|\nabla u_{3,k} - \nabla u_3\|_2^2)^q])^{\frac{1}{q}} \leqslant (q-1)^2 C_2 k^{-\alpha}, \quad q = 2, 3, \ldots. \tag{A.32}$$

Then, by (A.28), (A.30–A.32)(we conclude that there exists some C' that depends only on M such that

$$(E^{\mu}[|\,\|\nabla u_{3,k}\|_2^2 - \|\nabla u_3\|_2^2|^q])^{\frac{1}{q}} \leqslant (q-1)^2 C' k^{-\alpha}.$$

Moreover using that $\delta u_{3,k}(\phi_\omega)$ and $\delta u_3(\phi_\omega)$ have expressions by means of multiple stochastic integral we easily see that

$$(E^{\mu}[|\delta u_{3,k} - \delta u_3|^q])^{\frac{1}{q}} \leqslant (q-1)^2 C' k^{-\alpha}.$$

Combining these evaluations we obtain (A.24). □

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