Irreducible Character Degrees and Normal Subgroups

I. M. Isaacs* and Greg Knutson

*Department of Mathematics, University of Wisconsin, 480 Lincoln Drive,
Madison, Wisconsin 53706*

1. INTRODUCTION

Let G be a finite group and, as usual, write $\mathrm{cd}(G)$ to denote the set of degrees of the irreducible characters of G. This set of positive integers encodes a great deal of information about the structure of G, and we mention just a few of the many known results of this type. If G is nilpotent (or more generally, if G is an M-group), a result of K. Taketa asserts that the derived length $\mathrm{dl}(G)$ is at most equal to $|\mathrm{cd}(G)|$. (See Theorem 5.12 of [6].) It has been conjectured that the Taketa inequality $\mathrm{dl}(G) \leq |\mathrm{cd}(G)|$ holds for all solvable groups, but this remains unproved. (It is known, however that $\mathrm{dl}(G) \leq 2|\mathrm{cd}(G)|$ for all solvable groups G [4], and the Taketa inequality is known to hold if $|G|$ is odd [1].)

It is an old result of the first author that if $|\mathrm{cd}(G)| \leq 3$, then G is necessarily solvable and that $\mathrm{dl}(G) \leq |\mathrm{cd}(G)|$. (See Theorems 12.15 and 12.6 of [6].) The Taketa inequality is also known to hold for solvable groups G if $|\mathrm{cd}(G)| = 4$. (This result of S. Garrison is the principal theorem in his Ph.D. thesis [3], which remains unpublished.)

If more information about the set $\mathrm{cd}(G)$ is known that its cardinality, then, of course, one can expect to be able to deduce correspondingly more information about G. For example, J. Thompson proved that if there is some prime p that divides every member of $\mathrm{cd}(G)$ exceeding 1, then G has a normal p-complement. (See Corollary 12.2 of [6].)

Our goal in this paper is to study how the structure of a normal subgroup of G is influenced by the degrees of an appropriate subset of $\mathrm{Irr}(G)$. It seems reasonable that the characters that should be relevant to

controlling the structure of $N \lhd G$ are exactly those whose kernels do not contain N, and so we introduce some convenient notation. Given that $N \lhd G$, we write

$$\mathrm{Irr}(G|N) = \{\chi \in \mathrm{Irr}(G) \,|\, N \nsubseteq \ker(\chi)\}$$

and

$$\mathrm{cd}(G|N) = \{\chi(1) \,|\, \chi \in \mathrm{Irr}(G|N)\}.$$

If we take $N = G'$, the derived subgroup of G, we see that $\mathrm{Irr}(G|N)$ is exactly the set of nonlinear characters in $\mathrm{Irr}(G)$, and thus $|\mathrm{cd}(G|N)| = |\mathrm{cd}(G)| - 1$ in this case. Also, if G is solvable and $N = G'$, then $\mathrm{dl}(N) = \mathrm{dl}(G) - 1$, and hence the conjectured Taketa inequality, that $\mathrm{dl}(G) \leq |\mathrm{cd}(G)|$ for solvable groups, can be restated as $\mathrm{dl}(N) \leq |\mathrm{cd}(G|N)|$, where, of course, $N = G'$.

The first of our main results provides an upper bound for $\mathrm{dl}(N)$ in terms of $|\mathrm{cd}(G|N)|$, where N is an arbitrary solvable normal subgroup of an arbitrary finite group G. We do not need to assume that $N = G'$ or that G is solvable.

THEOREM A.   *Let $N \lhd G$, where N is solvable, and write $n = |\mathrm{cd}(G|N)|$. Then $|\mathrm{dl}(N)| \leq f(n)$ for some quadratic function f. Furthermore, if G is assumed to be solvable, then a linear upper bound exists.*

Returning to the case where $N = G'$, we know that if $|\mathrm{cd}(G|N)| \leq 2$, which corresponds to the situation where $|\mathrm{cd}(G)| \leq 3$, then N is solvable and $\mathrm{dl}(N) \leq \mathrm{cd}(G|N)$. We get exactly this result for arbitrary normal subgroups N of arbitrary finite groups G.

THEOREM B.   *Let $N \lhd G$ and suppose that $|\mathrm{cd}(G|N)| \leq 2$. Then N is solvable and $\mathrm{dl}(N) \leq |\mathrm{cd}(G|N)|$.*

Unfortunately, our proof of the solvability of N in the case where $|\mathrm{cd}(G|N)| = 2$ in Theorem B relies on a consequence the classification of simple groups.

If we assume some solvability hypothesis on the whole group G, we can push this one step farther.

THEOREM C.   *Let $N \lhd G$ and assume that G is p-solvable for all prime divisors p of $|N|$. If $|\mathrm{cd}(G|N)| = 3$, then $\mathrm{dl}(N) \leq 3$.*

Note that in the situation of Theorem C, the subgroup N is automatically solvable. If the whole group G is solvable and $|\mathrm{cd}(G)| = 4$, we can apply Theorem C to the subgroup $N = G'$, and we deduce that $\mathrm{dl}(G) \leq 4$. In other words, Theorem C includes the unpublished result of Garrison's thesis as a special case. It seems reasonable to conjecture that Theorem C

might remain true if the hypothesis that G is p-solvable for all prime divisors p of $|N|$ were replaced by the weaker assumption that N is solvable, but we have been unable to prove this. Perhaps it is even true when N is solvable that $\mathrm{dl}(N) \leq |\mathrm{cd}(G|N)|$, even when $|\mathrm{cd}(G|N)|$, exceeds 3.

There is at least one result already in the literature that provides control over the structure of a subgroup $N \lhd G$ in terms of $\mathrm{cd}(G|N)$. This result of Y. Berkovich [2] is the analog of the theorem of Thompson that we mentioned previously. Our results are dependent on Berkovich's theorem, and so for completeness, we present a simple proof of a somewhat strengthened version.

THEOREM D (Berkovich).   *Let $N \lhd G$ and suppose that every member of* $\mathrm{cd}(G|N')$ *is divisible by some fixed prime p. Then N is solvable and has a normal p-complement.*

As in Theorem B, the proof of the solvability of N relies on the classification of simple groups. The conclusion that N has a normal p-complement, however, is elementary.

Since $\mathrm{Irr}(G|G')$ is exactly the set of nonlinear irreducible characters of G, we see that in the case $N = G$, the conclusion in Theorem D that N has a normal p-complement is exactly Thompson's theorem. It is interesting to compare Theorem D with Berkovich's original result, which is slightly weaker. He showed that the intersection of the kernels of those nonlinear irreducible characters of G having p'-degree has a normal p-complement. In other words, Berkovich showed that $N \lhd G$ has a normal p-complement if all nonlinear members of $\mathrm{Irr}(G|N)$ have degree divisible by p. Theorem D obtains the same conclusion with a weaker hypothesis: we allow $\mathrm{Irr}(G|N)$ to contain nonlinear characters of degree not divisible by p, provided that they lie over linear characters of N.

We close this introduction with a brief explanation of how this paper came to be written. At the time of his death in 1996, Greg Knutson was Isaacs' Ph.D. student at the University of Wisconsin, Madison. Knutson had made substantial progress in his study of the set $\mathrm{cd}(G|N)$, and it seemed appropriate for Isaacs to continue that research and to write what might have become Knutson's thesis. The present paper is the result of that effort.

2. BERKOVICH'S THEOREM

In this section, we give a quick proof of our slightly generalized version of Berkovich's theorem. We begin with the elementary part of the argument.

(2.1) THEOREM. *Let $N \lhd G$ and suppose that every member of $\mathrm{cd}(G|N')$ is divisible by some fixed prime number p. Then N has a normal p-complement.*

Proof. Write $M = \mathbf{O}^p(N)$, let $P \in \mathrm{Syl}_p(M)$, and choose $S \in \mathrm{Syl}_p(G)$ such that $S \supseteq P$. Our goal is to show that p does not divide $|M|$, and so we assume that $P > 1$ and we work to obtain a contradiction. Note that $P = S \cap M \lhd S$, and thus $[P, S] < P$, and we can choose a nonprincipal S-invariant linear character λ of P.

Now S stabilizes λ^M, and thus it permutes the irreducible constituents of this character. But $\lambda^M(1) = |M : P|$ is not divisible by p, and so λ^M must have some S-invariant constituent $\alpha \in \mathrm{Irr}(M)$ with $\alpha(1)$ not divisible by p. Now α is invariant in MS, and the p-power $|MS : M|$ is relatively prime to $\alpha(1)$. Furthermore, the determinantal order $o(\alpha)$ is not divisible by p since $M = \mathbf{O}^p(M)$, and thus $o(\alpha)$ is also relatively prime to $|MS : M|$. We deduce that α extends to some character $\beta \in \mathrm{Irr}(MS)$. (See Corollary 6.28 of [6].)

Next, observe that $\beta^G(1) = \beta(1)|G : MS| = \alpha(1)|G : MS|$ is not divisible by p, and hence it has some constituent $\chi \in \mathrm{Irr}(G)$ with degree not divisible by p. By hypothesis, χ is not a member of $\mathrm{Irr}(G|N')$, and thus $N' \subseteq \ker(\chi)$, and so the irreducible constituents of χ_N are linear. The irreducible constituents of χ_M are therefore linear, and in particular, α is linear. Thus α is an extension of λ to M, and it follows that $o(\lambda)$ divides $o(\alpha)$, which, as we have seen, is not divisible by p, and therefore $o(\lambda)$ is not divisible by p. This is a contradiction since λ is a nontrivial linear character of a p-group, and this completes the proof. ∎

To complete the proof of Theorem D, we need the following known, and relatively easy consequence of the classification of finite simple groups.

(2.2) LEMMA. *Suppose that A acts on G via automorphisms and that $(|A|, |G|) = 1$. If $\mathbf{C}_G(A) = 1$, then G is solvable.*

Sketch of Proof. If $N \lhd G$ admits the action of A, then A has no nontrivial fixed points on N and on G/N. If $1 < N < G$, therefore, the result follows by induction on $|G|$, and we can thus assume that G is characteristically simple. Thus G is the direct product of nonabelian simple subgroups that are permuted by A.

Let S be one of the simple direct factors of G, let B be the stabilizer of S in A, and let T be a transversal for the right cosets of B in A. Then G is the direct product of the subgroups S^t as t runs over T. If $s \in \mathbf{C}_S(B)$, we see that $\prod_{t \in T} s^t$ is centralized by A, and so is trivial. It follows that s is trivial, and hence $\mathbf{C}_S(B) = 1$.

Since $(|S|, |B|) = 1$, it follows from the classification that the only possibilities for B and S are that (up to conjugacy in $\mathrm{Aut}(S)$) the action of B is

via field automorphisms on a group of Lie type defined over some field E. In this case, the fixed point subgroup of B in S is exactly the corresponding group of Lie type defined over the fixed field F of B acting on E. In particular, this fixed-point subgroup is nontrivial, and this is a contradiction. ∎

Proof of Theorem D. By Theorem 2.1, we know that N has a normal p-complement K, and we write $M = N' \cap K$. Then p does not divide $|M|$ and every member of $\mathrm{Irr}(G|M)$ has degree divisible by p. Let $P \in \mathrm{Syl}_p(G)$ and note that if $\theta \in \mathrm{Irr}(M)$ is P-invariant, then θ is extendible to some character $\psi \in \mathrm{Irr}(MP)$. Since p does not divide $|G : MP|\psi(1) = \psi^G(1)$, we see that ψ^G has an irreducible constituent χ with degree not divisible by p. It follows that $\chi \notin \mathrm{Irr}(G|M)$, and hence $M \subseteq \ker(\chi)$ and $\theta = 1_M$. Thus P fixes no nonprincipal irreducible characters of M, and hence $\mathbf{C}_M(P) = 1$ by Glauberman's theorem. (See Theorem 13.1 of [6].) It follows by Theorem 2.2 that M is solvable, and thus N is solvable, as required. ∎

3. AN ANALOG OF TAKETA'S THEOREM

The principal result of this section is the following theorem, whose proof is a variation on the standard Taketa argument used to show that $\mathrm{dl}(G) \leq |\mathrm{cd}(G)|$ when G is an M-group. To state the result, we shall use the notation m_p to denote the p-part of a positive integer m, where p is a prime number. In other words, m_p is the largest power of p that divides m.

(3.1) THEOREM. *Let $N \lhd G$, where N has an abelian normal p-complement for some prime p. If $\chi \in \mathrm{Irr}(G|N)$ is chosen such that $\chi(1)_p$ is as small as possible, then $N' \subseteq \ker(\chi)$. In particular, if $N > 1$, then $\mathrm{cd}(G|N') < \mathrm{cd}(G|N)$.*

Proof. Let $P/N \in \mathrm{Syl}_p(G/N)$ and choose an irreducible constituent ψ of χ_P with $\psi(1)$ as small as possible. By hypothesis, N has an abelian normal p-complement, and since P/N is a p-group, we see that P also has an abelian normal p-complement, and it follows that P is an M-group. (See Theorems 6.22 and 6.23 of [6], for example.) We can thus write $\psi = \lambda^P$, where λ is a linear character of some subgroup $Q \subseteq P$.

If η is an irreducible constituent of the character $(1_Q)^P$, we claim that $N \subseteq \ker(\eta)$. We can certainly assume that η is nonprincipal, and thus $\eta(1) < |P : Q| = \psi(1)$. Also, since P has an abelian normal p-complement, Ito's theorem guarantees that all irreducible characters of P have p-power degrees. (See Theorem 6.15 of [6].) In particular, $\eta(1)$ is a p-power, and

since p does not divide $|G:P|$, we conclude that $\eta(1)$ is exactly the p-part of $\eta^G(1) = \eta(1)|G:P|$. It follows that there exists an irreducible constituent ξ of η^G such that $\xi(1)_p \leq \eta(1)$.

As all irreducible constituents of χ_P have p-power degrees and $\psi(1)$ is the smallest of these degrees, we see that $\psi(1)$ divides $\chi(1)$. Therefore, $\xi(1)_p \leq \eta(1) < \psi(1) \leq \chi(1)_p$, and it follows from the choice of χ that $\xi \notin \mathrm{Irr}(G|N)$, and hence $N \subseteq \ker(\xi)$. But η is an irreducible constituent of ξ_P, and thus $N \subseteq \ker(\eta)$, as claimed. Since this holds for every irreducible constituent η of $(1_Q)^P$, we deduce that $N \subseteq \ker((1_Q)^P) \subseteq Q$, and thus $N' \subseteq Q' \subseteq \ker(\lambda)$.

Since $N' \lhd G$, it follows that N' is contained in the kernel of every irreducible constituent of $\lambda^G = \psi^G$. In particular, $N' \subseteq \ker(\chi)$, as required.

Finally, given that $N > 1$, we see that $\mathrm{cd}(G|N)$ is nonempty. But $\mathrm{cd}(G|N') \subseteq \mathrm{cd}(G|N)$, and this subset is missing all members of $\mathrm{cd}(G|N)$ that have the smallest possible p-part. This completes the proof. $\blacksquare$

We can now prove the case of Theorem B where $|\mathrm{cd}(G|N)| \leq 1$.

(3.2) COROLLARY. *Let $N \lhd G$ and suppose that $|\mathrm{cd}(G|N)| \leq 1$. Then $\mathrm{dl}(N) \leq |\mathrm{cd}(G|N)|$, and in particular, N is abelian.*

Proof. If $N = 1$, then $\mathrm{dl}(N) = 0$ and the desired inequality holds. We thus assume that $N > 1$, and since $|\mathrm{cd}(G|N)| = 1$ in this case, it suffices to show that N is abelian, and we do this by induction on $|N|$. If N is nonabelian, it has a nonlinear irreducible character ψ, and we choose a prime divisor p of $\psi(1)$. If $\chi \in \mathrm{Irr}(G)$ lies over ψ, then $N \nsubseteq \ker(\chi)$, and thus $\chi \in \mathrm{Irr}(G|N)$ and $\chi(1)$ is the unique member of $\mathrm{cd}(G|N)$. Also, $\psi(1)$ divides $\chi(1)$, and so p divides every member of $\mathrm{cd}(G|N)$. By Theorem D, therefore, N has a normal p-complement M. Also, $\psi(1)$ divides $|N|$, and so p divides $|N|$, and $M < N$.

Clearly, $\mathrm{Irr}(G|M) \subseteq \mathrm{Irr}(G|N)$, and thus $|\mathrm{cd}(G|M)| \leq 1$ and the inductive hypothesis guarantees that M is abelian. Thus N has an abelian normal p-complement, and hence $\mathrm{cd}(G|N') < \mathrm{cd}(G|N)$ by Theorem 3.1. It follows that $\mathrm{cd}(G|N')$ is empty, and therefore $N' = 1$ and N is abelian. This is a contradiction, and the theorem is proved. $\blacksquare$

We also have another easy consequence of Theorem 3.1.

(3.3) COROLLARY. *Let $N \lhd G$ and suppose that N is nilpotent. Then $\mathrm{dl}(N) \leq |\mathrm{cd}(G|N)|$.*

Proof. Since N is nilpotent, its derived length is the maximum of the derived lengths of its Sylow subgroups. Also, if $P \in \mathrm{Syl}_p(N)$, then $P \lhd G$ and $\mathrm{cd}(G|P) \subseteq \mathrm{cd}(G|N)$. It suffices, therefore, to prove the theorem in the case where N is a p-group, and we do this by induction on $|N|$.

Assuming that N is a p-group and that $N > 1$, we see from Theorem 3.1 that $\mathrm{cd}(G|N') < \mathrm{cd}(G|N)$, and thus $|\mathrm{cd}(G|N')| < |\mathrm{cd}(G|N)|$. Also, $N' < N$, and so by the inductive hypothesis, we have $\mathrm{dl}(N') \leq |\mathrm{cd}(G|N')| \leq |\mathrm{cd}(G|N)| - 1$, and it follows that $\mathrm{dl}(N) \leq |\mathrm{cd}(G|N)|$, as required. ∎

4. THE FITTING SUBGROUP

It is known that for every nonidentity solvable group G, the set $\mathrm{cd}(G/F)$ fails to contain the largest member of $\mathrm{cd}(G)$, where $F = \mathbf{F}(G)$ is the Fitting subgroup. In this section, we prove the analogous result for $\mathrm{cd}(G|N)$.

(4.1) THEOREM. *Let* $N \lhd G$ *and choose* $\chi \in \mathrm{Irr}(G|N)$ *such that* $\chi(1) = \max(\mathrm{cd}(G|N))$. *Then* $N \cap \ker(\chi)$ *is nilpotent. Also, if* K/M *is an abelian chief factor of* G, *where* $M \subseteq N \cap \ker(\chi)$, *then* K *is nilpotent.*

The proof we present for Theorem 4.1 is a minor variation on the argument used to prove Theorem 12.19 of [6]. (Although the nilpotence of $N \cap \ker(\chi)$ in Theorem 4.1 is immediate from Corollary 12.20 of [6], it does not seem possible to prove the entire result without repeating at least some part of the older proof.)

The key to the argument is the "vanishing-off" subgroup $\mathbf{V}(\chi)$ of a character $\chi \in \mathrm{Irr}(G)$. Recall that by definition, $\mathbf{V}(\chi)$ is the subgroup generated by all elements $x \in G$ such that $\chi(x) \neq 0$, and note that it is obvious that $\ker(\chi) \subseteq \mathbf{V}(\chi)$. We shall need the following variation on Lemma 12.18 of [6].

(4.2) LEMMA. *Let* $V = \mathbf{V}(\chi)$, *where* $\chi \in \mathrm{Irr}(G)$. *Suppose that* $K \lhd G$ *and that* $K \cap V \subseteq \ker(\chi)$. *Then* $K \subseteq \ker(\chi)$.

Proof. Since χ vanishes on $K - V$, we see that $|K[\chi_K, 1_K] = |K \cap V[\chi_{K \cap V}, 1_{K \cap V}] > 0$, where the inequality holds since $K \cap V \subseteq \ker(\chi)$. It follows that χ_K has a principal constituent, and thus $K \subseteq \ker(\chi)$. ∎

Proof of Theorem 4.1. Assuming that the theorem is false, we define normal subgroups K and M of G as follows. If $N \cap \ker(\chi)$ is not nilpotent, define $K = N \cap \ker(\chi) = M$. Otherwise, let K/M be an abelian chief factor of G, where $M \subseteq N \cap \ker(\chi)$ and K is not nilpotent. Note that our definitions imply that although K is not nilpotent, M is nilpotent in the case where $M < K$.

Since K is not nilpotent, we can choose a nonnormal Sylow p-subgroup P of K and we write $H = \mathbf{N}_G(P)$. If $K = M$, then by the Frattini argument, we have $G = MH$. If, on the other hand, $K > M$, then M is nilpotent and K/M is an abelian chief factor of G. In that situation,

$P \not\subseteq M$, and thus p must be the unique prime divisor of K/M, and $K = MP$. By the Frattini argument, we deduce that $G = KH = MPH = MH$, and thus $G = MH$ in all cases.

Since $G = MH$ and $M \subseteq \ker(\chi)$, it follows that χ_H is irreducible, and we write $\psi = \chi_H$. By definition, $M \subseteq N$, and thus $N = M(N \cap H)$. Since $N \not\subseteq \ker(\chi)$ and $M \subseteq \ker(\chi)$, we see that $N \cap H \not\subseteq \ker(\chi)$, and thus $N \cap H \not\subseteq \ker(\psi)$. It follows that $N \cap H$ is not contained in the kernel of any irreducible constituent of ψ^G, and in particular, N is not contained in any of these kernels. All irreducible constituents of ψ^G, therefore, lie in $\mathrm{Irr}(G|N)$, and hence each of them has degree at most $\max(\mathrm{cd}(G|N)) = \chi(1) = \psi(1)$. It follows that the restriction to H of each irreducible constituent of ψ^G must be ψ exactly, and thus by Lemma 12.17 of [6], we conclude that $\mathbf{V}(\psi)$ is normal in G. We shall show that $K \cap H \subseteq \mathbf{V}(\psi)$, and thus $K \cap H = K \cap \mathbf{V}(\psi) \triangleleft G$. This will yield a contradiction, however, since $K \cap H = \mathbf{N}_K(P)$ is not normal in K.

Consider the subgroup $X = M(K \cap \mathbf{V}(\psi))$, which is normal in G. Note that $M \subseteq X \subseteq K$, so that either $X = K$ or $X = M$, and we suppose first that $X = K$. Then since $K \cap \mathbf{V}(\psi) \subseteq H$, we have

$$K \cap H = M(K \cap \mathbf{V}(\psi)) \cap H = (M \cap H)(K \cap \mathbf{V}(\psi)) \subseteq \mathbf{V}(\psi),$$

where the containment holds because $M \cap H \subseteq \ker(\psi) \subseteq \mathbf{V}(\psi)$. We are done in this case, and so we can assume that $X = M$, so that $K \cap \mathbf{V}(\psi) \subseteq M$.

We now have $(K \cap H) \cap \mathbf{V}(\psi) \subseteq M \cap H \subseteq \ker(\psi)$, and we can apply Lemma 4.2 to the normal subgroup $K \cap H$ in the group H. We deduce that $K \cap H \subseteq \ker(\psi) \subseteq \mathbf{V}(\psi)$, as wanted. The proof is now complete. ∎

(4.3) COROLLARY. *Let* $F = \mathbf{F}(N)$, *where* $1 < N \triangleleft G$ *and* N *is solvable, and let* $m = \max(\mathrm{cd}(G|N))$. *Then* $m \notin \mathrm{cd}((G/F)|(N/F))$, *which is therefore a proper subset of* $\mathrm{cd}(G|N)$.

Proof. Note that m exists since $N > 1$. If we view irreducible characters of G/F as irreducible characters of G, we see that $\mathrm{Irr}((G/F)|(N/F)) \subseteq \mathrm{Irr}(G|N)$, and thus $\mathrm{cd}((G/F)|(N/F)) \subseteq \mathrm{cd}(G|N)$. It therefore suffices to show that if $\chi \in \mathrm{Irr}(G|N)$ with $\chi(1) = m$, then $F \not\subseteq \ker(\chi)$.

Suppose that $F \subseteq \ker(\chi)$. Since $\chi \in \mathrm{Irr}(G|N)$, we know that $N \not\subseteq \ker(\chi)$, and thus $F < N$, and we can choose a chief factor K/F of G with $K \subseteq N$. Since N is solvable, K/F is abelian, and thus Theorem 4.1 applies and K is nilpotent. This is a contradiction, however, since $K \triangleleft N$, but $K \not\subseteq F = \mathbf{F}(N)$. This completes the proof. ∎

Our next result includes part of Theorem A. If N is solvable, we bound both the Fitting height $h(N)$ and the derived length $\mathrm{dl}(N)$ in terms of $|\mathrm{cd}(G|N)|$.

(4.4) COROLLARY. *Let $N \triangleleft G$, where N is solvable, and write $n = |cd(G|N)|$. Then $h(N) \leq n$ and $dl(N) \leq n(n + 1)/2$.*

Proof. We proceed by induction on $|N|$. If $N = 1$, then $h(N) = 0 = dl(N)$ and there is nothing to prove. Otherwise, set $F = \mathbf{F}(N)$ and observe that $F > 1$. Then $|cd((G/F)|(N/F))| \leq n - 1$ by Corollary 4.3, and we have

$$h(N) = 1 + h(N/F) \leq 1 + (n - 1) = n,$$

by the inductive hypothesis. Also, $Irr(G|F) \subseteq Irr(G|N)$, and so $|cd(G|F)| \leq n$ and $dl(F) \leq n$ by Corollary 3.3. We know from the inductive hypothesis that $dl(N/F) \leq (n - 1)n/2$, and it follows that $dl(N) \leq n + (n - 1)n/2 = n(n + 1)/2$, as required. ∎

To complete the proof of Theorem A, we use a known argument, independent of the rest of this paper.

(4.5) THEOREM. *Let G be solvable and suppose $N \triangleleft G$ and $|cd(G|N)| = n$. Then $dl(N) \leq 3n$.*

Proof. Working by induction on $|N|$, it suffices to show that $|cd(G|M)| \leq n - 1$, where $M = N'''$. Certainly, $cd(G|M) \subseteq cd(G|N)$, and so it suffices to show that if $\chi \in Irr(G|N)$ with $\chi(1) = \min(cd(G|N))$, then $M \subseteq \ker(\chi)$, and thus $\chi \notin Irr(G|M)$. But $N \subseteq \ker(\psi)$ for all $\psi \in Irr(G)$ with $\psi(1) < \chi(1)$, and thus Theorem 6 of [5] applies. This result says exactly that $M = N''' \subseteq \ker(\chi)$, as required. ∎

We mention that the solvability of G is used in the proof of Theorem 6 of [5] so that the Fong–Swan theorem can be applied to lift a modular irreducible representation of G to the complex numbers.

We close this section with an easy, but technical, application of Theorem 4.1 that will be needed in what follows.

(4.6) LEMMA. *Let $m = \max(cd(G|N))$, where $N \triangleleft G$. Suppose $M \subseteq N$, where $M \triangleleft G$ and $\mathbf{F}(M)$ is a p-group. If $m \in cd((G/M')|(N/M'))$, then M is a p-group.*

Proof. We can assume that $N > 1$, so that m exists. By hypothesis, there exists $\chi \in Irr(G|N)$ with $\chi(1) = m$ and $M' \subseteq \ker(\chi)$. Then M' is nilpotent by Theorem 4.1, and hence $M' \subseteq \mathbf{F}(M)$ is a p-group. If some prime $q \neq p$ divides $|M/M'|$, then there exists a subgroup $K \subseteq M$ such that K/M' is an abelian q-group that is a chief factor of G. We deduce from Theorem 4.1 that K is nilpotent, and thus $K \subseteq \mathbf{F}(M)$ is a p-group, and this is a contradiction since $|K/M'|$ is divisible by $q \neq p$. It follows that $|M/M'|$ is not divisible by any prime different from p, and hence M is a p-group, as required. ∎

5. TWO DEGREES

In this section we study the situation where $|\mathrm{cd}(G|N)| = 2$. We begin with two easy general lemmas.

(5.1) LEMMA. *Let $N \lhd G$ and suppose that $a = \min(\mathrm{cd}(G|N))$. If $H \subseteq G$ and $|G : H| \le a$, then $N \subseteq H$. If, in fact, $|G : H| < a$, then $N \subseteq H'$.*

Proof. We can certainly assume that $H < G$, so that $(1_H)^G$ is reducible. If η is any irreducible constituent of $(1_H)^G$, therefore, we have $\eta(1) < |G : H| \le a$, and thus $\eta \notin \mathrm{Irr}(G|N)$. It follows that $N \subseteq \ker(\eta)$, and since η was arbitrary, we have $N \subseteq \ker((1_H)^G) \subseteq H$, as required. Finally, if $N \nsubseteq H'$, then N is not contained in the kernel of some linear character λ of H, and thus the irreducible constituents of λ^G lie in $\mathrm{Irr}(G|N)$. In this case, we have $a \le \lambda^G(1) = |G : H|$, and the last assertion of the lemma now follows. ∎

(5.2) LEMMA. *Let $\alpha \in \mathrm{Irr}(N)$, where $N \lhd G$. Suppose that α is invariant in G and let p be a prime that does not divide $o(\alpha)\alpha(1)$. Then there exists $\gamma \in \mathrm{Irr}(G)$, lying over α, such that $\gamma(1)$ is not divisible by p.*

Proof. Let $P/N \in \mathrm{Syl}_p(G/N)$. By Corollary 6.28 of [6], we can choose an extension β of α to P, and we observe that $\beta^G(1) = \beta(1)|G : P|$ is not divisible by p. The degree of some irreducible constituent γ of β^G is therefore not divisible by p, and the proof is complete since γ lies over α. ∎

(5.3) THEOREM. *Let $N \lhd G$ and suppose that $|\mathrm{cd}(G|N)| = 2$. If N is solvable, then $\mathrm{dl}(N) \le 2$.*

Proof. Assume that $\mathrm{dl}(N) \ge 3$ and work by induction on $|N|$. By Corollary 3.3, we see that N cannot be nilpotent, and thus $F < N$, where we have written $F = \mathbf{F}(N)$. If $M \subseteq N$ with $M \lhd G$, then $\mathrm{cd}((G/M)|(N/M)) \subseteq \mathrm{cd}(G|N)$, and hence if $M > 1$, the inductive hypothesis implies that N/M is metabelian, and so $N'' \subseteq M$. In other words, N'' is the unique minimal normal subgroup of G contained in N, and it follows that F is a p-group, where p is the unique prime divisor of $|N''|$.

Now let $M < N$, where $M \lhd G$. Since $\mathrm{cd}(G|M) \subseteq \mathrm{cd}(G|N)$, the inductive hypothesis guarantees that M' is abelian, and thus N/M' must be nonabelian. It follows by Corollary 3.2 that $|\mathrm{cd}((G/M')|(N|M'))| \ge 2$, and thus both members of $\mathrm{cd}(G|N)$ lie in $\mathrm{cd}((G/M')|(N|M'))$. In particular, if we write $\mathrm{cd}(G|N) = \{a, b\}$ with $a < b$, we see that $b \in$

cd$((G/M')|(N/M'))$. Since $\mathbf{F}(M) \subseteq \mathbf{F}(N)$ is a p-group, it follows by Lemma 4.6 that M is a p-group, and thus $M \subseteq F$. In other words, F is the unique normal subgroup of G that is maximal with the property of being properly contained in N. In particular, $N' \subseteq F$, and hence F is nonabelian. Also, N/F is a chief factor of G, and hence it is a q-group for some prime $q \neq p$. It follows that $\mathbf{O}^p(N) = N$.

Since F is nonabelian, we have $|\text{cd}(G|F)| \geq 2$, and hence there exists $\psi \in \text{Irr}(G|F)$ with $\psi(1) = a$. Let α be an irreducible constituent of ψ_F, so that α is nonprincipal, and let T be the stabilizer of α in G. Then $|G : T|$ divides $\psi(1) = a$, and hence $T \supseteq N$ by Lemma 5.1, and thus α is invariant in N. Also, since F is a p-group and N/F is a q-group with $q \neq p$, it follows by Corollary 6.28 of [6] that α extends to N and that a unique extension β can be chosen so that $o(\beta) = o(\alpha)$. But $o(\beta)$ is not divisible by p since $N = \mathbf{O}^p(N)$, and thus α is a nonprincipal irreducible character of the p-group F such that $o(\alpha)$ is not divisible by p. Therefore α cannot be linear, and so p divides $\alpha(1)$, and thus p divides a. Also, since β is uniquely determined by α, it is fixed by every element of G that fixes α.

Now let λ be a nontrivial linear character of N with $F \subseteq \ker(\lambda)$, and let S be the stabilizer of λ in G. Note that p divides neither $\lambda(1) = 1$ nor $o(\lambda)$, which is a q-power, and so by Lemma 5.2, there exists $\mu \in \text{Irr}(S)$ lying over λ and such that p does not divide $\mu(1)$. Let $\chi = \mu^G$ and note that $\chi \in \text{Irr}(G)$ by the Clifford correspondence.

As χ lies over $\lambda \in \text{Irr}(N)$ and λ is nonprincipal, we see that $N \not\subseteq \ker(\chi)$ and $\chi \in \text{Irr}(G|N)$. It is not possible, however, that $\chi(1) = b$ since otherwise, Theorem 4.1 would imply that N is nilpotent, which is not the case. (Theorem 4.1 applies because $F \subseteq \ker(\lambda) \subseteq \ker(\chi)$ and N/F is an abelian chief factor of G.) It follows that $|G : S| \mu(1) = \chi(1) = a$, and thus $|G : S|$ is divisible by the full p-part of a.

Now consider the character $\lambda\beta \in \text{Irr}(N)$, and observe that the characters β and $\lambda\beta$ uniquely determine λ by Gallagher's theorem. (See Corollary 6.17 of [6]). Let U be the stabilizer of $\lambda\beta$ in G and note that U stabilizes $\alpha = (\lambda\beta)_F$, and thus U also stabilizes β. It follows that U stabilizes λ, and thus $U \subseteq S$, and we conclude that $|G : U|$ is divisible by the full p-part of a.

Let $\delta \in \text{Irr}(U)$ lie over $\lambda\beta$ and note that δ^G is irreducible, and in fact, $\delta^G \in \text{Irr}(G|N)$ since $\lambda\beta$ is nonprincipal. Also, $\delta^G(1) = \delta(1)|G : U|$, and the p-part of this integer exceeds the p-part of a because $\delta(1)$ is a multiple of $\alpha(1)$, which is divisible by p. It follows that $\delta^G(1) \neq a$, and we conclude that $\delta^G(1) = b$, and hence p divides b. Thus a and b are each divisible by p, and therefore N has a normal p-complement by Theorem D. This is a contradiction, however, since $N = \mathbf{O}^p(N)$ and yet p divides $|N|$, and this completes the proof. ∎

To complete the proof of Theorem B, we need to be able to prove Theorem 5.3 without assuming that N is solvable. To do this, we shall appeal to the solvability part of Theorem D, and thus indirectly to the classification of simple groups.

Proof of Theorem B. We have $N \triangleleft G$ and $|\text{cd}(G|N)| \leq 2$, and our task is to show that N is solvable and that $\text{dl}(N) \leq |\text{cd}(G|N)|$. By Theorem 3.2, we can assume that $|\text{cd}(G|N)| = 2$, and we write $\text{cd}(G|N) = \{a, b\}$ with $a < b$. By Theorem 5.3, we can assume that N is not solvable, and thus by Theorem D, the degrees a and b are relatively prime.

Let $\mathscr{A} \subseteq \text{Irr}(N)$ be the set of irreducible constituents of characters χ_N, where $\chi \in \text{Irr}(G|N)$ has degree a. Similarly, let $\mathscr{B}$ be the set of irreducible characters of N that lie under characters in $\text{Irr}(G|N)$ that have degree b. Then $\mathscr{A}$ and $\mathscr{B}$ are nonempty and $\mathscr{A} \cup \mathscr{B} = \text{Irr}(N) - \{1_N\}$. Also, if $\alpha \in \mathscr{A}$ and $\beta \in \mathscr{B}$, then $\alpha(1)$ and $\beta(1)$ must be relatively prime since these degrees divide a and b, respectively. In particular $\mathscr{A} \cap \mathscr{B}$ consists only of linear characters.

Work by induction on $|N|$. If $1 < M < N$, then since $\text{cd}(G|M)$ and $\text{cd}((G/M)|(N/M))$ are each contained in $\text{cd}(G|N)$, we deduce that M and N/M are both solvable, which is not the case. Thus M does not exist, and in particular, since $N' > 1$, we have $N' = N$. Each member of $\mathscr{A}$ and of $\mathscr{B}$ is therefore nonlinear, and it follows that $\mathscr{A}$ and $\mathscr{B}$ are disjoint. Also, if $\chi \in \text{Irr}(G|N)$ and we write $M = \ker(\chi_N)$, then $M < N$ and $M \triangleleft G$. We deduce that $M = 1$, and thus χ_N is faithful.

Let χ_1 and χ_2 be characters of degree a in $\text{Irr}(G|N)$ and assume that no constituent of $(\chi_1)_N$ is the complex conjugate of an irreducible constituent of $(\chi_2)_N$. Then $(\chi_1 \chi_2)_N$ has no principal constituent, and it follows that $\chi_1 \chi_2$ is a sum of characters in $\text{Irr}(G|N)$, with (say) u irreducible constituents of degree a and v irreducible constituents of degree b, counting multiplicities. We thus have $a^2 = ua + vb$, and we deduce that a divides v. But $ab > a^2$, and thus $v < a$, and we conclude that $v = 0$. This shows that all irreducible constituents of $\chi_1 \chi_2$ have degree a.

Now let $\chi, \psi \in \text{Irr}(G|N)$ with $\chi(1) = a$ and $\psi(1) = b$. Since the irreducible constituents of χ_N and of ψ_N have different degrees, we see that $(\chi\psi)_N$ can have no principal constituent, and thus $\chi\psi$ can be written as a sum of (say) r members of $\text{Irr}(G|N)$ of degree a and s of degree b, counting multiplicities. Thus $ab = ra + sb$, and hence a divides s and b divides r. It follows from this that either $r = 0$ or $s = 0$.

Suppose $s = 0$. Then $\chi\psi$ is a sum of b irreducible characters of degree a, and we let χ be any one of these irreducible constituents. Then $\bar{\chi}\xi$ has ψ as an irreducible constituent of degree b, and it follows from the earlier calculation that some irreducible constituent of $(\bar{\chi})_N$ is the complex

conjugate of a constituent of ξ_N. It follows that ξ_N is of the form $e\chi_N$ for some rational number e, and since this is true for every choice of ξ as an irreducible constituent of $\chi\psi$, we deduce that $\chi_N\psi_N = f\chi_N$, where necessarily $f = \psi(1)$. Now ψ_N is a faithful character of N, and so its value is different from f on every nonidentity element of N. It follows that $\chi(n) = 0$ for $1 \neq n \in N$, and this is impossible, since χ_N has no principal constituent. It follows that $s > 0$ and $r = 0$, and thus all constituents of $\chi\psi$ have degree b. In particular, if $\alpha \in \mathscr{A}$ and $\beta \in \mathscr{B}$, then all irreducible constituents of $\alpha\beta$ lie in $\mathscr{B}$.

Now let $\alpha_1, \alpha_2 \in \mathscr{A}$. if $\alpha_1\alpha_2$ has an irreducible constituent $\beta \in \mathscr{B}$, then $\overline{\alpha_1}\beta$ has α_2 as a constituent, and that contradicts the result of the previous paragraph. It follows that the set $\mathscr{A} \cup \{1_N\} \subseteq \mathrm{Irr}(N)$ has the property that the product of any two of its members is a sum of members of this set.

Let $\chi \in \mathrm{Irr}(G|N)$ have degree a and note that all irreducible constituents of all powers $(\chi_N)^e$ lie in the set $\mathscr{A} \cup \{1_N\}$. But χ_N is faithful, and so the Burnside–Brauer theorem applies. (See Theorem 4.3 of [6].) This result guarantees that every irreducible character of N is a constituent of some power of χ_N, and we deduce that $\mathrm{Irr}(N) = \mathscr{A} \cup \{1_N\}$. This is a contradiction since $\mathscr{B}$ is nonempty and disjoint from this set. $\blacksquare$

6. THE EXCEPTIONAL CASE

Let $\chi \in \mathrm{Irr}(G|N)$, where $N \lhd G$. Suppose that X/Y is an abelian chief factor of G, where $X \subseteq N$, and let p be the prime divisor of $|X/Y|$. In this situation, we say that X/Y is a *reducing p-section* for χ in N if the irreducible constituents of χ_Y have smaller degrees than the irreducible constituents of χ_X. (Of course, all irreducible constituents of each of these restrictions have equal degrees.) As we shall see, if G is p-solvable and $\chi(1) = \min(\mathrm{cd}(G|N))$, then as a general rule, a reducing p-section for χ in N will be central in N. This is not always true, however, and our goal in this section is to deduce a number of properties of the exceptional case, where X/Y is not central. It will turn out, for example, that no exception can occur unless $p = 2$ and $\chi(1)$ is even.

We begin with an example. Let $G = GL(2, 3)$ and take $N = SL(2, 3)$, so that $\mathrm{cd}(G|N) = \{2, 3, 4\}$. Let $\chi \in \mathrm{Irr}(G)$ be faithful of degree 2 and take X to be the quaternion normal subgroup of N of order 8 and let $Y = \mathbf{Z}(X)$, so that X/Y is a chief factor of order 4 of G. Now χ_X is irreducible and χ_Y reduces, and thus X/Y is a reducing 2-section for χ in N. But N acts nontrivially on X/Y, and so we are in the exceptional case.

(6.1) THEOREM.   *Let* $N \lhd G$, *where* G *is* p-*solvable, and let* $\chi \in \mathrm{Irr}(G|N)$ *with* $\chi(1) = \min(\mathrm{cd}(G|N))$. *Assume that* X/Y *is a reducing* p-*section for* χ *in* N *but that* X/Y *is not central in* N. *The following then occur.*

(a) $p = 2$.

(b) $\chi(1)$ *is even.*

(c) $|\mathrm{cd}(G|N)| \geq 3$.

(d) *If* $|\mathrm{cd}(G|N)| = 3$, *then* $\mathrm{cd}(G|N)$ *consists of two powers of* 2 *and an odd number. In this case,* N/X *is abelian.*

Proof. Since X/Y is an elementary abelian p-group, we can write $|X/Y| = p^e$, where e is a positive integer, and we can view X/Y as an irreducible G-module of dimension e over the field $F = GF(p)$. In fact, if E is the field of G-endomorphisms of X/Y, then E is an extension field of F and X/Y can be viewed as a vector space over E with dimension $e/|E : F| \leq e$. This makes X/Y into an absolutely irreducible module for G over E, and as such, it corresponds to an irreducible Brauer character φ of G at the prime p, where $\varphi(1) = \dim_E(X/Y) \leq e$.

Because G is p-solvable, we can apply the Fong–Swan theorem to find $\alpha \in \mathrm{Irr}(G)$ such that $\alpha(x) = \varphi(x)$ for all p-regular elements $x \in G$. In particular, $\alpha(1) = \varphi(1) \leq e$. We are assuming that X/Y is a p-group that is a chief factor of G, acted on nontrivially by $N \lhd G$. It follows that there exists a p-regular element $x \in N$ that acts nontrivially on X/Y. Thus $\alpha(x) = \varphi(x) \neq \varphi(1) = \alpha(1)$, and we see that $x \notin \ker(\alpha)$. Thus $N \nsubseteq \ker(\alpha)$, and $\alpha \in \mathrm{Irr}(G|N)$. In particular, if we set $a = \min(\mathrm{cd}(G|N))$, we see that $a \leq \alpha(1) \leq e$.

Now let θ be an irreducible constituent of χ_X, write $f = \theta(1)$, and let t be the number of distinct conjugates of θ in G. Then $a = \chi(1)$ is a multiple of ft, and in particular, $ft \leq a$. Also, since X/Y is a reducing section for χ, we see that θ_Y reduces, and it follows from the fact that X/Y is a p-group that θ is induced from a character of some subgroup U, with $Y \subseteq U < X$. (This follows, for instance, from Theorem 6.22 of [6].) Since $U \lhd X$, we see that θ vanishes on $X - U$, and we let V be the unique smallest subgroup of X containing Y such that θ vanishes on $X - V$. (Thus $V = Y\mathbf{V}(\theta)$.) We now know that $Y \subseteq V < X$.

Since θ uniquely determines V, we see that V is normalized by the stabilizer of θ in G, and thus the number of distinct conjugates of V is at most t. Since X/Y is a chief factor of G and $Y \subseteq V < X$, it follows that the intersection of the G-conjugates of V must be exactly Y, and therefore, $p^e = |X : Y| \leq |X : V|^t$. We can obtain control over the index $|X : V|$ by observing that since θ vanishes on elements of $X - V$, we have

$$|X| = |X|[\theta, \theta] = |V|[\theta_V, \theta_V] \leq |V|\theta(1)^2 = |V|f^2,$$

and so $|X : V| \leq f^2$.

Combining the various inequalities we have found, we obtain

$$p^{ft} \le p^a \le p^e \le |X:V|^t \le f^{2t},$$

and thus $p^f \le f^2$. Since $2 \le p$, this yields at $2^f \le f^2$, and thus $f \in \{2, 3, 4\}$, and if f is equal to either 2 or 4, then all of the previous inequalities are actually equalities and in particular, $p = 2$. Furthermore, if $f = 3$, then $p^3 \le 3^2$, and in this case too, $p = 2$. But the latter situation is impossible since θ_Y is reducible, and thus p must divide $\theta(1) = f$. We have definitely established, therefore, that $f \in \{2, 4\}$, that $p = 2$, and that all of the equalities in the previous paragraphs are equalities. In particular, conclusions (a) and (b) both hold. (Recall that f divides $a = \chi(1)$, and thus $\chi(1)$ is even.)

Let T be the stabilizer of θ in G, so that χ is induced from some character $\xi \in \mathrm{Irr}(T)$, lying over θ. Then $ft = a = \chi(1) = |G:T| \xi(1) = t\xi(1)$, and so $\xi(1) = f$ and ξ is an extension of θ to T. Also, $|G:T| = t = a/f < a$, and hence $N \subseteq T'$ by Lemma 5.1. But $X < N$ since X/Y is abelian, but is not a central factor of N, and thus $X < T'$, and we deduce that T/X is nonabelian. It follows that there exists a nonlinear character $\beta \in \mathrm{Irr}(T/X)$, and we see that $\xi\beta \in \mathrm{Irr}(T)$ lies over θ. It follows that $(\xi\beta)^G \in \mathrm{Irr}(G|N)$, and thus $\mathrm{cd}(G|N)$ contains some proper multiple ra of a, where $r = \beta(1)$. We see now that a and ra are two distinct even members of $\mathrm{cd}(G|N)$, and so to show that $|\mathrm{cd}(G|N)| \ge 3$, it suffices to show that some member of this set is odd. If not, then N has a normal 2-complement by Theorem D, and this subgroup necessarily centralizes the 2-group X/Y. This would contradict the previously observed fact that some p-regular element of N must act nontrivially on X/Y. (Recall that $p = 2$.) Thus $\mathrm{cd}(G|N)$ does contain an odd number m, and so $|\mathrm{cd}(G|N)| \ge 3$, proving conclusion (c).

We assume now that $|\mathrm{cd}(G|N)| = 3$, so that we have $\mathrm{cd}(G|N) = \{a, ra, m\}$. In particular, r is unique, and thus $\mathrm{cd}(T/X) = \{1, r\}$. In this case T/X must be metabelian, and since $N \subseteq T'$, we deduce that N/X is abelian, as asserted in (d). To complete the proof, we must show that a and r are each powers of 2.

Now X/V is a module for T/X over $GF(2)$, and since $|X/V| = f^2$, we see that the dimension of this module is $2\log_2(f)$, and this is equal to f since f is 2 or 4. As $N \subseteq T$, we see that $\mathbf{O}^2(N)$ acts on X/V, and we claim that this action is nontrivial. Otherwise, $[X, \mathbf{O}^2(N)] \subseteq V$, and thus $[X, \mathbf{O}^2(N)] \subseteq Y$ because $[X, \mathbf{O}^2(N)] \lhd G$ and the intersection of the G-conjugates of V is Y. It follows that the 2-regular elements of N act trivially on X/Y, and we know that this is not the case.

The module X/V for T/X yields a Brauer character γ (at the prime 2), and we know that $\gamma(1) = f$ and that $\gamma(x) \ne \gamma(1)$ for some 2-regular

element $x \in N$. We deduce by the Fong–Swan theorem that there exists $\beta \in \mathrm{Irr}(T/X)$ with $N/X \not\subseteq \ker(\beta)$ and $\beta(1) \leq f$. Viewing $\beta \in \mathrm{Irr}(T)$, we see that $N \not\subseteq \ker(\beta)$, and so each irreducible constituent of β^G lies in $\mathrm{Irr}(G|N)$. We conclude that $a \leq \beta^G(1) = |G:T|\,\beta(1) \leq tf = a$. It follows that $\beta(1) = f$, and we observe also that $\beta(1) \in \mathrm{cd}(T/X) = \{1, r\}$. Thus $r = f$ is a power of 2, as desired.

Next, we claim that the product of the t conjugates of θ is an irreducible character of X. Writing δ to denote this product, we observe that $\delta(x) = 0$ unless x is in V and in every G-conjugate of V, and thus δ vanishes on $X - Y$. Also, we know that $[\theta_V, \theta_V] = \theta(1)^2$, and it follows that θ_V is a multiple of a linear character. Therefore, θ_Y is a multiple of a linear character, and since a similar conclusion holds for each conjugate of θ, it follows that δ_Y is a multiple of a linear character. Thus $|\delta(y)| = \delta(1)$ for all elements $y \in Y$, and therefore $|X|[\delta, \delta] = |Y|[\delta_Y, \delta_Y] = |Y|\delta(1)^2$. But $|X:Y| = |X:V|^t = f^{2t} = \delta(1)^2$, and thus $[\delta, \delta] = 1$ and δ is irreducible, as claimed.

It follows that $f^t = \delta(1)$ must divide some member of $\mathrm{cd}(G|N)$, and we conclude that f^t divides $ra = fa = f^2 t$. Thus $f^{t-2} \leq t$, and hence $2^{t-2} \leq t$ and $2^t \leq 4t$. It follows that $t \leq 4$. If $t = 3$, however, then since $f \in \{2, 4\}$, we see that $f^t = f^3$ does not divide $f^2 t$, and this is a contradiction. We conclude that $t \neq 3$, and thus $t \in \{1, 2, 4\}$ and $a = ft$ is a power of 2. This concludes the proof of (d). ∎

Note that we actually proved more than we stated in part (d) of Theorem 6.1. We showed, for example, that the smallest member of $\mathrm{cd}(G|N)$ is a divisor of 16 and that the other power of 2 in this set is at most 64. It is also possible to obtain more precise information about the odd member of $\mathrm{cd}(G|N)$, but we shall not need to do so.

The following result extracts from Theorem 6.1 the information we need to prove Theorem C.

(6.2) THEOREM. *Let* $K \subseteq N \triangleleft G$, *where* $K \triangleleft G$ *and* G *is* p-*solvable for some prime* p. *Assume that* K *has a normal Sylow* p-*subgroup* E *and that* $\mathbf{O}^p(K) = K$. *Suppose that* $|\mathrm{cd}(G|N)| = 3$ *and assume that* $a = \min(\mathrm{cd}(G|N))$ *lies in* $\mathrm{cd}(G|E)$. *Then* N/E *is abelian and* $E' \subseteq \mathbf{Z}(K)$, *so that* E *is metabelian.*

Proof. Suppose $\chi \in \mathrm{Irr}(G)$ and $\chi(1) = a$, and let α be an irreducible constituent of χ_E. Assuming that α is nonprincipal, we work to show that N/E is abelian and that $E' \subseteq \mathbf{Z}(K)$.

Observe that the stabilizer of α in G has index dividing $\chi(1) = a$, and thus by Lemma 5.1, this stabilizer contains N, and in particular α is invariant in K. Since $E \in \mathrm{Syl}_p(K)$, it follows from Corollary 6.28 of [6] that α extends to some character $\beta \in \mathrm{Irr}(K)$, and we see that the

determinantal order $o(\beta)$ cannot be divisible by p since $\mathbf{O}^p(K) = K$. It follows that $o(\alpha)$ is not divisible by p, and thus α cannot be linear since we are assuming that it is nonprincipal. It follows that $\alpha(1) > 1$, and thus, since the p-group E is an M-group, α is induced from a proper subgroup of E. We conclude that α reduces upon restriction to some maximal subgroup of E, and thus $\alpha_{E'}$ is reducible.

Now consider a G-chief series containing E' and E among its terms. There clearly must be some chief factor X/Y with $E' \subseteq Y < X \subseteq E$ such that α_Y reduces, but α_X is irreducible. Then X/Y is a reducing p-section for χ in N, and we argue that X/Y cannot be centralized by N. In fact, we claim that K acts nontrivially on X/Y. To see why this is so, observe that K/E acts coprimely on the abelian group E/E', and thus by Fitting's theorem, $E/E' = [(E/E'),(K/E)] \times \mathbf{C}_{E/E'}(K/E)$. However, $[E, K] = E$ since E is a normal Sylow p-subgroup of K and $\mathbf{O}^p(K) = K$, and thus the first factor in the Fitting decomposition is the whole group E/E'. It follows that $\mathbf{C}_{E/E'}(K/E)$ is trivial, and thus K/E has no nontrivial fixed points on X/E'. We conclude (since this is a coprime action) that K/E has no nontrivial fixed points on X/Y, and in particular, the action of K on X/Y is nontrivial, as claimed.

Now X/Y is a reducing p-section for χ in N, but it is not central in N, and thus Theorem 6.1 applies. In particular, by Theorem 6.1(d), we see that N/X is abelian, and thus N/E is abelian, as required. Also, we know that $p = 2$ and that two of the three members of $\mathrm{cd}(G|N)$ are powers of 2, while the remaining degree is odd.

Next, we consider a p-complement Q in K, and we show that it acts trivially on E'. Since $(|Q|, |E'|) = 1$ and E' is a p-group, it suffices to show that Q acts trivially on E'/E'', and in fact, it suffices to consider a G-chief series having E' and E'' as terms, and to show that Q centralizes each chief factor U/V, where $E'' \subseteq V < U \subseteq E'$. We need to prove, in other words, that $[U, Q] \subseteq V$.

It suffices to show that $[U, Q] \subseteq \ker(\psi)$ for every character $\psi \in \mathrm{Irr}(G)$ such that $V \subseteq \ker(\psi)$. Of course, there is nothing to prove unless $\psi \in \mathrm{Irr}(G|N)$. Also, if $\psi(1)$ is odd, then the irreducible constituents of χ_E must be linear since E is a 2-group, and in this case, $[U, Q] \subseteq U \subseteq E' \subseteq \ker(\psi)$, as desired. We may assume, therefore, that $\psi(1)$ is a power of 2.

Let γ be an irreducible constituent of ψ_K, and note that γ_E is irreducible since $\gamma(1)$ is a power of 2 and $|K : E|$ is odd. Also, $V \subseteq \ker(\gamma)$ and U/V is central in E/V. Since γ_E is irreducible, it follows that $U \subseteq \mathbf{Z}(\gamma)$. (In other words, γ_U is a multiple of a linear character.) Therefore $[U, Q] \subseteq \ker(\gamma)$, and since this holds for all choices of γ, we deduce that $[U, Q] \subseteq \ker(\psi)$, as desired.

We have shown that Q centralizes E', and thus $\mathbf{C}_K(E')$ is a normal subgroup of K having p-power index. Since $\mathbf{O}^p(K) = K$, however, we

conclude that $E' \subseteq \mathbf{Z}(K)$, and thus E' is abelian and E is metabelian, as required. ∎

There is an amusing consequence of the fact that the exceptional case cannot occur in groups of odd order. Although this corollary will not be used in what follows, it seems possible that some application might eventually be found for it, and so we digress briefly to present it here. To state our result, we use the notation N^∞ to denote the unique normal subgroup of a group N, minimal such that the corresponding factor group is nilpotent. Note that N^∞ is the final term in the lower central series for N, and thus $[N^\infty, N] = N^\infty$.

(6.3) COROLLARY. *Let* $N \lhd G$, *where* G *is solvable, and let* $a = \min(\mathrm{cd}(G|N))$. *Assume either that* a *is odd, or that* $|N|$ *is odd, and write* $M = N^\infty$. *Then* $a \notin \mathrm{cd}(G|M)$.

Proof. Let $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = a$. Assuming (as we may) that $N > 1$, we see that $M < N$, and thus working by induction on $|N|$, we can suppose that $M^\infty \subseteq \ker(\chi)$. We want to prove that $M \subseteq \ker(\chi)$, and so we consider an irreducible constituent α of χ_M.

First, assume that α is linear. Observe that the stabilizer in G of α has index dividing $\chi(1) = a$, and thus by Lemma 5.1, α is invariant in N. It follows that $[M, N] \subseteq \ker(\alpha)$, and since this holds for all irreducible constituents of χ_M, we deduce that $[M, N] \subseteq \ker(\chi)$. But $M = N^\infty$, and thus $[M, N] = M$ and $M \subseteq \ker(\chi)$, as required.

We complete the proof by deriving a contradiction in the case where α is nonlinear. Since $M^\infty \subseteq \ker(\alpha)$, we see that α can be viewed as a character of the nilpotent group M/M^∞, and thus by the argument that we used in the proof of the previous theorem, $\alpha_{M'}$ is reducible. If β is an irreducible constituent of $\alpha_{M'}$, choose any prime divisor p of $\alpha(1)/\beta(1)$ and let U/M' be the p-complement of M/M'. It is easy to see that α_U must be reducible, and thus there exists a G-chief factor X/Y with $U \subseteq Y < X \subseteq M$ such that X/Y is a reducing p-section for χ in N. Furthermore, $p \neq 2$ since we are assuming that either a or $|N|$ is odd, and thus by Theorem 6.1, the section X/Y is central in N.

Let Q/M be the p-complement of the nilpotent group N/M, and note that the coprime action of Q/M on M/U must have nontrivial fixed points since the action of Q/M on the section X/Y is trivial. But $M/U = [(M/U), (Q/M)] \times \mathbf{C}_{M/U}(Q/M)$ by Fitting's theorem, and we know that the second factor is nontrivial, and we deduce that $[(M/U), (Q/M)] < M/U$. It follows that there exists a nontrivial chief factor M/K of G, centralized by Q, and such that $U \subseteq K$.

Now let P/M denote the Sylow p-subgroup of N/M. Then P/M acts on the p-group M/K, and thus $[(M/K), (P/M)] < M/K$. Since M/K is a

G-chief factor, it follows that $[(M/K),(P/M)] = 1$, and thus P centralizes M/K. We now know that $N = PQ$ acts trivially on M/K, and thus $[M, N] \subseteq K$. This is not the case, however, since $[M, N] = M$, and this is the desired contradiction. ∎

Corollary 6.3 clearly yields an alternative proof (when it applies) of the fact that if $N \lhd G$ and N is solvable, then the Fitting height of $h(N) \leq |\mathrm{cd}(G|N)|$. (See Corollary 4.4.) This "top-down" argument, however, works only when G is solvable and $|N|$ is odd, while the "bottom-up" proof in Section 4 works in all cases.

Corollary 6.3 should be compared with the result of T. R. Berger [1], which makes a stronger oddness assumption, but obtains a very much stronger conclusion. In the case where $|G|$ is odd, Berger proved that $a \notin \mathrm{cd}(G|N')$, where $N \lhd G$ and $a = \min(\mathrm{cd}(G|N))$. Berger's conclusion does not remain valid, however, under the weaker hypotheses of Corollary 6.3. If G is the semidirect product of Q_8 acting faithfully on a nonabelian group N of order 27 and exponent 3, for example, we see that $\min(\mathrm{cd}(G|N)) = 3$. In this situation, N' is not contained in the kernels of the irreducible characters of G of degree 3.

7. THREE DEGREES

In this section, we prove Theorem C. Recall that the assumption is that $N \lhd G$, where G is p-solvable for all prime divisors p of $|N|$, and $|\mathrm{cd}(G|N)| = 3$. Our goal is to prove that $\mathrm{dl}(N) \leq 3$.

Proof of Theorem C. Write $\mathrm{cd}(G|N) = \{a, b, c\}$ with $a < b < c$ and assume that $\mathrm{dl}(N) > 3$. Working by induction on $|N|$ and following the usual argument, we consider $M \subseteq N$ with $M \lhd G$. Then $\mathrm{cd}((G/M)|(N/M)) \subseteq \mathrm{cd}(G|N)$, and thus if $M > 1$, the inductive hypothesis yields that $\mathrm{dl}(N/M) \leq 3$ and $N''' \subseteq M$. We conclude that N''' is the unique minimal normal subgroup of G contained in N, and thus $F = \mathbf{F}(N)$ is a p-group, where p is the unique prime divisor of $|N'''|$. Also, by Corollary 3.3, we know that N is not a p-group, and thus $F < N$ and N does not have a normal p-complement. By Theorem D, therefore, p fails to divide at least one of a, b, and c.

Suppose $M < N$ with $M \lhd G$. Since $\mathrm{cd}(G|M) \subseteq \mathrm{cd}(G|N)$, we have $M''' = 1$, and thus M'' is abelian. Also, since $\mathrm{dl}(N) > 3$, we see that $\mathrm{dl}(N/M'') > 2$, and thus $|\mathrm{cd}((G/M'')|(N/M''))| > 2$ by Theorem 5.3. It follows that $\mathrm{cd}((G/M'')|(N/M'')) = \mathrm{cd}(G|N)$, and in particular, $c \in \mathrm{cd}((G/M'')|(N/M''))$. Since $\mathbf{F}(M') \subseteq F$ is a p-group, we can apply Lemma 4.6 to the subgroup $M' \lhd G$ to deduce that M' is itself a p-group. This shows that $M' \subseteq F$ whenever $M < N$ with $M \lhd G$.

Now let $K = \mathbf{O}^p(N)$ and write $E = K \cap F$, so that $E = \mathbf{F}(K) = \mathbf{O}_p(K)$. We claim that $E \in \mathrm{Syl}_p(K)$, and to prove this, we consider separately the cases where $K < N$ and $K = N$. If $K < N$, then $K' \subseteq F$ by the result of the previous paragraph, and so K/E is abelian. Since $E = \mathbf{O}_p(K)$, it follows that K/E is a p'-group, and thus $E \in \mathrm{Syl}_p(K)$, as desired.

Now suppose that $K = N$ and let N/M be a chief factor of G, so that N/M is a q-group for some prime $q \neq p$. Then $F \subseteq M$, and we know that $M' \subseteq F$, so that M/F is abelian. Since $F = \mathbf{O}_p(M)$, it follows that M/F is a p'-group, and hence N/F is also a p'-group. Since we are assuming that $K = N$, we have $E = F$, and thus $E \in \mathrm{Syl}_p(K)$ in this case too.

Since $\mathbf{O}^p(K) = K$ and E is a normal Sylow p-subgroup of K, we can apply Theorem 6.2. Because $\mathrm{dl}(N) > 3$, we see that either N/E is non-abelian, or else E is not metabelian, and thus Theorem 6.2 guarantees that $a \notin \mathrm{cd}(G|E)$.

Assume until further notice that $K < N$. As we have seen, this implies that $K' \subseteq F$, and thus $K' \subseteq E$ and K/E is abelian. Now K is not a p-group since otherwise, N would be a p-group, and we have seen that this is not the case. But $\mathbf{F}(K)$ is a p-group, and thus by Lemma 4.6, we cannot have $c \in \mathrm{cd}((G/K')|(N/K'))$, and therefore $c \notin \mathrm{cd}((G/E)|(N/E))$. We deduce that $\mathrm{cd}((G/E)|(N/E)) \subseteq \{a, b\}$, and in particular, N/E is metabelian by Theorem 5.3. It follows that E is nonabelian. Also, we claim that $K' = E$ since otherwise, $K' < E$ and p divides $|K/K'|$. This cannot happen, however, since $K = \mathbf{O}^p(K)$.

We have seen that $a \notin \mathrm{cd}(G|E)$. If also $a \notin \mathrm{cd}(G|K)$, then $E' = K'' = 1$ by Theorem 5.3, and E is abelian, which is a contradiction. It follows that there exists $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = a$, such that $K \nsubseteq \ker(\chi)$. Since we know that $E \subseteq \ker(\chi)$ and that K/E is abelian, we see that an irreducible constituent λ of χ_K is linear and nonprincipal. The stabilizer of λ in G has index dividing $\chi(1) = a$, and thus λ is invariant in N by Lemma 5.1. Since $E \subseteq \ker(\lambda)$ and K/E is a p'-group, while N/K is a p-group, it follows that λ extends to N. An extension of λ to N, however, is a linear character whose order is a multiple of $o(\lambda)$. This order has a nontrivial p'-part, and we conclude that $\mathbf{O}^{p'}(N) < N$.

Let $L = \mathbf{O}^{p'}(N)$ and observe that $F \subseteq L$ since F is a p-group. Since $L < N$, we know by an earlier argument that $L' \subseteq F$, and thus L/F is abelian. But $\mathbf{O}^{p'}(L) = L$, and thus L/F must be a p-group, and we deduce that L is a p-group, and hence $L = F$ and $|N/F|$ is not divisible by p. It follows that $FK = N$ and $N/E = (F/E) \times (K/E)$, and we draw a number of conclusions. First, we observe that K acts trivially on F/E. Also, we know that K/E is abelian but that N/E is nonabelian, and so we deduce that F/E is nonabelian. Finally, since K/E is abelian and F/E is a p-group, we see that $\mathrm{cd}(N/E)$ consists entirely of p-powers.

Now fix $f \in \{a, b, c\}$ such that f is not divisible by p and suppose, for the moment, that $f \notin \mathrm{cd}(G|E)$. (We know, for example, that this would be the situation if $f = a$.) Let $\chi \in \mathrm{Irr}(G)$ have degree f, and note that $E \subseteq \ker(\chi)$, and so the irreducible constituents of χ_N have degrees lying in $\mathrm{cd}(N/E)$. The degrees of these constituents, therefore, are p-powers dividing f, and hence χ_N is a sum of linear characters, and $N' \subseteq \ker(\chi)$. This shows that $f \notin \mathrm{cd}(G|N')$, and thus $|\mathrm{cd}(G|N')| < 3$ and N' is metabelian by Theorem 5.3. we are assuming that $\mathrm{dl}(N) > 3$, however, and this contradiction shows that $f \in \mathrm{cd}(G|E)$. In particular, $f \neq a$, and thus p divides a.

Again let $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = f$, and note that χ_E must have linear constituents since E is a p-group and p does not divide f. By the previous paragraph, we can choose χ with $E \nsubseteq \ker(\chi)$, and we let μ be an irreducible constituent of χ_E, so that μ is nonprincipal and linear. Let T be the stabilizer of μ in G and note that $|G : T|$ divides $\chi(1) = f$, so that p does not divide $|G : T|$.

We claim now that μ extends to T. Since E is a p-group, it suffices by Theorem 6.26 of [6] to show that μ extends to P, where $P \in \mathrm{Syl}_p(T)$. To produce an extension of μ to P, we consider the Clifford correspondent $\xi \in \mathrm{Irr}(T)$ of χ with respect to μ, so that ξ_E is a multiple of μ and $\xi^G = \chi$. It follows that p does not divide $\xi(1)$, and thus some irreducible constituent of ξ_P has degree not divisible by p, and hence is linear. This linear character is an extension of μ to P, as desired, and it follows that μ extends to T, as claimed.

Let $\nu \in \mathrm{Irr}(T)$ be an extension of μ. Then ν^G is irreducible by the Clifford correspondence, and thus $\nu^G \in \mathrm{Irr}(G|N)$ since ν extends $\mu \neq 1_E$. It follows that $|G : T| = \nu^G(1) \in \mathrm{cd}(G|N) = \{a, b, c\}$. Also, we know that $|G : T|$ is not divisible by p, and thus $|G : T| \neq a$. Since $F \triangleleft G$ is a p-group, we see that $F \subseteq T$, and thus T/E contains the nonabelian subgroup F/E. Therefore, T/E is nonabelian, and has a nonlinear irreducible character β, which we can view as a character of T. (In fact, since F/E is a nonabelian normal p-subgroup of T/E, we can choose β so that $\beta(1)$ is divisible by p.) Now $\beta\nu \in \mathrm{Irr}(T)$ lies over μ, and thus $(\beta\nu)^G$ is irreducible and lies in $\mathrm{Irr}(G|N)$. The degree of this character is $\beta(1)|G : T|$, which exceeds $|G : T|$, and we deduce that $|G : T| \neq c$. The only remaining possibility is that $|G : T| = b$, and hence p does not divide b. Also, we must have $\beta(1)|G : T| = c$, and thus $\beta(1) = c/b$, which is therefore an integer. Furthermore, since β was an arbitrary nonlinear irreducible character of T/E, we deduce that $\mathrm{cd}(T/E) = \{1, c/b\}$, and hence T/E is metabelian. (We mention that since β could have been chosen to have degree divisible by p, we can conclude that p divides c/b and thus p divides c and $f = b$. We will not need this information, however.)

Now $F \subseteq T$ and T/E is metabelian, while F/E is nonabelian. It follows that $F/E \nsubseteq (T/E)'$, and thus there exists a linear character $\tau \in \mathrm{Irr}(T)$ with $E \subseteq \ker(\tau)$ and such that $F \nsubseteq \ker(\tau)$. It follows that τ^G is a character of degree $|G:T| = b$, and all of its irreducible constituents lie in $\mathrm{Irr}(G|N)$. There are just two possibilities, therefore: either τ^G is irreducible, or else it is a sum of irreducible characters of degree a. If the latter situation occurs, however, then a would divide $\tau^G(1) = b$, and this is impossible since p divides a but p does not divide b. We conclude, therefore, that τ^G is irreducible.

Now write $\sigma = \tau_F$, and recall that σ is nonprincipal by the choice of τ, and also $E \subseteq \ker(\sigma)$. Let S be the stabilizer of σ in G and note that $S \supseteq T$ since σ extends to T. Since $|G:T|$ is not divisible by p, we see that T contains a full Sylow p-subgroup of S, and thus σ extends to this Sylow subgroup. Because F is a p-group, it follows by Theorem 6.26 of [6] that σ extends to some character $\delta \in \mathrm{Irr}(S)$. Then δ^G is irreducible and has degree $|G:S|$. Also, δ^G lies in $\mathrm{Irr}(G|N)$ since σ is nonprincipal, and it follows that $|G:S| \in \{a, b, c\}$. Also, $|G:S|$ divides $|G:T| = b$ and we have observed that a does not divide b since p divides a. We deduce that $|G:S| = b$, and thus $S = T$.

Recall that K centralizes F/E, and thus K stabilizes σ since $E \subseteq \ker(\sigma)$. Thus $K \subseteq S = T$, and K stabilizes μ. Since μ is a linear character of the p-group E and p does not divide $|K:E|$, we deduce that μ extends to K. Since μ is nonprincipal and has p-power order, its extension to K has order divisible by p, and this contradicts the fact that $K = \mathbf{O}^p(K)$. This contradiction arose from the assumption that $K < N$, and so we can now assume that $K = N$ and $E = F$.

Since $a \notin \mathrm{cd}(G|F)$, we see that $\mathrm{cd}(G|F) \subseteq \{b, c\}$. Thus F has derived length at most 2, and we conclude that N/F must be nonabelian. As before, we let N/M be a chief factor of G and we write q to denote the unique prime divisor of $|N/M|$, so that $q \neq p$ and $F \subseteq M$. Then M/F is abelian, and all Sylow subgroups of N/F except possibly the Sylow q-subgroups are abelian. If there also exists a G-chief factor N/L that is not a q-group, it follows that all Sylow subgroups of N/F are abelian. In this case, $ML = F$ and since M/F and L/F are abelian, we deduce that N/F is nilpotent, and thus it is abelian. We know that this is not the case, however, and so we conclude that L does not exist and that $N = \mathbf{O}^r(N)$ for all primes $r \neq q$.

Let R/F be the normal q-complement of M/F, and suppose that $a \notin \mathrm{cd}(G|R)$. Then $R'' = 1$, so that R' is abelian, and hence $\mathrm{dl}(N/R') > 2$. We therefore have $|\mathrm{cd}((G/R')|(N/R'))| = 3$, and thus $c \in \mathrm{cd}((G/R')|(N/R'))$. Lemma 4.6 now applies, and we conclude that R is a p-group, and so $R = F$ in this case.

If $R > F$, therefore, there exists $\psi \in \mathrm{Irr}(G)$ with $\psi(1) = a$ and $R \not\subseteq \ker(\psi)$. But $F \subseteq \ker(\psi)$ and R/F is abelian, and thus ψ_R has a nontrivial linear constituent λ with multiplicative order not divisible by q. Since the stabilizer of λ in G has index dividing a, we know by Lemma 5.1 that this stabilizer must contain N, and it follows that λ extends to N since N/R is a q-group. Therefore, $\mathbf{O}^r(N) < N$ for all primes r dividing $o(\lambda)$. This is not the case, however, and we deduce that $R = F$, so that N/F is a nonabelian q-group.

Now choose $e \in \{a, b, c\}$ having the smallest possible q-part. Since F' is abelian, we see that $\mathrm{dl}(G/F') > 2$, and therefore $\mathrm{cd}((G/F')|(N/F')) = \{a, b, c\}$. It follows that there exists $\chi \in \mathrm{Irr}(G|N)$ with $\chi(1) = e$, and such that $F' \subseteq \ker(\chi)$. For every such choice of χ, Theorem 3.1 guarantees that $N' \subseteq \ker(\chi)$. (Observe that Theorem 3.1 applies because F/F' is an abelian normal q-complement in N/F'.) As $F \subseteq N' \subseteq \ker(\chi)$, we deduce that $e \in \mathrm{cd}((G/F)|(N/F))$, and by Corollary 4.3, therefore, we have $e \neq c$.

We showed in the previous paragraph that if $\chi(1) = e$ and $F' \subseteq \ker(\chi)$, where $\chi \in \mathrm{Irr}(G)$, then $N' \subseteq \ker(\chi)$. This cannot happen for every character $\chi \in \mathrm{Irr}(G)$ of degree e, however, or else $e \notin \mathrm{cd}(G|N')$, and thus N' is metabelian, which is not the case. It follows that $e \in \mathrm{cd}(G|F')$, and in particular, $e \neq a$ since we know that $a \notin \mathrm{cd}(G|F)$. We deduce that $e = b$, and thus the q-part of a exceeds the q-part of b, and a does not divide b. Also, $b = e \in \mathrm{cd}(G|F')$, and hence there is some irreducible character of G of degree b that has nonlinear constituents upon restriction to F. We conclude from this that p divides b.

Since $N' < N$, we know that $N'' \subseteq F$, and thus $a \notin \mathrm{cd}(G|N'')$, and $\mathrm{cd}(G|N'') \subseteq \{b, c\}$. If p divides c, then since p also divides b, Theorem D would imply that N' has a normal p-complement. But the unique minimal normal subgroup of G contained in N is a p-group, and thus the p-complement of N' would have to be trivial. In this situation, N' is a p-group, and thus N/F is abelian. We know that this is not the case, however, and we conclude that p does not divide c.

Now let $\alpha \in \mathrm{Irr}(F)$ be nonlinear. (Note that α exists because we know that F is nonabelian.) Since irreducible characters of G of degree a have F in the kernel and we know that p does not divide c, we see that all irreducible characters of G lying over α must have degree b. Since N/F is a nonabelian q-group, it follows that some irreducible character of N lying over α has degree divisible by q. (This is clear if α does not extend to N. On the other hand, if α extends to $\gamma \in \mathrm{Irr}(N)$, let $\beta \in \mathrm{Irr}(N/F)$ be nonlinear, and view $\beta \in \mathrm{Irr}(N)$. Then $\beta\gamma$ is an irreducible character of N lying over α and having degree divisible by $\beta(1)$, which is divisible by q.) It follows that q divides $b = e$, and thus q divides all three members of $\mathrm{cd}(G|N)$.

Let T be the stabilizer of α in G, so that $|G:T|$ divides b and all irreducible characters of T lying over α have degree $b/|G:T|$. Given any prime $r \neq p$, we know from Lemma 5.2 that some irreducible character of T lying over α has degree not divisible by r, and it follows that $b/|G:T|$ must be a power of p.

We claim now that $T \cap N = F$. To see why this is so, write $D = T \cap N \lhd T$, and suppose that $D > F$, so that there exists a nonprincipal linear character τ of D/F, which we view as a character of D. Let S be the stabilizer of τ in T and let $Q/F \in \mathrm{Syl}_q(S/F)$. Since α is invariant in Q, there is an extension δ of α to Q, and we can choose δ such that $o(\delta) = o(\alpha)$ is a p-power. Observe that δ_D is the unique extension of α to D that has p-power determinantal order, and thus δ_D is invariant in T. Now $\tau\delta_D$ is an irreducible character of D that is invariant in S, and, in fact, S is the full stabilizer of $\tau\delta_D$ in T because δ_D and $\tau\delta_D$ uniquely determine τ.

Let $\xi \in \mathrm{Irr}(S)$ lie over $\tau\delta_D$. Then ξ^T is irreducible by the Clifford correspondence, and ξ^T lies over α. It follows that $\xi(1)|T:S| = \xi^T(1) = b/|G:T|$ is a power of p, and in particular, $\xi(1)$ is not divisible by q. There is thus an irreducible constituent η of ξ_Q such that q does not divide $\eta(1)$. But Q/F is a q-group, and it follows that η_F is irreducible, and hence $\eta_F = \alpha$. Since also $\delta \in \mathrm{Irr}(Q)$ and $\delta_F = \alpha$, we must have $\eta = \nu\delta$, where ν is some linear character of Q with $F \subseteq \ker(\nu)$. Also, η lies over $\tau\delta_D$, and so $\nu_D\delta_D = \eta_D = \tau\delta_D$, and hence $\nu_D = \tau$ and τ extends to Q. Applying Theorem 6.26 of [6] in the group S/F, we deduce that τ extends to S, and we let $\sigma \in \mathrm{Irr}(S)$ be an extension of τ. Thus σ is linear, and since τ is nonprincipal, all irreducible constituents of σ^G lie in $\mathrm{Irr}(G|N)$.

Recall that $\xi(1)|T:S| = b/|G:T|$. We have $\sigma(1) = 1 < \alpha(1) \leq \xi(1)$, and thus

$$\sigma^G(1) < \xi^G(1) = \xi(1)|G:S| = \xi(1)|T:S|\,|G:T| = b,$$

and it follows that all irreducible constituents of σ^G have degree a. Thus a divides $\sigma^G(1) = |G:S|$, which is a divisor of b. This is a contradiction, however, since a does not divide b, and we conclude that $T \cap N = F$, as claimed.

We have now shown that the stabilizer in N of every nonlinear irreducible character α of F is exactly F, and thus if we let $U \in \mathrm{Syl}_q(N)$, we know that the stabilizer in U of α is trivial, and this holds for all nonlinear irreducible characters α of F. Recall that $F' > 1$ and let ϵ be an arbitrary nonprincipal linear character of F'. If V is the stabilizer in U of ϵ, then V permutes the irreducible constituents of ϵ^F, which has p-power degree. It follows that V stabilizes some irreducible constituent of ϵ^F, and we note that such a constituent must be nonlinear since F' is not contained in its

kernel. It follows that $V = 1$, and thus the action of U on the (nontrivial) group of linear characters of F is Frobenius. Now $U \cong N/F$, and so U is a nonabelian q-group, which, as we have just seen, is a Frobenius complement. We deduce that $q = 2$ and that N/F is a generalized quaternion group.

Now let $C = \mathbf{C}_G(N/F)$ so that $C \triangleleft G$ and G/C is isomorphic to a subgroup of the automorphism group of the generalized quaternion group N/F. If $|N/F| > 8$, then NC/C, which corresponds to the inner automorphisms of N/F, is nonabelian, and thus $|\mathrm{cd}((G/C)|(NC/C))| \geq 2$ and thus at least two of a, b, and c are degrees of irreducible characters of G/C. But G/C is a 2-group in this case, and so its order is not divisible by p. Since b is divisible by p, we conclude that $\mathrm{Irr}(G|N)$ contains an irreducible character of degree c with C in its kernel. But $F \subseteq C$, and this is a contradiction by Corollary 4.3.

We are left with the case where $N/F \cong Q_8$, and thus G/C is isomorphic to a subgroup of the symmetric group S_4 that contains the Klein subgroup (corresponding to NC/C). If G/C is a 2-group, it has a linear character such that NC/C is not in the kernel, and otherwise, it has an irreducible character of degree 3 with NC/C not in the kernel. It follows that $\mathrm{cd}(G|N)$ contains either 1 or 3. We have seen, however, that each member of $\mathrm{cd}(G|N)$ is divisible by $q = 2$, and this is our final contradiction. ∎

REFERENCES

1. T. R. Berger, Characters and derived length in groups of odd order, *J. Algebra* **39** (1976), 199–207.
2. Y. G. Berkovich, Degrees of irreducible characters and normal p-complements, *Proc. Amer. Math. Soc.* **106** (1989), 33–34.
3. S. Garrison, "On Groups with a Small Number of Character Degrees," Ph.D. Thesis, University of Wisconsin, Madison, 1973.
4. D. Gluck, Bounding the number of character degrees of a solvable group, *J. London Math. Soc.* (2) **31** (1985), 457–462.
5. I. M. Isaacs, Character degrees and derived length of a solvable group, *Canad. J. Math.* **27** (1975), 146–151.
6. I. M. Isaacs, "Character Theory of Finite Groups," Academic Press, New York, 1976.