A second strain gradient elasticity theory with second velocity gradient inertia – Part I: Constitutive equations and quasi-static behavior

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A multi-cell homogenization procedure with four geometrically different groups of cell elements (respectively for the bulk, the boundary surface, the edge lines and the corner points of a body) is envisioned, which is able not only to extract the effective constitutive properties of a material, but also to assess the “surface effects” produced by the boundary surface on the near bulk material. Applied to an unbounded material in combination with the thermodynamics energy balance principles, this procedure leads to an equivalent continuum constitutively characterized by (ordinary, double and triple) general-ized stresses and momenta. Also, applying this procedure to a (finite) body suitably modelled as a simple material cell system, in association with the principle of the virtual power (PVP) for quasi-static actions, an equivalent structural system is derived, featured by a (macro-scale) PVP having the typical format as for a second strain gradient material model. Due to the surface effects, the latter model does work as a combination of two subsystems, i.e. the bulk material behavior as a Cauchy continuum, and the boundary surface operating as a membrane-like boundary layer, each subsystem being in (local and global) equi-librium by its own. Further, the applied (ordinary) boundary traction splits into two (response-depen-dent) parts, i.e. the “Cauchy traction” transmitted to the bulk material and the “Gurtin–Murdoch traction” acting, together with all other boundary tractions, upon the boundary layer. The role of the boundary layer as a two-dimensional manifold enclosing a Cauchy continuum is elucidated, also with the aid of a discrete model. A strain gradient elasticity theory is proposed which includes a minimum total potential energy principle featuring the relevant boundary-value problem for quasi-static loads and its (unique) solution. A simple application is presented. Two appendices are included, one reports the proof of the global equilibrium of the boundary layer, the other is concerned with double and triple stresses. The paper is complemented by a companion Part II one on dynamics. Previous findings by the author [Polizzotto, C., 2012. A gradient elasticity theory for second-grade materials and higher order inertia. Int. J. Solids Struct. 49, 2121–2137] are improved and extended.

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1. Introduction

Strain gradient elasticity theories emerged in the literature, together with the analogous nonlocal theories, as analytical means capable to account for long range interaction forces within materials microstructure, and to capture experimentally detected phenomena (as length scale effects, wave dispersion, strain localization, etc.) which would be ignored by a classical theory, see e.g. Kröner (1967), Krumhansl (1968), Mindlin (1964, 1965) and Mindlin and Eshel (1968). Strain gradient theories do not consider independent rotations of the material particles, hence they are nonpolar in nature and couple stresses may there appear only in a “constrained” form (i.e. resulting from the rotation gradient of the continuum). Within this framework, of particular interest is the landmark paper by Mindlin (1965), who formulated a second strain gradient elasticity theory for statics, as well as for dynamics (Mindlin, 1964; Mindlin and Eshel, 1968). In spite of the excessive number of material constants therein required, this theory remains a prototype one for its conceptual foundations and its capability to capture length scale effects. Toupin et al. (1963), by a study on mono- and di-atomic lattice models, pointed out the ability of a second strain gradient theory to interpret the atomic structure of the matter.

Subsequently, Aifantis and co-workers proposed an isotropic elasticity theory based on the first strain gradient which required only three material constants (i.e. the two classical Lamé constants and a length scale parameter), see Aifantis (1992), Atlam and Aifantis (1992) and Ru and Aifantis (1993). This theory was then extended to dynamics with higher order inertia (Askes and Aifantis, 2006, 2011) for which only one additional length scale parameter is required. This elasticity model (here referred to as...
the Aifantis model) has been employed to address a variety of structural problems showing that no strain singularities arise at crack tips and dislocation cores, and that dispersion effects in wave propagation can be captured (see the review paper by Askes and Aifantis, 2011). However, as shown by Lazar and Maugin (2005), singularities of the double stress cannot be removed with the mentioned Aifantis model.

An extension of the above first strain gradient theory to a second strain gradient one (featured by four material constants, i.e. the two Lamé constants and two length scale parameters) was advanced by Polizzotto (2003). A similar theory, called strain gradient theory of bi-Helmholtz type, was independently elaborated by Lazar et al. (2006) and applied to a series of dislocation problems within an infinite domain with the outstanding result that no singularities of any kind arise at the dislocation core and that therefore this material model possesses high regularization capacities. This outcome was largely confirmed by a number of applications to defect interaction problems (Zhang et al., 2006), dislocation analysis (Lazar and Maugin, 2006; Lazar, 2013) and disclination analysis (Deng et al., 2007). However, no attention has been paid to the boundary conditions and the surface effects that typically characterize the latter strain gradient models.

The role of the velocity gradient and of the inherent higher order inertia terms in the motion equations has been addressed in a series of studies dealing with the wave motion and the related dispersion phenomena. This topic has been covered extensively in the literature, but here we limit ourselves to mention Altan and Aifantis (1997), Georgiadis et al. (2000), Askes et al. (2002), Askes et al. (2007), Metrikine and Askes (2002), Askes and Aifantis (2006, 2009) and Papargyri-Beskou et al. (2009), see also the review paper by Askes and Aifantis (2011) and the literature therein. It emerges from the latter literature that the higher order inertia models are able to describe realistically wave dispersion phenomena. Metrikine and Askes (2002) and Askes and Aifantis (2006, 2009) advanced the concept of “dynamically consistent” gradient model, namely a model endowed with gradient enhancements in both its stiffness and inertia characteristics, which makes it able to remove singularities of the strain field arising e.g. in the vicinity of a sharp crack tip, and to realistically describe the dispersive characteristics of the wave propagation in a nonhomogeneous medium. However, the higher order inertia terms appearing in the governing equations were introduced heuristically in these studies, and their relationship to the kinetic energy is left unclarified. Lazar and Anastasia-dis (2007) and Agiasofitou and Lazar (2009) provided a useful insight on the conservation and balance laws for gradient elastodynamics. For an overview on the historical developments of gradient elasticity and higher order inertia with related applications see Askes and Aifantis (2011).

Fried and Gurtin (2006) addressed a first strain gradient theory combined with first velocity gradient inertia and cast it within a general isotropic visco-elasticity theory based on thermodynamics principles. An analogous elasticity theory was presented by Polizzotto (2012) with some new findings on surface effects, whereby the boundary surface works as a membrane-like boundary layer. This is in global and local equilibrium by its own and encloses a bulk material working as a classical Cauchy continuum, to which some part of the boundary traction is directly transmitted; additionally, besides the inertia body forces distributed in the bulk volume, inertia surface forces arise within the boundary layer.

The purpose of the present study is to extend the findings by Polizzotto (2012) to second strain gradient elasticity with second velocity gradient inertia. In contrast to the previous study (and to that of Mindlin, 1965 as well) – where a strain gradient model was considered from the beginning as a primitive object – here instead the start will be taken from a material model conceived as a distribution of cell elements of simple material, in a fashion quite similar to Mindlin (1964) and Germain (1973). We shall use a (nonstandard) multi-cell homogenization procedure with four geometrically different types of cell elements, respectively for the bulk, the boundary surface, the edge lines and the corner points. The use of such different types of cell elements in the homogenization process makes it possible to macroscopically take into account the “surface effects”, that is, the effects occurring in the presence of a boundary surface due to the different ways in which in a real material the microstructure may operate at qualitatively different locations, either within the bulk sufficiently far from the boundary surface, or very close to the latter, or in particular to a surface singularity like an edge line or a corner point.

Through such homogenization procedures and with the aid of thermodynamics arguments, the considered cell system is transformed into an equivalent structural system formed up by a continuum enclosed within a boundary layer. This equivalent structural system happens to exhibit the typical features of a second strain gradient elasticity model with second velocity gradient inertia. Its study leads to results that generalize and improve those previously derived by the author (Polizzotto, 2012) for first strain gradient elasticity models with first velocity gradient inertia.

The usefulness of second strain gradient models has been abundantly assessed in the literature (quoted previously), but it is not so for second velocity gradient inertia models, to the author’s knowledge. However, second velocity gradients, which here arise in a natural way from the adopted Taylor expansion techniques, do have effects on the structural behavior which are worthy of study by their own. We believe that the latter models may improve the assessed capacity of first velocity gradient models in describing realistically dispersion phenomena.

For space reasons, the subject is treated in two distinct parts, i.e. Part I devoted to the constitutive equations (for both elasticity and inertia) and the quasi-static behavior, Part II devoted to the dynamic behavior. The outline of the present Part I paper is as follows. In Section 2, by a homogenization procedure applied to an unbounded material modelled as a continuous distribution of cell elements, the constitutive forms of the internal energy and of the kinetic energy for the equivalent continuum are derived together with the constitutive equations of the related sets of generalized stresses and momenta. A particular choice of the internal energy and of the kinetic energy leads to an isotropic material model (here called extended Aifantis model) featured by, besides the two Lamé constants, four length scale parameters (two for statics and two for inertia).

In Section 3, a multi-cell homogenization procedure is advanced whereby four geometrically different groups of cell elements are used, respectively for the bulk volume, the boundary surface, the edge line and the corner points of a body. With the latter procedure, applied to a (finite) body mechanically described through the principle of the virtual power (PVP) for quasi-static actions, the body is transformed into a (macro-scale) equivalent structural system, which happens to exhibit the characteristics of a second strain gradient material model, associated to a (macro-scale) PVP with a form as known from the literature (Mindlin, 1965; Germain, 1973; Gurtin, 2001).

In Section 4 a few explanations are given in regard to the content and the invariance requisite of the (macro-scale) PVP enforced for the previously derived second strain gradient material. In Section 5, by the latter PVP the equilibrium equations relating the generalized stresses to the external (quasi-static) forces are derived, which are suitably re-interpreted in Section 6. Like in the case of first strain gradient (Polizzotto, 2012), it is found that—also within a second strain gradient model—the bulk material behaves as a classical Cauchy continuum under a total (Cauchy) stress \(\mathbf{T} = \{\mathbf{T}_{ij}\}\) (depending on the generalized stresses), and that the boundary surface behaves as a membrane-like boundary layer endowed with a multiple
located at \( \mathbf{C}_0 \): 

\[ t = \mathbf{t}_c + \mathbf{t}_{\text{GM}}, \]

where \( \mathbf{t}_c := \mathbf{n} \cdot \mathbf{T} \) is the Cauchy traction, which is transmitted to the bulk material, whereas \( \mathbf{t}_{\text{GM}} \) (here called Gurtin–Murdoch traction) is supported by the boundary layer. The latter layer finds itself in global and local equilibrium under this Gurtin–Murdoch traction, as well as under all other boundary external actions including the surface higher order tractions (\( t^{(1)}, t^{(2)} \)); the edge line tractions (\( f, f^{(1)} \)); and the corner point forces \( \mathbf{F}_c, (c = 1, 2, \ldots, N) \). Local (two-dimensional) equilibrium equations must be satisfied at every point of the boundary layer, the latter being subjected to the surface stresses \( \Sigma = (\Sigma_{ij}) \) and \( \Sigma^{(1)} = (\Sigma^{(1)}_{ijkl}) \) (also depending on the generalized stresses), according to the principles of surface mechanics (Gurtin and Murdoch, 1975, 1978). The surface tractions \( \mathbf{n} \cdot \Sigma \) and \( \mathbf{n} \cdot \Sigma^{(1)} \) describe the mutual interactions between the surface microstructures. A discrete model of the boundary layer enables one to ascertain the micro-scale mechanisms through which it does operate as a two-dimensional manifold substituting the classical purely geometrical concept of boundary surface. Similar concepts are also found to hold for the edge line behaving as a rod.

In Section 7 the obtained results are particularized for a first strain gradient material. In Section 8 a total potential energy principle is shown to characterize the relevant boundary-value problem for quasi-static loadings. Section 9 is devoted to a simple application to a bar under end tractions exhibiting the Toupin–Gazis puckering effect (Toupin et al., 1963). Conclusions are drawn in Section 10. The paper is ended by two appendices. Appendix A reports the proof of the global equilibrium conditions of the boundary layer. Appendix B is devoted to some questions regarding higher order strains, double and triple stresses.

Notation. A compact notation is used, with boldface letters denoting vectors or tensors of any order. The scalar product between vectors or tensors is denoted with as many dots as the number of contracted index pairs. For instance, denoting by \( \mathbf{u} = \{ u_i \}, \mathbf{v} = \{ v_i \}, \mathbf{e} = \{ e_i \}, \sigma = \{ \sigma_{ij} \}, \tau = \{ \tau_{ij} \} \) and \( \mathbf{A} = \{ A_{ijkl} \} \) some vectors and tensors, one can write: \( \mathbf{u} \cdot \mathbf{v} = u_i v_i, \sigma : \tau = \sigma_{ij} \tau_{ij}, A : e = \{ A_{ijkl} e_{kl} \}, A : \mathbf{r} = \{ A_{ijkl} r_{kl} \}, A^\tau : e = \{ A_{ijkl}^\tau e_{kl} \}. \) The summation rule for repeated indices holds and the subscripts denote components with respect to an orthogonal Cartesian co-ordinate system, say \( \mathbf{x} = (x_1, x_2, x_3). \) The tensor product is simply indicated as, for instance, \( \mathbf{u} \otimes \mathbf{v} = (u_i v_j) \delta_{ij}. \) An upper dot over a symbol denotes its (material) time derivative, \( \mathbf{u} = \mathbf{d}u/dt. \) The symbol \( \nabla \) denotes the spatial gradient operator, i.e. \( \nabla \mathbf{u} = (\partial u_i / \partial x_j), \nabla^\tau \) is the symmetric part of \( \nabla, \) and \( \Delta \) is the Laplacian operator. The projection operator \( \mathbf{P}(\mathbf{n}) \) associated to a plane of unit normal \( \mathbf{n} \) is defined as \( \mathbf{P}(\mathbf{n}) := I - \mathbf{n} \otimes \mathbf{n} = (\delta_{ij} - n_i n_j) \delta_{ij}. \) The symbol \( \mathbf{V}_{(\mathbf{n})} \) denotes the tangential gradient over a plane of normal \( \mathbf{n}, \) i.e. \( \mathbf{V}_{(\mathbf{n})} := \mathbf{P}(\mathbf{n}) \cdot \nabla, \) or \( \partial_{(\mathbf{n})} \mathbf{e} = (\delta_{ij} - n_i n_j) \partial_i \mathbf{e}. \) The symbol \( \mathbf{e} = \{ e_i \} \) indicates the alternating third order tensor. The symbol := means equality by definition. Other symbols will be defined in the text at their first appearance.

2. Equivalent continuum and related constitutive equations

Let us consider a real material occupying a continuous unbounded domain, which exhibits defects and inhomogeneities emerging as properties of an inherent microstructure. The material is assumed to be statistically representable by geometrically equal cell elements (or reference volume elements), say \( V_c, \) appended to every point \( \mathbf{x} \) of the domain, each cell being formed up by simple material. The characteristic dimension of \( V_c \) is so chosen that \( V_c \) is sufficiently small to capture small scale defects and inhomogeneities, but it is large enough such as defects and inhomogeneities may be there distributed in a statistically homogeneous manner

(see e.g. Krajcinovic, 1996). The generic cell \( V_c \) located at \( \mathbf{x} \) constitutes a micro-continuum with displacements \( \mathbf{u}(\mathbf{x} + \mathbf{r}), \) (primed symbols denote micro-scale quantities within the cells, \( \mathbf{r} \) is the local position vector).

Assuming isothermal conditions for simplicity, the internal energy balance principle (or first thermodynamics principle) can be enforced as in the following:

\[
\dot{\epsilon}(\mathbf{x}) = \frac{1}{V_c} \int_{V_c} \sigma(\mathbf{x} + \mathbf{r}) : \dot{\epsilon}(\mathbf{x} + \mathbf{r}) \, dV(\mathbf{r}) \quad \forall \mathbf{x}
\]

where \( dV \) means volume measure; moreover \( \sigma(\mathbf{x}) \) denote the (nonsymmetric, in general) stress and the strain at the micro-scale within the cell located at \( \mathbf{x}. \) Eq. (1) realizes a homogenization of the cell system whereby, at every time \( t, \) the internal energy rate density at \( \mathbf{x}, \dot{\epsilon}(\mathbf{x}), \) measured at the macro-scale per unit volume of the equivalent (or homogenized) continuum, is postulated to be equal to the mean value of the micro-scale stress power within the cell located at the same point. For simplicity of notation, the reference to the current time \( t \) is omitted in (1).

Analogously, as discussed by Polizzotto (2012), the inertial energy balance principle (parallel to the internal energy balance principle, but concerned solely with inertial actions) can be enforced by writing

\[
k(\mathbf{x}) = \frac{1}{V_c} \int_{V_c} \pi(\mathbf{x} + \mathbf{r}) : \mathbf{v}(\mathbf{x} + \mathbf{r}) \, dV(\mathbf{r}) \quad \forall \mathbf{x}
\]

where \( \pi \) and \( \mathbf{v} \) are the micro-scale momentum and velocity within the cell. According to (2), the kinetic energy rate density at \( \mathbf{x}, k(\mathbf{x}), \) measured at the macro-scale per unit volume of the equivalent continuum is postulated to be equal to the mean value of the micro-scale momentum power within the cell attached to the same point.

Let us remark that, within classical continuum mechanics, the mentioned inertial energy balance principle (or anything equivalent) is not required. In fact, in the latter case, the momentum, say \( \mathbf{p} \), is related to the velocity by the classical law \( \mathbf{p} = \rho \mathbf{v}. \) Instead, within the present framework of higher order inertia, the momentum-velocity relation is unknown a priori and its determination can be achieved treating it just like an additional constitutive equation (or set of equations) for the inertial-type state variables. The inertial energy balance principle plays then a role analogous to the one played by the familiar internal energy balance principle.

Next, following Mindlin (1964) and Germain (1973), let the strain field \( \epsilon(\mathbf{x} + \mathbf{r}) \) and the velocity field \( \mathbf{v}(\mathbf{x} + \mathbf{r}) \) within the cell \( V_c \) be approximated by a Taylor series expansion till e.g. the second order terms, i.e.

\[
\gamma(\mathbf{x} + \mathbf{r}) = \epsilon(\mathbf{x}) + \mathbf{r} \cdot \nabla \epsilon(\mathbf{x}) + \frac{1}{2} \mathbf{r} \cdot \nabla \nabla \epsilon(\mathbf{x})
\]

\[
\mathbf{v}(\mathbf{x} + \mathbf{r}) = \mathbf{v}(\mathbf{x}) + \mathbf{r} \cdot \nabla \mathbf{v}(\mathbf{x}) + \frac{1}{2} \mathbf{r} \cdot \nabla \nabla \mathbf{v}(\mathbf{x})
\]

Substituting (3) into (1) and (2) and omitting the reference to \( \mathbf{x} \) as an argument gives:

\[
\dot{\epsilon} = \mathbf{\sigma} : \dot{\mathbf{\sigma}} + \sigma^{(1)} : \mathbf{\nabla} \dot{\mathbf{\sigma}} + \sigma^{(2)} : \mathbf{\nabla} \mathbf{\nabla} \dot{\mathbf{\sigma}} \quad \forall \mathbf{x}
\]

\[
k = \pi : \mathbf{v} + \pi^{(1)} : \mathbf{\nabla} \mathbf{v} + \pi^{(2)} : \mathbf{\nabla} \mathbf{\nabla} \mathbf{v} \quad \forall \mathbf{x}
\]

where we have introduced the (macro-scale) generalized stresses defined as

\[
\mathbf{\sigma}(\mathbf{x}) := \frac{1}{V_c} \int_{V_c} \mathbf{\sigma}(\mathbf{x} + \mathbf{r}) \, dV(\mathbf{r}) \quad \text{(ordinary stress)}
\]

\[
\sigma^{(1)}(\mathbf{x}) := \frac{1}{V_c} \int_{V_c} \mathbf{r} \sigma(\mathbf{x} + \mathbf{r}) \, dV(\mathbf{r}) \quad \text{(double stress)}
\]

\[
\sigma^{(2)}(\mathbf{x}) := \frac{1}{V_c} \int_{V_c} \frac{1}{2} \mathbf{r} \mathbf{r} \sigma(\mathbf{x} + \mathbf{r}) \, dV(\mathbf{r}) \quad \text{(triple stress)}
\]

as well as the (macro-scale) generalized momenta given by
Eq. (4) states that the internal energy \( e \) associated to the equivalent continuum must have a dependence on the set \( \{ \varepsilon, \nabla \varepsilon, \nabla^2 \varepsilon \} \), and that analogously the kinetic energy \( \kappa \), also associated to the equivalent continuum, must have a dependence on the set \( \{ \psi, \nabla \psi, \nabla^2 \psi \} \). Indeed, such features are recognized to characterize a second strain gradient material with second velocity gradient inertia. Here we assume that the latter dependences can be adequately specified by assigning the free energy \( \psi \) and the kinetic energy \( \kappa \) as the functions

\[
\psi = \psi(\varepsilon, \nabla \varepsilon, \nabla^2 \varepsilon), \quad \kappa = \kappa(\psi, \nabla \psi, \nabla^2 \psi)
\]

both to be continuous and convex with respect to their own arguments. The internal energy \( e \) can be easily derived from \( \psi \) through a Legendre transform (but \( e = \psi \) in the present case).

According to Noll (1963), the internal energy balance law (4)\( _1 \) can be regarded as associated to the (noninertial) actions (stresses, body forces, tractions) arising from the exterior bodies belonging the near world (as the solar system), whereas the inertial energy balance law (4)\( _2 \) can be regarded as associated to the inertial actions (momentum, inertia forces) arising from the totality of bodies belonging to the remote universe (the fixed stars). Hence, by (4)\( _1 \), written in terms of free energy, we can express the generalized stresses in the energetic form:

\[
\sigma = \frac{\partial \psi}{\partial \varepsilon}, \quad \sigma^{(1)} = \frac{\partial \psi}{\partial \nabla \varepsilon}, \quad \sigma^{(2)} = \frac{\partial \psi}{\partial \nabla^2 \varepsilon}
\]

which constitute the elasticity laws for the generalized stresses. Let us note that the stresses \( \sigma^{(1)} \) and \( \sigma^{(2)} \) are work-conjugate, respectively, to the first and second strain gradients; furthermore, as shown in Appendix B, they are associated with double forces (or dipoles) having one lever arm and, respectively, triple forces (or quadrupoles) having two lever arms.

Analogously, by the inertial energy balance law (4)\( _2 \), we can in turn derive the constitutive equations (or inertia laws) for the generalized momenta, which can be cast in the energetic form:

\[
\pi = \frac{\partial \kappa}{\partial \varepsilon}, \quad \pi^{(1)} = \frac{\partial \kappa}{\partial \nabla \varepsilon}, \quad \pi^{(2)} = \frac{\partial \kappa}{\partial \nabla^2 \varepsilon}
\]

The latter momenta are work conjugate of the velocity and the first and second velocity gradients, respectively.

In order that the constitutive laws (8) be objective, it is required that the energy balance law (4) be invariant under change of observer, that is, (4)\( _1 \) has to remain unchanged under the superposition of any (infinitesimal) rigid-body motion of the form (Noll, 1963; Truesdell and Noll, 1965; Gurtin, 2001):

\[
\mathbf{v}'(x, t) = \mathbf{a}'(t) + \mathbf{W}'(t) \cdot \mathbf{x}
\]

where \( \mathbf{a}'(t) \) is an arbitrary translation vector, and \( \mathbf{W}'(t) \) an arbitrary skew-symmetric rotation tensor. Since \( \mathbf{e} \rightarrow \mathbf{W}', \nabla \varepsilon \rightarrow \mathbf{0} \) and \( \nabla^2 \varepsilon \rightarrow \mathbf{0} \) correspondingly, we must have \( \sigma : \mathbf{W}' = 0 \vee \nabla^2 \varepsilon \), hence the stress tensor \( \sigma \) has to be symmetric, and thus \( \psi \) can be identified with the standard symmetric strain tensor. This is the reason why in writing (1) we have used the (symmetric) strain tensor in place of the distortion \( \nabla \mathbf{e} \).

It has to be noted that the invariance requisite mentioned above for the validity of (8) is not required for the validity of (9), since any quantity having a dynamic significance (like the kinetic energy, the velocity and their consequences) needs only to be evaluated with respect to a Galilean reference observer, that is, one being fixed, or uniformly moving, with respect to the fixed stars. Alternatively, we can state that, on operating an arbitrary change of observer through a transformation as \( \mathbf{x}' = \mathbf{c}(t) + \mathbf{Q}(t) \cdot \mathbf{x} \), where \( \mathbf{c}(t) \) is a vector and \( \mathbf{Q}(t) \) a rotation tensor \((\mathbf{Q}^T \cdot \mathbf{Q} = \mathbf{Q} \cdot \mathbf{Q}^T = \mathbf{I})\), we expect that the fixed stars move together with the moving observer (Noll, 1963; Truesdell and Noll, 1965; Malvern, 1969).

Let us remark that (8) and (9) hold true independently of the extent of the domain occupied by the material. In the present case of compatible elasticity, in accord to Mindlin (1965) Eq. (8) provides in total \( 6 + 18 + 30 = 54 \) independent generalized stress variables, which are work-conjugate of as many strain and strain gradient variables, that is the variables \( \varepsilon = \{ e_{ij} \} \), \( \nabla \varepsilon = \{ \partial_{ij} e_{ij} \} \), and \( \nabla^2 \varepsilon = \{ \partial_{ij} \partial_{kl} e_{ij} \} \), respectively. Analogously, (9) provides in total \( 3 + 9 + 18 + 30 = 50 \) independent generalized momentum variables, which are work-conjugate of as many velocity and velocity gradient variables, that is the variables \( \psi = \{ v_j \} \), \( \nabla \psi = \{ \partial_{ij} v_j \} \), and \( \nabla^2 \psi = \{ \partial_{ij} \partial_{kl} v_j \} \), respectively.

In the following, for practical reasons, we shall take \( \psi \) and \( \kappa \) in the special forms:

\[
\psi := \frac{1}{2} \mathbf{C} :: [\mathbf{e} \varepsilon + e_{s1}^2 (\nabla \mathbf{e})^2 : \mathbf{e} \varepsilon + e_{s2}^2 (\nabla \mathbf{e}) : (\nabla \mathbf{e})] \quad \kappa := \frac{1}{2} \mu [\mathbf{v} \cdot \mathbf{v} + e_{d1}^2 \mathbf{v} : \mathbf{v} : e_{d2}^2 \nabla \mathbf{v} : \nabla \mathbf{v} : \nabla \mathbf{v}]
\]

where \( \mathbf{C} \) denotes the usual fourth-order tensor of isotropic elasticity, i.e., denoting by \( \lambda, \mu \) the classical Lamé constants,

\[
\mathbf{C} = \{ C_{ijkl} \}, \quad C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{j} + \delta_{kj} \delta_{i}) \quad (12)
\]

Also, \( e_{s1}, e_{s2} \) and \( e_{d1}, e_{d2} \) are length scale parameters for statics and \( e_{d1}, e_{d2} \) analogous parameters for inertia effects; often the equalities \( e_{d1} = e_{d1} \) and \( e_{d2} = e_{d2} \) are taken for practical purposes (Lazar et al., 2006). With \( \psi \) and \( \kappa \) as above, the stresses (8) and the momenta (9) possess the following constitutive forms:

\[
\sigma = \mathbf{C} \cdot \varepsilon, \quad \sigma^{(1)} = e_{s1}^2 \nabla \mathbf{e} \varepsilon, \quad \sigma^{(2)} = e_{s2}^2 \nabla \varepsilon \nabla \mathbf{e} \varepsilon \quad \pi = \rho \mathbf{v}, \quad \pi^{(1)} = e_{d1}^2 \nabla \mathbf{v} \varepsilon, \quad \pi^{(2)} = e_{d2}^2 \nabla \mathbf{v} \nabla \mathbf{v} \varepsilon \quad (13)
\]

The characteristic feature of the latter constitutive equations consists in the circumstance whereby the ordinary stress \( \sigma \) obeys the Hooke’s law and the ordinary momentum \( \pi \) obeys the classical motion quantity \( \rho \mathbf{v} \), whereas the double and triple stress and momentum tensors are expressed as the first and the second gradient of the related ordinary stress and momentum tensors, respectively. Such a feature constitutes a distinctive character of the so-called Aifantis model for first strain gradient elasto-statics (where \( e_{s1} = e_{s2} = e_{d2} = 0 \) (Aifantis, 1992; Aifantis and Aifantis, 1992; Ru and Aifantis, 1993), and subsequently for dynamics \( e_{s2} = e_{d2} = 0 \), (Askes and Aifantis, 2006, 2011)). For second strain gradient elasto-statics \( e_{d1} = e_{d2} = 0 \), the above material model was advanced by Polizzotto (2003) and then independently elaborated as the strain gradient theory of bi-Helmholtz type by Lazar et al. (2006), Zhang et al. (2006), Deng et al. (2007), Shodja et al. (2012) and Lazar (2013), as already referenced in the Introduction. On relaxing the hypothesis of simple material within the cells, the preceding developments may be extended to other types of material, like Cosserat, polar and multipolar materials (Eringen, 1966; Green and Rivlin, 1964), but this point is skipped for brevity.

3. Multi-cell homogenization procedure leading to a second strain gradient model

In Section 2 some properties of the material microstructure have been extracted and brought up to the macrostructural level in the absence of perturbations due to the presence of any circumventing boundary surface. In order to evaluate the effects produced by an existing boundary surface (hereafter called surface effects), here we consider a body of finite domain, say \( V \). Within \( V \), every particle can be thought of to incorporate an equally distributed
atomic network (like in a crystal lattice), whereby the long range actions can manifest themselves in the same manner at all locations within the body, but sufficiently far from the boundary surface. Instead, in a particle being close, or even attached, to the boundary surface, the mentioned atomic network is in some way modified by the cutting surface and the long range actions do take place in some altered manner due to the absence of the atomic attractions from the outside (Toupin et al., 1963). The same holds for a particle being close, or attached, to an edge line, or to a corner point.

We can address this very complex material behavior in a schematic way through a (nonstandard) multi-cell homogenization procedure. For more generality, the body is assumed to possess some edge line(s) and corner points. The generic subdomain of the latter body is conceived (Fig. 1) as the union of a core volume, say \( B \), and a circulating boundary layer, say \( B_s \), the latter being generated by the totality of straight segments of constant length \( c_0 \) in the direction of the (generalized) normal \( \mathbf{n} \), with one end on \( \partial B \) at all points of this surface. The length parameter \( c_0 \) is taken equal to a certain number of inter-particle spaces, such that \( B_s \to \partial B \) at the limit as \( c_0 \to 0 \). \( B_s \) is formed up, in general, by three parts, say \( B_s = B_{s1} \cup B_{s2} \cup B_{s3} \), where \( B_{s1} \) includes the regular points of \( \partial B \), whereas \( B_{s2} \) and \( B_{s3} \) include the singular points of \( \partial B \), i.e. the points of the edge lines (globally denoted \( \Gamma(B) \)), and the corner points, say \( x_c \). (\( c = 1,2, \ldots, N(B) \)), respectively. Four different types of cell elements are employed for this homogenization, that is: (i) the cell elements \( V_e \) within \( B \), each taken in the form of a sphere of radius \( c_0 \) and centered at the generic point \( x \in B \); (ii) the cell elements \( L_e \) within \( B_{s1} \), each in the form of a linear segment normal to the surface, of length \( c_0 \) and externally attached to \( \partial B \); (iii) the cell elements \( A_e \) within \( B_{s2} \), each in the form of a circular sector lying on a plane orthogonal to the edge line, of radius \( c_0 \) and with the center lying on the edge line; (iv) the cell elements \( V^c_e \) attached to the corner points \( x_c \), each taken as just a portion of a sphere \( V_c \), in the form of a spherical sector.

Let us note that a cell \( A_c \) can be thought of as being generated by a linear segment of length \( c_0 \) with one end fixed at the related point \( x \in \Gamma(B) \), external to \( \partial B \) and rotating in all possible directions of the generalized normal to \( \partial B \) drawn from \( x \), see Fig. 1(b). In an analogous way can be generated a cell \( V^c_e \) attached to the corner point \( x_c \), see Fig. 1(c).

The nonstandard cell elements attached to the boundary layer \( B_s \), i.e. the cells \( \{ L_e, A_e, V^c_e \} \), are motivated by the physical circumstance that the external actions on the boundary surface can in practice include not only the classical contact forces (i.e. ordinary tractions), but also long range forces, that is, multi-pole forces applied at points within some boundary layer circulating the bulk material. In the case of the unbounded domain treated in Section 2 no elements of the latter sort were used due to the absence of a boundary surface; in any case the homogenization there accomplished, in which only domain-type elements were used, can obviously be thought of as a particular case of a multi-cell homogenization procedure.

The evaluation of the mentioned surface effects has to be achieved separately either for the quasi-static behavior of the material, or for its dynamic behavior. The latter case will be treated in the Part II paper, whereas the former one is treated here. For this aim, the above multi-cell homogenization procedure is applied in combination with the classical principle of the virtual power (PVP) enforced at the micro-scale of the cell system. At every point of the body, the relevant virtual power contribution is computed as a mean value of the analogous micro-scale contribution computed over the cell element attached to the considered point. Denoting by primed symbols the micro-scale quantities within the cells, and by a superposed tilde the virtual kinematic variables, we can write the (macro-scale) internal virtual power for the subdomain \( B \in V \) as:

\[
L_{\text{int}} := \int_V \frac{1}{V_c} \int_{V_e} \mathbf{\sigma}(\mathbf{x}+\mathbf{r}) : \mathbf{\varepsilon}(\mathbf{x}+\mathbf{r}) \, dV/\partial(r) \, dV(\mathbf{x})
\]

Fig. 1. Geometrical sketch showing the decomposition of a (finite) body into a core domain \( B \) and a boundary layer \( B_s = B_{s1} \cup B_{s2} \cup B_{s3} \) of constant thickness \( c_0 \); (a) cell element distribution, i.e. (i) spherical cells \( V_e \) of radius \( c_0 \) within \( B \), (ii) linear cells \( L_e \) of length \( c_0 \) within \( B_{s1} \), (iii) plane cells \( A_e \) shaped as circular sectors of radius \( c_0 \) within \( B_{s2} \), (iv) solid cells \( V^c_e \) shaped as spherical sectors of radius \( c_0 \) at the corner points \( x_c \in B_{s3} \); (b) and (c) details of the edge line rod and of the corner point junctions.
and the companion (macro-scale) external virtual power as:

\[
L_{\text{ext}} := \int_{\Gamma_{1\text{B}}} \int_{V} b_{\text{i}}(x) \cdot \dot{v}(x) \, d\nu(x) \, d\nu(r) \\
+ \int_{\Gamma_{1}B} \int_{V} t_{1}(x) \cdot \dot{v}(x) \, d\nu(x) \, ds \\
+ \int_{\Gamma_{1}B} \int_{V} f_{e}(x + P(s) \cdot r) \cdot \dot{v}(x + P(s) \cdot r) \, d\nu(r) \, ds \\
+ \sum_{c=1}^{N} \int_{\Gamma_{c}} \int_{V} F_{c}(x + r) \cdot \dot{v}(x + r) \, d\nu(r) \\
\]

where \( \dot{v} \) denotes virtual velocity and \( \dot{\ell} = \nabla \dot{v} \) gives the related virtual strain rates. Also, the micro-scale stresses \( \sigma \) represent the actions at points within \( V_{c} \), from the surrounding material; analogously, the micro-scale surface tractions \( t \) represent the actions at points within every \( L_{c} \), from the external ambient. In a similar fashion can the analogous body forces \( b_{\text{i}} \), line tractions \( f \) and point forces \( F \) be interpreted. Moreover, I and \( s \) denote arc-lengths along the normal to \( \partial B \) and, respectively, the edge line \( \Gamma(B) \).

In view of \( c_{0} \) being relatively small, let the velocity \( v \) be expanded by a Taylor series up to a sufficient number of terms, such that we can replace the virtual kinematic variables of (14) and (15) in the following:

\[
\begin{align*}
\ddot{v}(x + r) &= \dddot{v}(x) + \dddot{v}(x) \cdot r + \dfrac{3}{2} \dddot{v}(x) \cdot r \cdot r + \nabla^{2} \dddot{v}(x), & \forall x \in B \\
\dot{v}(x + l) &= \dot{v}(x) + l \dot{v}(x) + \frac{1}{2} l^{2} \dddot{v}(x), & \forall x \in \partial B \\
\dot{v}(x + P(s) \cdot r) &= \dot{v}(x) + P(s) \cdot r \cdot \nabla_{(x)}(\dot{v}(x)), & \forall x \in \Gamma(B) \\
\dot{v}(x + r) &= \dot{v}(x), & (c = 1, 2, \ldots, N(B))
\end{align*}
\]

Here, \( \nabla_{(x)} \) denotes the line-transversal gradient over the edge line with unit tangent vector \( s \), that is, the tangential gradient over a plane orthogonal to the edge line. Hence, substituting (16), into (14), we can write:

\[
L_{\text{int}} = L_{\text{int}}(\dot{v}, B) := \int_{B} \left[ \sigma \cdot \ddot{v} + \sigma(1) \cdot \dddot{v}(1) + \sigma(2) \cdot \dddot{v}(2) \right] 
\]

where we have posed \( \ell \cdot \dddot{v} = \nabla \dddot{v}, \dddot{v} = \nabla \dddot{v} \), whereas the stresses \( \sigma \), \( \sigma(1) \) and \( \sigma(2) \) denote the generalized stresses defined in Section 2, Eq. (8).

Next, let us introduce the positions

\[
\begin{align*}
\mathbf{b}^{(0)}(x) &:= \frac{1}{2} \int_{\partial B} \mathbf{b} \cdot \dot{v}(x + r) \, d\nu(r) \\
\mathbf{b}^{(1)}(x) &:= \frac{1}{2} \int_{\partial B} \mathbf{b} \cdot \dot{v}(x + r) \, d\nu(r) \\
\mathbf{b}^{(2)}(x) &:= \frac{1}{2} \int_{\partial B} \mathbf{b} \cdot \dddot{v}(x + r) \, d\nu(r)
\end{align*}
\]

and

\[
\begin{align*}
\mathbf{b} &:= \mathbf{b}^{(0)} - \nabla \cdot \mathbf{b}^{(1)} + \nabla \cdot \mathbf{b}^{(2)} \\
\mathbf{t} &:= \mathbf{n} \cdot (\mathbf{b}^{(1)} - \nabla \cdot \mathbf{b}^{(2)}) - (\nabla_{(\partial B)} + \nabla_{\partial B}) \cdot (\mathbf{n} \cdot \mathbf{b}^{(2)}) \\
f_{e} &:= \nabla_{(\partial B)} \cdot \mathbf{b}^{(2)} \\
f_{b} &:= \nabla \cdot \mathbf{b}^{(2)}
\end{align*}
\]

where \( \nabla_{(\partial B)} \) means area measure and moreover we have set

\[
\begin{align*}
t(x) &:= \frac{1}{2} \int_{\partial B} \mathbf{t}(x + l) \, d\nu(x) \\
t^{(1)}(x) &:= \frac{1}{2} \int_{\partial B} \mathbf{t}(x + l) \, d\nu(x) \\
t^{(2)}(x) &:= \frac{1}{2} \int_{\partial B} \mathbf{t}^{(1)}(x + l) \, d\nu(x)
\end{align*}
\]

With the operations achieved so far (the details have been in most part skipped for brevity), the mean virtual power contributions from every cell are replaced by the macroscopic contributions given by some equivalent short and long range actions (as \( \sigma^{(1)}, \sigma^{(2)} \ldots \)) at the generic point \( x \) through the related work-conjugate continuum deformation modes (as \( x^{(1)}, x^{(2)} \ldots \)). In particular, the body forces within the cells are replaced, at the macroscopic level, by some equivalent external actions like body forces \( (b) \) within \( B \), surface tractions \( (t) \), \( (t^{(1)}) \) on \( \partial B \), line tractions \( (f) \), \( (f^{(1)}) \) on \( \Gamma(B) \), and point forces \( (F) \), as shown by (18), (19), (21)–(23). All this implies that, by the multi-cell homogenization process, the original cell system has been transformed into an equivalent structural system which happens to exhibit the same features of a second strain gradient material model and is characterized by the internal and external virtual powers of (17) and (20), respectively. The latter quantities coincide with the analogous quantities provided by Mindlin (1965) in all, except for the contributions from the edge line \( \Gamma(B) \).

To this concern, we remark that, as on the boundary surface \( \partial B \) the normal derivatives \( \partial_{n}v \) and \( \partial_{n}^{2}v \) are the independent parts of the gradients \( \nabla v \) and \( \nabla^{2}v \), likewise on the edge line \( \Gamma(B) \) (where \( \nabla v = \partial_{n}v + \nabla_{(\partial B)}v \)) the line-transversal gradient \( \nabla_{(\partial B)}v \) has to be regarded as the independent part of \( \nabla v \), for the tangential part \( \partial_{n}v = s \cdot \nabla v \) can be computed with the values of \( v \) over \( \Gamma(B) \). This is not mentioned in the Mindlin’s work (see the Appendix of the paper by Mindlin, 1965).

On equating (17)–(20) and considering the equivalent external actions of (20) as primitive external load parameters, we obtain the PVP cast in its (macro-scale) format typical of a second strain gradient material model. This operation will be achieved in next section, see Eq. (24).

We close this section stating that the inertia forces are presumed to be incorporated within the body forces and, as we shall see in the Part II paper, within the surface and the edge line forces as well.

### 4. The principle of the virtual power for quasi-static actions

In this section we consider a second strain gradient elastic material occupying a (three-dimensional) domain \( V \) of boundary surface \( S = \partial V \) and subjected to some specified quasi-static external actions. Our aim here is to write out – independently of the constitutive properties of the material – the equilibrium equations relating the (ordinary, double and triple) stresses to the external actions using for this purpose the principle of the virtual power (PVP). Following Fried and Gurtin (2006), as already done by Polizzotto (2012) we shall apply the latter principle not to the whole body, but rather to any portion of it, even infinitesimal, say \( B \subseteq V \). Also, we assume that the material possesses the same
constitutive features as the equivalent system gained at the end of the preceding section. Therefore, considering the generic subdomain \( B \) coinciding with the domain \( B \) of the preceding section, the above PVP can be considered coincident with the (macro-scale) PVP that, as announced at the end of the preceding section, can be obtained by equating (17)–(20), that is:

\[
L_{\text{int}}(\mathbf{v}, B) = L_{\text{ext}}(\mathbf{v}, B) \quad (24)
\]

This equality has to hold for any \( B \subseteq V \), as well as for any choice of the virtual kinematic variables satisfying the compatibility conditions

\[
\mathbf{e} = \mathbf{e}(\mathbf{v}) := \nabla \mathbf{v}, \quad \mathbf{e}(1) = \nabla \mathbf{v}(\mathbf{y}), \quad \mathbf{e}(2) = \nabla \nabla \mathbf{v}(\mathbf{y}) \quad \text{in } B \quad (25)
\]

The two sides of (24), coinciding with the right hand sides of (17) and (20), respectively, exhibit a functional dependence on \( B \) and on the virtual velocity field, \( \mathbf{v} \), there explicitly pointed out.

Eq. (24), together with the latter constraints, constitutes the (macro-scale) PVP as derived in the preceding section by means of the multi-cell homogenization procedure. It does exhibit the same format known from the literature (Mindlin, 1965; Germain, 1973) for a second strain gradient material model. We might have started directly from the latter format, but the adopted homogenization has transformed the various geometrical manifolds each as a condensed aggregate of cell elements, and in particular the original boundary layer, \( B \), finds itself condensed over the surface \( \partial B \).

4.1. A few explanations on the physical content of the PVP

Before going on with the mathematical exploitation of (24), it is appropriate to point out the precise physical meanings of the several ingredients appearing in (17) and (20). Namely, the generic particle within \( B \) possesses three intrinsic deformation modes described by the tensors \( \mathbf{e} = \{\mathbf{e}_{xy}\} \), \( \mathbf{e}(1) = \{\mathbf{e}_{xy}(1)\} := \{\partial_{x} \mathbf{e}_{xy}\} \) and \( \mathbf{e}(2) = \{\mathbf{e}_{xy}(2)\} := \{\partial_{y} \mathbf{e}_{xy}\} \), all symmetric with respect to the index pair \((i, j)\), but \( \mathbf{e}(i) \) also with respect to \((k, l)\), such that (as stated before), they possess \( 6 + 18 + 30 = 54 \) independent components in total, all being derivable from a sufficiently continuous velocity field \( \mathbf{v} \). The ordinary stress tensor \( \mathbf{\sigma} = \{\mathbf{\sigma}_{xy}\} \), the double stress tensor \( \mathbf{\sigma}(1) = \{\mathbf{\sigma}_{xy}(1)\} \) and triple stress tensor \( \mathbf{\sigma}(2) = \{\mathbf{\sigma}_{xy}(2)\} \) possess analogous symmetry features, hence the same number of independent components. These stresses represent the short and long range actions of the surrounding material upon the considered particle.

Within the surface integral on the right hand side of (20) we can distinguish the ordinary traction \( \mathbf{t} = \{t_{i}\} \) which works through \( \mathbf{v} \), the double (or dipole) traction \( \mathbf{t}(1) = \{t_{i}(1)\} \) with one lever arm in the normal direction and working through \( \partial_{i} \mathbf{v} \), and the triple (or quadrupole) traction \( \mathbf{t}(2) = \{t_{i}(2)\} \) with two lever arms both in the normal direction and working through \( \partial_{i}^{2} \mathbf{v} \). Both these three tractions are measured per unit area of surface \( \partial B \) on which they are applied. Analogously, within the line integral on the right hand side of (20) we can distinguish the ordinary line force \( \mathbf{f} = \{f_{j}\} \) working through \( \mathbf{v} \) and the double line force \( \mathbf{f}(1) = \{f_{i}(1)\} \) of direction \( j \) and one lever arm of direction \( k \), working through the line-transversal gradient \( \nabla_{(l)s} \mathbf{v} = \{\partial_{s} \mathbf{v}(l)\} \), both these two tractions being measured per unit length of the edge line, on which they act. The surface tractions on \( \partial B \), together with the line tractions on \( \Gamma B \) and the corner point forces, describe macroscopically the short and long range actions of the external ambient upon the body across, respectively, the boundary surface, the edge line and the corner points.

A paramount question about the higher order stresses previously introduced regards the physical interpretation of their indices. This interpretation can be readily established looking at the detailed expression of the stress power within \( L_{\text{int}}(\mathbf{v}, B) \) of (17), that is,

\[
\mathcal{W} := C_{g} \mathbf{g} + C_{g}^{(1)} \partial_{i} \mathbf{g} + C_{g}^{(2)} \partial_{i} \partial_{j} \mathbf{g}
\]

(26)

The point of departure is the accepted standard meaning of the index pair \((i, j)\) of the (symmetric) ordinary stress \( \mathbf{\sigma}_{ij} \), namely, \( j \) is the direction of the force, \( i \) as the direction normal to the plane on which the force acts (but, due to the symmetry, inverted roles for the latter indexes can also be accepted). This obviously has to hold also for the double and triple stresses, which in addition need further specifications to denote the lever arm(s) attached to them. This role has to be necessarily played by the extra indexes, either \( k \) for the \( \mathbf{\sigma}(k) \), stress components, or \((k, l)\) for the \( \mathbf{\sigma}(kl) \) stress components. Therefore, looking at the expression \( \mathbf{\sigma}(kl) \), we can say that it represents a double force of direction \( j \) acting on a plane of normal \( i \) and having a lever arm of direction \( k \). The same holds for the expression \( \mathbf{\sigma}(klm) \) but this time the force is a quadrupole force thus having two lever arms with directions \( k \) and \( l \), respectively.

The above rule, followed within the present work, can be changed somewhat, as explained in Appendix B. Namely, looking at the double stress \( \mathbf{\sigma}(kl) \) (or the triple stress \( \mathbf{\sigma}(klm) \)) the meanings of the indexes \( k \) and \( i \) can be interchanged, in the sense that the first index \( k \) represents the normal to the plane on which the force of direction \( j \) acts, the second (or third) index \( i \) the direction of the (or of one) lever arm. The latter rule, seemingly advanced by Mindlin (1964, 1965) (and for this reason referred to as the “Mindlin rule” in the following), has gained some popularity within the literature, see e.g. Lazar and Maugin (2005); Lazar et al. (2006), but here the previous one is followed. The convenience of one choice with respect to the other mainly depends on the adopted index contraction rules.

4.2. Invariance requisites

For the objectivity of any consequence derived from the PVP (24), it is required that \( L_{\text{int}}(\mathbf{v}, B) \) remain unchanged under change of observer, that is, after a transformation as in (10). As we have verified in Section 2, this invariance requisite is guaranteed if, and only if, the ordinary stress \( \mathbf{\sigma} \) is symmetric (and thus the related strain \( \epsilon \) can also be considered symmetric). This requisite is assumed satisfied in the following.

The equality (24) implies that \( L_{\text{int}}(\mathbf{v}, B) \) is also invariant. The consequences of this fact can be readily obtained by imposing \( \mathbf{v} = \mathbf{v}' \) and \( \mathbf{W}' = \mathbf{v}' - \mathbf{v}' \), where \( \mathbf{\phi}' = \mathbf{\phi}'(t) \) is an arbitrary rotation vector. With the latter choice, it is \( \partial_{i} \mathbf{v}' = \mathbf{n} \cdot \mathbf{W}' \), \( \partial_{i}^{2} \mathbf{v}' = \mathbf{0} \), and \( \nabla_{(s)} \mathbf{v}' = i \mathbf{W}' \). Then, the following two identities can be shown to hold:

\[
\mathbf{R}(\mathbf{B}) := \int_{B} \mathbf{b} d\mathbf{v} + \int_{\partial B} \mathbf{t} d\mathbf{s} + \int_{\Gamma B} \mathbf{f} d\mathbf{s} + \sum_{\Gamma} \mathbf{F}_{C} = \mathbf{0}
\]

(27)

and

\[
\mathbf{M}(\mathbf{B}) := \int_{B} \mathbf{b} \times d\mathbf{v} + \int_{\partial B} (\mathbf{t} \times \mathbf{n} + \mathbf{t}^{(1)}(1)) d\mathbf{s} + \int_{\Gamma B} (\mathbf{f}^{(1)}(1) \times \mathbf{n}) d\mathbf{s} + \sum_{\Gamma} \mathbf{X}_{C} \times \mathbf{F}_{C} = \mathbf{0}
\]

(28)

Here, \( \mathbf{n} \) denotes the unit vector normal to \( \partial B \). Also, the moments are taken with respect to the axes origin and \( \mathbf{f}^{(1)}(1) \) denotes the line-transversal component of \( \mathbf{f}^{(1)} \), namely, its component on a plane orthogonal to the edge line, i.e. \( \mathbf{f}^{(1)}(1) := \mathbf{n} \cdot \mathbf{f}^{(1)} \). Eqs. (27) and (28) are the linear and angular momentum balance equations of the body with (closed) domain \( B \). It is worth noting that the absence of the triple surface tractions \( \mathbf{t}^{(2)} \) from (28) is due to the fact that quadrupole forces are always self-equilibrated.
5. Mathematical consequences of the PVP

In this rather lengthy section the mathematical consequences of the PVP are derived. To obtain the latter consequences, let the space integrals $I_0$, $I_1$, $I_2$ shown by (17) be transformed by applying the standard divergence theorem where needed. We can then write, remembering (25):

$$I_2 = \int_B \sigma^{(2)} : \nabla \hat{\varepsilon} \, d\nu = - \int_B \nabla \cdot \sigma^{(2)} : \hat{\varepsilon} \, d\nu + \int_{\partial B} n \cdot \sigma^{(2)} : \hat{\varepsilon} \, d\alpha$$

Using the latter result, we can also write:

$$I_1 + I_2 = - \int_B \nabla \cdot (\sigma^{(1)} - \sigma^{(2)}) : \hat{\varepsilon} \, d\nu + \int_{\partial B} n \cdot \sigma^{(2)} : \hat{\varepsilon} \, d\alpha$$

With the decomposition $\nabla \hat{\varepsilon} = n \partial_\alpha \hat{\varepsilon} + \nabla_{\mid \alpha} \hat{\varepsilon}$, Eq. (30) can be rewritten as

$$I_1 + I_2 = - \int_B \nabla \cdot (\sigma^{(1)} - \sigma^{(2)}) : \hat{\varepsilon} \, d\nu + \int_{\partial B} n \cdot (\sigma^{(1)} - \sigma^{(2)}) : \hat{\varepsilon} \, d\alpha + \int_{\partial B} \nabla \cdot \sigma^{(2)} : \hat{\varepsilon} \, d\alpha$$

Next, by the surface divergence theorem, the integral $I_1$ here above can be transformed as

$$I_1 = - \int_{\partial B} \left( \nabla \cdot (\nabla_{\mid \alpha} \hat{\varepsilon}) + Hn \right) : \hat{\varepsilon} \, d\alpha + \int_{\partial B} \left[ n (n \cdot \sigma^{(2)}) \right] : \hat{\varepsilon} \, d\alpha$$

where $d\alpha$ denotes length measure on the edge line and $H := - \nabla_{\mid \alpha} \hat{\varepsilon}$ equals twice the mean curvature at the (regular) points of $\partial B$.

Furthermore, the symbol $[\sigma]_e$ denotes the jump of a quantity ($\sigma$) being discontinuous across $\Gamma(B)$ passing from one of the two surfaces mutually intersecting on $\Gamma(B)$. More precisely, let $s$ denote a unit vector tangential to $\Gamma(B)$ (e.g. according to the integration sense), and let $n_+, n_-$ denote the outward normals at points of $\partial B$ close to $\Gamma(B)$. The vector $n_+$ is attached to the “positive” side surface, i.e. the one such that the external normal agrees with the positive (i.e. anticlockwise) rotation sense around $s$, whereas $n_-$ is attached to the “negative” side surface, i.e. the other one. Also, we let $v_+ := s \times n_+$, $v_- := s \times n_-$, such that the unit vectors $v_+$ and $v_-$ are both normal to $\Gamma(B)$ and lie, respectively, on the tangent planes of $\partial B$ on the positive and negative side surfaces.

It thus follows that the integral $I_{\text{int}}(\nabla, B) = I_0 + I_1 + I_2$ can be rewritten as:

$$I_{\text{int}}(\nabla, B) = \int_B (\sigma - \nabla \cdot \sigma^{(1)} + \nabla \cdot \sigma^{(2)}) : \nabla \hat{\varepsilon} \, d\nu$$

$$+ \int_{\partial B} \left[ n (\sigma^{(1)} - \nabla \cdot \sigma^{(2)}) - \left( \nabla \cdot (\nabla_{\mid \alpha} \hat{\varepsilon}) + Hn \right) \right] : \nabla \hat{\varepsilon} \, d\alpha$$

$$+ \int_{\partial B} \nabla \cdot (\nabla_{\mid \alpha} \hat{\varepsilon}) \, d\alpha + \int_{\partial B} \nabla \cdot \sigma^{(2)} : \hat{\varepsilon} \, d\alpha$$

Next, with the positions:

$$\mathbf{T} := \sigma - \nabla \cdot \sigma^{(1)} + \nabla \cdot \sigma^{(2)} \quad \text{(Total Cauchy stress)}$$

we can write the equalities

$$J_1 := \int_B \mathbf{T} : \nabla \hat{\varepsilon} \, d\nu = - \int_B \nabla \cdot \mathbf{T} : \hat{\varepsilon} \, d\nu + \int_{\partial B} \mathbf{n} \cdot \sigma^{(2)} : \hat{\varepsilon} \, d\alpha$$

and, using the decomposition $\nabla \hat{\varepsilon} = \mathbf{n} \partial_\alpha \hat{\varepsilon} + \nabla_{\mid \alpha} \hat{\varepsilon}$,

$$J_2 := \int_{\partial B} \mathbf{n} \cdot \sigma^{(2)} : \hat{\varepsilon} \, d\alpha = \int_{\partial B} \mathbf{n} (\mathbf{n} \cdot \partial_\alpha \hat{\varepsilon}) + \int_{\partial B} \mathbf{n} \cdot \nabla_{\mid \alpha} \hat{\varepsilon} \, d\alpha$$

$$= \int_{\partial B} \mathbf{n} (\mathbf{n} \cdot \partial_\alpha \hat{\varepsilon}) + \int_{\partial B} \mathbf{n} \cdot (\nabla \cdot (\nabla_{\mid \alpha} \hat{\varepsilon}) + Hn) \cdot \hat{\varepsilon} \, d\alpha$$

Using Mindlin (1965), we can write:

$$\partial_\alpha \nabla \hat{\varepsilon} = \nabla (\partial_\alpha \hat{\varepsilon}) - \nabla (\nabla \cdot \partial_\alpha \hat{\varepsilon})$$

$$= \mathbf{n} \partial_\alpha \mathbf{n} \cdot \nabla_{\mid \alpha} \hat{\varepsilon} + \mathbf{K} \cdot \nabla_{\mid \alpha} \hat{\varepsilon}$$

where $\mathbf{K} := - \nabla_{\mid \alpha} \mathbf{n}$ is the (symmetric) Weingarten curvature tensor at points of $\partial B$, possessing the properties $H = K_{\alpha\beta}, \mathbf{n} \cdot \mathbf{K} = 0$ (Sokolnikoff, 1951)). Therefore, substituting from (38), the integral $J_3$ of (33) becomes:

$$J_3 = \int_{\partial B} \mathbf{n} (\mathbf{n} \cdot \sigma^{(2)} : \nabla \mathbf{n} \cdot \partial_\alpha \hat{\varepsilon}) \, d\alpha + \int_{\partial B} \mathbf{n} (\mathbf{n} \cdot \sigma^{(2)} : \nabla \cdot (\nabla_{\mid \alpha} \hat{\varepsilon}) + Hn) \, d\alpha$$

$$+ \int_{\partial B} \mathbf{K} \cdot (\mathbf{n} \cdot \sigma^{(2)} : \nabla_{\mid \alpha} \hat{\varepsilon}) \, d\alpha$$

By the surface divergence theorem, the integral $J_3$ of (39) can be expanded as follows:

$$J_3 = - \int_{\partial B} \left( \nabla \cdot (\nabla_{\mid \alpha} \hat{\varepsilon}) + Hn \right) : \nabla \cdot (\mathbf{n} \cdot \sigma^{(2)}) \cdot \partial_\alpha \hat{\varepsilon} \, d\alpha$$

$$+ \int_{\partial B} \left[ \mathbf{n} (\mathbf{n} \cdot \sigma^{(2)} : \partial_\alpha \hat{\varepsilon}) \right] \, d\alpha$$

Noting that $\partial_\alpha \mathbf{n} \cdot \mathbf{v} = \mathbf{n} \cdot \nabla \mathbf{v}$ and that at any point of the edge line a decomposition as $\nabla \mathbf{v} = s \mathbf{v} + \nabla_{\mid \alpha} \mathbf{v}$ holds true, and that therefore it is, everywhere on $\Gamma(B)$:

$$\left[ \mathbf{n} (\mathbf{n} \cdot \sigma^{(2)} : \partial_\alpha \hat{\varepsilon}) \right] = \left[ \mathbf{n} (\mathbf{n} \cdot \nabla \cdot \sigma^{(2)}) \right] \cdot \nabla \mathbf{v}$$

then the integral $J_3$ of (40) can be written as follows:

$$J_3 = - \int_{\partial B} \left( \nabla \cdot (\nabla_{\mid \alpha} \hat{\varepsilon}) + Hn \right) : \nabla \cdot (\mathbf{n} \cdot \sigma^{(2)}) \cdot \partial_\alpha \hat{\varepsilon} \, d\alpha$$

$$+ \int_{\partial B} \left[ \mathbf{n} (\mathbf{n} \cdot \sigma^{(2)} : \partial_\alpha \hat{\varepsilon}) \right] \cdot \nabla \mathbf{v} \, d\alpha$$

Again by the surface divergence theorem we can write the integral $J_3$ of (39) in the form:

$$J_3 = - \int_{\partial B} \left( \nabla \cdot (\nabla_{\mid \alpha} \hat{\varepsilon}) + Hn \right) : \mathbf{v} \, d\alpha$$

$$+ \int_{\partial B} \left[ \mathbf{v} \cdot \mathbf{K} \cdot (\mathbf{n} \cdot \sigma^{(2)}) \right] \cdot \nabla \mathbf{v} \, d\alpha$$

Moreover, with the decomposition $\nabla \mathbf{v} = s \partial_\alpha \mathbf{v} + \nabla_{\mid \alpha} \mathbf{v}$ (already used before), we can write the integral $J_4$ of (33) as:
for (42) and $j'_0$ of (43) into (39) and then adding the latter to the other obtained results, after some ordering, we can compute the integral $L_{\text{int}}(\Psi, B) = f_1 + f_2 + f_3 + f_4$ and cast it in the form:

$$L_{\text{int}}(\Psi, B) = -\iint_B \nabla \cdot T \; \Psi \, dv + \int_{\partial B} n \cdot T \; \Psi \, da$$

Next, let us introduce the stress tensors defined as:

$$\Sigma := S + K (n \cdot \sigma^2) \quad \text{(Surface stresses)}$$

$$\Sigma^{(1)} := nn \cdot \sigma^2 \quad \text{(Edge line stress)}$$

$$\Phi_c := \left( \sum_{c=1}^{N(B)} \lambda_c \Pi_c \right) \quad \text{at the corner points}$$

Using the latter notations and substituting (45) and (20) into (24), after some reordering we can obtain, as a consequence of the PVP under discussion, the following equality:

$$-\iint_B \left( \nabla \cdot T + b \right) \Psi \, dv - \iint_{\partial B} \left[ \nabla \cdot \left( \frac{\nabla}{\nabla + H n} \right) \Sigma + t - n \cdot T \right] \Psi \, da$$

$$-\iint_{\partial B} \left[ \nabla \cdot \left( \frac{\nabla}{\nabla + H n} \right) \Sigma^{(1)} + t^{(1)} - n \cdot \Sigma \right] \partial_n \Psi \, da$$

$$+ \iint_B \left( \nabla \cdot \sigma^2 \right) - ss \frac{n}{n} \cdot \sigma^2 + \left[ \nabla \cdot \left( \frac{n}{n} \cdot \sigma^2 \right) \right] \cdot \nabla \cdot \Psi \, dv$$

$$+ \sum_{c=1}^{N(B)} \left( \sum_{c} \lambda_c \left[ \nabla \cdot \sigma^2 \right] \right) \Psi_c$$

This is an identity holding for arbitrary choices of the virtual velocity fields, hence it implies that a set of equilibrium equations be satisfied. The latter equations are reported in next section.

6. Cauchy continuum enclosed within a membrane-like boundary layer

6.1. Equilibrium equations

The identity reported at the end of the previous section, there obtained as a direct consequence of the PVP, leads to the following equilibrium equations:

$$\nabla \cdot T + b = 0 \quad \text{in} \ B$$

$$\left( \nabla \cdot \left( \frac{\nabla}{\nabla + H n} \right) \Sigma + t - n \cdot T \right) = 0 \quad \text{on} \partial B$$

$$\nabla \cdot \Sigma^{(1)} - t^{(1)} = 0 \quad \text{on} \Gamma(B)$$

$$\Phi_c = F_c, \quad \forall c = 1, \ldots, N(B)$$

where the total stress $T$, the surface stresses $\Sigma$ and $\Sigma^{(1)}$, the edge line stress $\Sigma$ and the corner point tractions $\Phi_c$, $\forall c = 1, \ldots, N(B)$ have been defined in the preceding section.

The latter equations coincide with the analogous ones obtained by Mindlin (1965), but they are cast here in a form more suitable to point out their particular mathematical structure and physical implications. Namely, there are four groups of equations, of which (50) is related to the bulk material with degrees of freedom (DoFs) $v$; the equations of (51) are related to the boundary layer with DoFs $v$, $\partial_n v$ and $\partial_n^2 v$, respectively; the equations of (52) are related to the edge line $\Gamma(B)$ meant as a rod with DoFs $v$ and $\nabla \cdot v$, respectively; finally (53) is related to the corner points $x_c$ meant as junctions with DoFs $v_c$, $\forall c = 1, \ldots, N(B)$.

In particular, Eq. (50) is recognized as the classical equilibrium equations in which the total stress $T$ plays the role of Cauchy stress. Also, Eqs. (51), (52) can be interpreted with the aid of the principles of surface mechanics (Gurtin and Murdoch, 1975, 1978) as the equilibrium equations for a material surface subjected to surface stresses $\Sigma$ (dimension force per unit area) and $\Sigma^{(1)}$ (dimension moment per unit area) and to surface body force densities defined as:

$$b_1 := t - n \cdot T \quad b^{(1)}_1 := t^{(1)} - n \cdot \Sigma$$

The quantity $b_1$ (dimensionally, force per unit area) coincides with the so-called Gurtin–Murdoch traction, $b^{(1)}_1$, previously introduced (see also Polizzotto, 2012), but here a new notation is used to articulate its role as surface body force. An analogous role is played by $b^{(1)}_2$ (dimensionally, moment per unit area). Eq. (51) is a finite balance equation between internal and external triple surface tractions demanded by the DoFs $\partial_n^2 v$ of $\partial B$.

Eq. (52) (where $s - \Pi$ denotes the traction over a plane orthogonal to the edge line) can be interpreted as the equilibrium equation of a rod subjected to the stress $\Sigma$ and to a lineal body force density defined as:

$$b_L := f - [v \cdot \Sigma]$$

Eq. (53) collects a discrete set of equilibrium equations for the corner point junctions.
6.2. Physical consequences of the equilibrium equations

The above positions and interpretations enable us to state that a body made up by second strain gradient material manifests strong surface effects, which make it operate as a combination of two material subsystems, that is, the bulk material within $B$ with a stress state described by the total (Cauchy) stress $\boldsymbol{T}$, and the boundary layer $S_B$ (geometrically coinciding with $\partial B$) with a complex stress state described by the surface stresses $\Sigma$ and $\Sigma^{(1)}$, the edge line stress $\Pi$ on $\Gamma(B)$ and the corner point tractions $\Phi$. Additionally, we can state that the ordinary boundary traction $\mathbf{t}$ decomposes into two parts, i.e. $\mathbf{t} = \mathbf{t}_c + \mathbf{t}_{GM}$, of which $\mathbf{t}_c := n \cdot \mathbf{T}$ (called Cauchy traction) is sustained by the bulk material, whereas $\mathbf{t}_{GM} = \mathbf{b}$ (called Gurtin–Murdoch traction) is sustained by the boundary layer itself. The above complies with ideas by Forte and Vianello (1988) about the existence of surface stresses within strain gradient materials; it also substantiates a conjecture previously advanced by the author (Polizzotto, 2003) and generalizes similar findings for first strain gradient materials (Polizzotto, 2012).

For a further explanation, let us consider Fig. 2(a)–(c) giving a schematic picture of the manner in which the body works. For more graphical clarity, we let an arrow also represent either a dipole force or a quadrupole one according to whether it is marked with one or two crossing strokes, respectively. When the boundary layer $S_B$ is separated from the bulk material, the latter finds itself in equilibrium under the body forces $\mathbf{b}$ distributed within $B$, and the Cauchy traction $\mathbf{t}_c$ applied upon the skin of $B$, as shown in Fig. 2(b) through a Cauchy tetrahedron. Analogously, the boundary layer $S_B$ considered as an unique rigid object (of which a portion is shown in Fig. 2(c)) finds itself in equilibrium under the externally applied actions, i.e. the (a priori unknown) Gurtin–Murdoch traction $\mathbf{t}_{GM} = \mathbf{b}_s = \mathbf{t} - \mathbf{t}_c$ and the higher order tractions $\mathbf{t}^{(1)}$ and $\mathbf{t}^{(2)}$ distributed over $\partial B$, the line tractions $\mathbf{f}$ and $\mathbf{f}^{(1)}$ distributed over $\Gamma(B)$, and the point forces $\mathbf{F}_c$.

The above implies that some equations of global equilibrium must hold for each subsystem, just like (27) and (28), namely:

- For the (isolated) boundary layer $S_B$:

$$
\mathbf{R}(S_B) = \int_{\partial B} \mathbf{b}_s \, da + \int_{\Gamma(B)} \mathbf{f} \, ds + \sum_{c=1}^{N(B)} \mathbf{F}_c = \mathbf{0}
$$

$$
\mathbf{M}(S_B) = \int_{\partial B} (\mathbf{x} \times \mathbf{b}_s + \mathbf{n} \times \mathbf{t}^{(1)}) \, da + \int_{\Gamma(B)} (\mathbf{x} \times \mathbf{f} + \mathbf{c} : \mathbf{f}^{(1)}) \, ds + \sum_{c=1}^{N(B)} \mathbf{x}_c \times \mathbf{F}_c = \mathbf{0}
$$

- For the (isolated) bulk material within $B$:

$$
\mathbf{R}(B) = \int_B \mathbf{b} \, dv + \int_{\partial B} \mathbf{t}_c \, da = \mathbf{0}
$$

$$
\mathbf{M}(B) = \int_B \mathbf{x} \times \mathbf{b} \, dv + \int_{\partial B} \mathbf{x} \times \mathbf{t}_c \, da = \mathbf{0}
$$

Eqs. (58) and (59) are of particular relevance since, from the whole set of external actions, they retain only the body forces $\mathbf{b}$ within $B$ and the Cauchy tractions $\mathbf{t}_c$ on $\partial B$ as the only external actions pertaining to the bulk material considered isolated from the boundary layer, indeed in the same way as for a classical Cauchy continuum. The latter equations can be considered as a consequence of (27), (28), (56) and (57). Balance equations similar to (56) and (57) were provided by Wu (1992).

Although Eqs. (56) and (57) may be considered self-evident (for any part of an equilibrated system has to be equilibrated too), a direct proof of the latter equations will be given in Appendix A. Here we want to point out the role played by the various stresses previously introduced for the equilibrium of a small element of boundary layer considered isolated as the one of Fig. 2(c). This is accomplished in next subsection.

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**Fig. 2.** Geometrical sketch representing: (a) The combination of the Cauchy continuum and the boundary layer; (b) A Cauchy tetrahedron close to the boundary surface; (c) An isolated element of boundary layer subjected to the relevant applied actions.
6.3. The boundary layer and the inherent microstructure

In Fig. 3(a)–(c) a small element of boundary layer is decomposed into three idealized surface microstructures, say $S_{b_1}$, $S_{b_2}$, and $S_{b_3}$, respectively associated with the DoFs attached to $\partial B$, i.e. $v$, $\partial_\nu v$, $\partial_{\nu n} v$. Every surface microstructure is a few inter-particle spaces thick, say $2h$. The three surface microstructures can be distinguished as follows:

(i) First level surface microstructure ($S_{b_1}$), endowed with DoFs $v$ (velocity) and subjected to the surface stress $\Sigma$, which is in equilibrium under ordinary forces, Fig. 3(a). The traction $t$ combines with the negative Cauchy traction, $-n \cdot T$ (coming from the bulk material), to give $t_{\text{ext}} = b_0 = t - n \cdot T$ supported by the boundary layer itself as a surface body force. The surface stress $\Sigma$ enters into the (two-dimensional) field equilibrium equations of $S_{b_1}$ and interacts with $S_{b_1}$ by the traction $n \cdot \Sigma$.

(ii) Second level surface microstructure ($S_{b_2}$), endowed with DoFs $\partial_\nu v$ and subjected to the surface stress $\Sigma_1$, which is in equilibrium under double forces with lever arm in the normal direction, Fig. 3(b). The double traction $t^{(1)}$ decomposes into a part equal to $n \cdot \Sigma$ (sustained by $S_{b_2}$) and a part $b_1 = t^{(1)} - n \cdot \Sigma$ supported by $S_{b_2}$ as a surface body force. The surface stress $\Sigma_1$ enters into the (two-dimensional) field equilibrium equations of $S_{b_2}$ and interacts with $S_{b_2}$ by the traction $n \cdot \Sigma_1$.

(iii) Third level surface microstructure ($S_{b_3}$), which is endowed with DoFs $\partial_{\nu n} v$ and is in equilibrium under quadrupole forces with two lever arms, both in the normal direction, Fig. 3(c). The triple traction $t^{(2)}$ equilibrates $-n \cdot \Sigma^{(2)}$, the action of $S_{b_3}$ on $S_{b_1}$.

The above indicates that the concept of boundary surface enclosing a Cauchy continuum, encountered within classical continuum mechanics, has to be replaced – within the context of higher-grade materials – with the more complex concept of boundary layer, which nevertheless also encloses a Cauchy continuum. In this context, we can observe that the external actions over the boundary surface of a second strain gradient material in general consist of ordinary, double and triple actions. In order to investigate upon the effectiveness of the latter kind of tractions as boundary actions, let the boundary layer $S_b$ be analyzed at the microscale through the discrete model sketched in Fig. 4(a)–(d). This model consists of a distribution of particle triplets, each of which is displayed in the normal direction $n$ to $S_b$. The totality of these particles constitutes a set of three discrete layers of global thickness equal to a few inter-particle spaces, say $2h$.

At this point, it is paramount to remark that the boundary layer $S_b$ does not have a specific constitutive equation of its own, since no energy is allowed to be stored within it. (Note: The surface energy envisioned by Polizzotto (2003) is not a strain power, since it equals, with the present notation, to the quantity $t_{\text{ext}} \cdot u$.) The boundary layer has to be considered as infinitely rigid against any change of configuration that would not preserve the displacement continuity between $S_b$ and the adjacent bulk material, but infinitely soft against any configuration change that instead would preserve this continuity. This implies that the deformation of $S_b$ is just guided by the bulk material and that therefore – in contrast to conventional discrete models of the literature – no spring like ligaments have to be inserted between two near particles. These in fact move with velocities dictated by the bulk material, say $v_0$, $v_1$, $v_2$ for the outer, middle and third particles of a same particle triplet.

The ordinary traction $t$ has as discrete counterpart an ordinary force per unit area, say $F$, which can be considered as applied on the outer particle of a particle triplet, as shown in Fig. 4(b). The discrete power per unit area correspondingly produced is $P = v_0 - v_1$, which becomes equal to $t \cdot v$, the related continuum power density, at the limit for $h \to 0$, $F \to t$ and $v_0 \to v$.

Analogously, the double traction $t^{(1)}$ has as discrete counterpart two ordinary equal, but opposite, forces per unit area, say $F$ and $-F$, which can be considered as applied, respectively, at the outer and middle particles of a same particle triplet, Fig. 4(c). The discrete power per unit area correspondingly accomplished is

$$F \cdot (v_0 - v_1) = (hF) \cdot \left( \frac{v_0 - v_1}{h} \right) = t^{(1)} \cdot \partial_\nu v$$

where the limit value on the right hand side is obtained for $h \to 0$ and $hF \to t^{(1)}$. The latter limit $t^{(1)} = \{t^{(1)}_j\}$ specifies a double (or dipole) traction with the same direction of $F$ and one lever arm in the direction of the normal $n$, and works through the velocity normal derivative $\partial_\nu v$, equal to the analogous limit of $(v_0 - v_1)/h$. (See Appendix B for the definitions of double and triple forces.)

Likewise, the triple traction $t^{(2)}$ has as discrete counterpart four equal ordinary forces per unit area, say $F$, of which two can be considered as applied, respectively, at the outer and third particles of a same particle triplet, whereas the other two forces, but taken with opposite sense, are both applied on the middle particle, as shown in Fig. 4(d). (Note: the relevant parallelogram of a quadrupole force here degenerates into a linear segment of length $2h$, see Appendix B). The resultant discrete power per unit area is
F' \cdot (v_0 + v_2 - 2v_1) = (h^2 F') \cdot \left(\frac{v_0 + v_2 - 2v_1}{h^2}\right) \to \mathbf{t}^{(2)} \cdot \partial_{nn}^2 \mathbf{v} \quad (61)

where the limit on the right hand side is obtained for \( h \to 0 \) and \( h^2 F' \to \mathbf{t}^{(2)} \). The latter limit \( \mathbf{t}^{(2)} = \{ \mathbf{t}_n^{(2)} \} \) specifies a triple (or quadrupole) traction with the same direction of \( \mathbf{F}' \) and two lever arms both in the direction of the normal \( \mathbf{n} \), and works through the velocity second normal derivative \( \partial_{nn}^2 \mathbf{v} \), coinciding with the analogous limit of \( (v_0 + v_2 - 2v_1)/h^2 \).

The above picture substantiates the ways in which the boundary tractions do work for the formation of the external power in (20). Analogous considerations may be developed for the edge line rod and the corresponding contribution in (20), but for space reasons this point is not further pursued here. All this makes it evident that, within the mechanics of strain gradient materials, the traditional concept of a purely geometrical boundary surface \( S = \partial B \) has to be replaced with that of boundary layer.

Remark 1. It is worth noting that the boundary layer does not constitute a true material surface in the sense of Curtin and Murdoch (1975, 1978) as no specific constitutive equation is assigned to it (whereas the inherent inertia forces are incorporated into the surface body forces). It has also to be noted that, whereas the latter quoted authors consider only ordinary forces and disregard the surface curvature effects on the surface equilibrium equations, here instead we take into account the latter effects together with the higher order stresses and tractions.

7. Particularization to first strain gradient models

It is worth reporting here the simplifications induced with the passage from the second strain gradient model addressed in the preceding sections to the first strain gradient one previously treated by Polizzotto (2012). For this purpose, we have just to set \( \kappa_2 = 0 \) and disregard all terms with a factor of the form \((\cdot)^{(2)}\), as well as the double line traction \( \mathbf{r}^{(1)} \) and the corner point forces \( \mathbf{F}_c \). This leads to the vanishing of \( \sigma^{(2)} \), \( \Sigma^{(1)} \), \( \Pi \), \( \Phi_\kappa \), whereas, in accord to (Polizzotto, 2012), \( \mathbf{T} \) and \( \Sigma \) become

\[
\mathbf{T} = \mathbf{\sigma} - \nabla \cdot \mathbf{\sigma}^{(1)}, \quad \Sigma = \mathbf{n} \cdot \mathbf{\sigma}^{(1)} \quad (62)
\]

and the equilibrium Eqs. (50)–(53) reduce to

\[
\begin{align*}
\nabla \cdot \mathbf{T} + \mathbf{b} &= \mathbf{0} \quad \text{in } B \\
(\nabla (\mathbf{\Sigma} + \mathbf{H} \mathbf{n}) \cdot \mathbf{n}) + \mathbf{t} - \mathbf{n} \cdot \mathbf{T} &= \mathbf{0} \quad \text{on } \partial B
\end{align*}
\]

(63)

Also, Eqs. (27) and (28) transform into

\[
\begin{align*}
\mathbf{R}(\mathcal{B}) &= \int_B \mathbf{b} \cdot d\mathbf{v} + \int_{\partial B} \mathbf{t} \cdot d\mathbf{a} + \int_{\Gamma_B} \mathbf{f} \cdot ds = \mathbf{0} \\
\mathbf{M}(\mathcal{B}) &= \int_B \mathbf{x} \times \mathbf{b} \cdot d\mathbf{v} + \int_{\partial B} (\mathbf{x} \times \mathbf{t} + \mathbf{n} \times \mathbf{t}^{(1)}) \cdot d\mathbf{a} + \int_{\Gamma_B} \mathbf{x} \times \mathbf{f} \cdot ds = \mathbf{0}
\end{align*}
\]

(64)

whereas Eqs. (56) and (57) can be written

\[
\begin{align*}
\mathbf{R}(\mathcal{S}_B) &= \int_{\partial \mathcal{S}_B} \mathbf{b} \cdot d\mathbf{a} + \int_{\Gamma_B} \mathbf{f} \cdot ds = \mathbf{0} \\
\mathbf{M}(\mathcal{S}_B) &= \int_{\partial \mathcal{S}_B} (\mathbf{x} \times \mathbf{b} + \mathbf{n} \times \mathbf{t}^{(1)}) \cdot d\mathbf{a} + \int_{\Gamma_B} \mathbf{x} \times \mathbf{f} \cdot ds = \mathbf{0}
\end{align*}
\]

(65)

but instead Eqs. (58) and (59) remain unchanged.

Remark 2. On comparing (64) with Eqs. (27) and (28) of the quoted paper (Polizzotto, 2012) (where the double traction \( \mathbf{t}^{(1)} \) is denoted with the symbol \( \mathbf{m} \)), we can observe that the compared expressions coincide in all, except for the corner point forces \( \mathbf{F}_c \). These in fact are correctly omitted by (64), but instead appear in the counterpart equations given by Polizzotto (2012). The same holds for (65) as compared with Eqs. (47) and (48) of the same quoted paper. The reason why the corner point forces \( \mathbf{F}_c \) were introduced by Polizzotto (2012) stems from the author’s tacit will.
to show that corner point forces, if any, must be included in the set of external actions under which the boundary layer $S_b$ finds itself in (global) equilibrium. However, for a first strain gradient material model, these corner forces may exist only in the case of stress singularities in the vicinity of the corner points. Therefore, as far as the latter circumstance has to be excluded, all the terms related to the corner point forces should be omitted throughout the quoted paper (Polizzotto, 2012).

8. Boundary-value problem and related variational principle

Let us consider a body of volume $V$ and boundary surface $S = \partial V$, with an edge line $\Gamma$ and corner points $x_c, (c = 1, 2, \ldots, N)$, and let it be subjected to body forces $f$ in $V$, to surface tractions $(\mathbf{t}, \mathbf{i}^{(1)}, \mathbf{i}^{(2)})$ on $S_t \subset S$, to line tractions $f f^{(1)}$ on $\Gamma$ and to point forces $F_{xc}, (c = 1, 2, \ldots, N)$ on the corner points, all forces acting in a quasi-static manner. The body is restrained at the points of $S_b = S \setminus S_t$ by the constraints

$$\mathbf{u} = \mathbf{u}_0, \quad \partial_n \mathbf{u} = \mathbf{g}, \quad \partial_{n_m} \mathbf{u} = \mathbf{h} \quad \text{on} \quad S_b$$

(66)

The material is a second strain gradient material like the one described in the preceding sections, it thus obeys the constitutive Eqs. (12), (13) and (34), which are equivalent to:

$$\mathbf{T} = \mathbf{C} - \mathbf{C}_0 \Delta \mathbf{u} + \mathbf{C}_1 \Delta \mathbf{u}, \quad \mathbf{C} = \mathbf{C} : \varepsilon$$

(67)

where $\mathbf{C}$ is given by (12). The material also obeys the equilibrium Eqs. (50)-(53), (but $B = V, \partial B = S_t, \mathbf{\Gamma}(B) - \mathbf{\Gamma}, \mathbf{N}(B) = N$), as well as the compatibility condition $\varepsilon = \nabla \cdot \mathbf{u}$ in $V$ and the displacement constraints (66).

After some straightforward substitutions and transformations, the totality of field equations can be cast in the following form:

$$\mathcal{L}(\mathbf{u} - \mathbf{r}_1 \mathbf{u} + \mathbf{r}_2 \mathbf{u})^2 = - \mathbf{b} \quad \text{in} \quad V$$

(68)

where $\mathcal{L}$ denotes the differential operator of classical elasticity theory, namely

$$\mathcal{L}\mathbf{u} = \mu \Delta \mathbf{u} + (\lambda + \mu) \nabla \cdot \mathbf{u}$$

(69)

$\mathbf{U}$ being any vector field (Sokolnikoff, 1956). The solution to the latter set of differential equations with the accompanying boundary conditions, if exists, is unique and is characterized by a minimum condition of the related total potential energy, say $\Omega(u)$. This is defined as follows:

$$\Omega(u) := \int_V \Psi(\mathbf{u}, \varepsilon, \nabla \varepsilon, \nabla \varepsilon) \mathrm{d} V - \int_V \mathbf{b} : \mathbf{u} \mathrm{d} V - \int_{S_t} (\mathbf{t} : \mathbf{u} + \mathbf{i}^{(1)}, \partial_n \mathbf{u} + \mathbf{i}^{(2)} ; \partial_{n_m} \mathbf{u}) \mathrm{d} \alpha$$

$$- \int_{\Gamma} (\mathbf{f} : \mathbf{u} + \mathbf{f}^{(1)} ; \nabla_{(\mathbf{u})} \mathbf{u}) \mathrm{d} s - \int_{\Gamma} \{ \mathbf{f} : \mathbf{u} + \mathbf{f}^{(1)} ; \nabla_{(\mathbf{u})} \mathbf{u} \} \mathrm{d} s$$

(70)

The validity of the latter minimum principle can be easily assessed by the exploitation of the developments and notations of Sections 5 and 6. For this purpose, let $u$ denote the a solution of the boundary-value problem outlined above, and let $\mathbf{u}$ denote any virtual variation of configuration from the one associated to $u$, satisfying the conditions $\mathbf{u} = \partial_n \mathbf{u} = \partial_{n_m} \mathbf{u} = 0$ on $S_b$, such that $\mathbf{u} + \mathbf{u}$ complies with the constraints of (66).

Then, expanding $\Omega(u + \mathbf{u})$ by a Taylor series, we can write:

$$\Omega(u + \mathbf{u}) = \Omega(u) + \delta \Omega(u) + \frac{1}{2} \delta^2 \Omega(u)$$

(71)

where $\delta \Omega(u)$ is the first variation of $\Omega(u)$ which remembering (17) and (20), can be written as

$$\delta \Omega(u) = -L_{ext}(\mathbf{u}, V) - L_{ext}(\mathbf{u}, V)$$

(72)

where $\delta \Omega(u)$ is the second variation of $\Omega(u)$, i.e.

$$\delta^2 \Omega(u) = \int_V \delta^2 \psi \mathrm{d} V \geq 0$$

(73)

Due to the positive definiteness of $\psi$ of (11), $\delta^2 \Omega(u)$ proves to be positive for any not identically vanishing $u$, but null if, and only if, $\mathbf{u} = 0$. Therefore, if the reference field $\mathbf{u}$ minimizes $\Omega(u)$, then $\delta \Omega(u) = 0$ and thus the PVP is satisfied, which implies that all the field and boundary equilibrium equations are satisfied correspondingly. Conversely, if the reference field $\mathbf{u}$ solves the governing equations and thus $\delta \Omega(u) = 0$, we have:

$$\Omega(u + \mathbf{u}) = \Omega(u) + \frac{1}{2} \delta^2 \Omega(u) \geq \Omega(u) \quad \forall \mathbf{u}$$

(74)

where the equality sign on the right side holds if, and only if, $\mathbf{u} = 0$, which means that $u$ minimizes $\Omega$ and constitutes the unique solution of the problem. The theorem is thus proved. QED

The latter theorem generalizes analogous theorems given by Mindlin (1965) and Polizzotto (2003) to solids whose boundary surface exhibits edge lines and corner points.

9. Application to a bar subjected to end tractions

Toupin et al. (1963) studied the so-called “puckering” effects occurring close to the boundary surface of a crystal, whereby the deformation pattern is the superposition of two distinct contributions; one reflects the overall deformation and is caused by the ordinary external traction, the other is confined to a zone close to the boundary surface and is caused by double and triple external tractions. For this purpose, the latter authors used mono- and di-atomic lattice models, that is, one (or two) string(s) of atoms with the nearest and next to nearest particle interactions. Here we consider a continuum equivalent of the latter discrete problem. For this purpose, let us consider a bar $0 < x < L$ of second strain gradient elastic material like the one studied in the preceding sections. The solving equilibrium equation in terms of displacement $u(x)$ is readily derived from (68), that is

$$u^2 - u_T^2 = 0$$

(75)

This admits the general solution

$$u(x) = N_1 e^{x/c_1} + N_2 e^{-x/c_1} + N_1 e^{x/c_2} + N_4 e^{-x/c_2} + N_1 x + B$$

(76)

where we have $c_1^2 = c_1^2 + c_2^2$ and $c_2^2 = c_1^2 + c_2^2$ and $N_1, N_2, N_3, N_4, A, B$ are constants to be evaluated by the boundary conditions. (For $c_1 = c_2$ the general solution changes somewhat formally, but we skip this point for brevity.) By (51) and noting that only the stress components $\sigma_{xx} = \sigma, \Sigma_{xx} = \Sigma$ and $\Sigma_{x_{yy}} = \Sigma_{11}^1$ are meaningfull here, the latter boundary conditions read:

$$u = u^0 = u_T = 0 \quad \text{at} \quad x = 0 \quad \Sigma = \Sigma^{(1)}, \quad \Sigma^{(1)} = \Sigma^{(2)} \quad \text{at} \quad x = L$$

(77)

Note that $H = 0$ and $\nabla \phi = 0$ in the present case, whereas $\mathbf{S} = \mathbf{S}_u, \mathbf{S}^{(1)} = \mathbf{S}_{uu}^T, \mathbf{S}^{(1)} = \mathbf{S}_{uu}^T$, where $E =$ Young modulus. Skipping the (straightforward) computational details for brevity, the displacement response can be cast in the form

$$u(x) = \frac{1}{E} \left[ u_0(x) + \mu_1 u_1(x) + \mu_2 u_2(x) \right]$$

(78)

where $\mu_1 := \Sigma^{(1)}/(IL), \mu_2 := \Sigma^{(2)}/(IL^2)$. In the above, $u_0(x), u_1(x)$ and $u_2(x)$ denote the displacement profiles caused by unit external tractions, that is, $\mathbf{f}/E = 1, \mathbf{f}^{(1)}/E = 1, \mathbf{f}^{(2)}/E = 1,$ and $\mathbf{f}^{(2)}/(EL) = 1, \mathbf{f} = \mathbf{f}^{(2)} = 0$ for $u_0(x); \mathbf{f}^{(1)}/(EL) = 1, \mathbf{f} = \mathbf{f}^{(2)} = 0$ for $u_1(x); \mathbf{f}^{(2)}/(EL^2) = 1, \mathbf{f} = \mathbf{f}^{(2)} = 0$ for $u_2(x).$ The latter displacement profiles are plotted in Fig. 5 by the dashed $(u_0)$, dot-dashed $(u_1)$ and dotted $(u_2)$ lines. It can be seen that, whereas $u_0$ (caused by the ordinary traction) is distributed all over...
the bar, instead \( u_1 \) and \( u_3 \) (caused by the double and triple tractions, respectively) prove to be meaningful only within a portion of the bar close to the free end. When the three types of external tractions act altogether, with say \( E_1/E = 1, \mu_1 = 2.5, \mu_2 = 0.25 \), the displacement response \( u(x) \), given by (78), is as in the solid line plot of Fig. 5. Indeed, the puckering effect proves to be smeared throughout a wider region near the free end, meaning that the bulk material and the boundary layer do co-operate with each other in determining the actual response to the loads.

10. Summary and conclusions

In this Part I paper, a class of second strain gradient elastic materials has been addressed and the inherent surface effects have been assessed. The results herein achieved are summarized as in the following

- A homogenization procedure has been applied to an (unbounded) distribution of geometrically equal cell elements in combination with the internal energy balance principle, or the parallel inertial energy balance principle. By this, the constitutive forms of the free energy and, respectively, of the kinetic energy have been obtained for the inherent equivalent continuum, which exhibits the features of a second strain gradient elastic material with second velocity gradient inertia. The constitutive equations of a related set of generalized (ordinary, double and triple) stresses and momenta have been obtained in this way.

- A wider multi-cell homogenization procedure with four geometrically different types of cell elements has been used in combination with the principle of the virtual power (PVP) enforced for the cell system. This made it possible taking into account macroscopically the surface effects, i.e. the effects produced by the presence of a boundary surface (with its singularities as edge lines and corner points) on the behavior of the near material particles, which in fact coalesce to form up a membrane-like boundary layer with specific mechanical (and inertial) characteristics. In particular, the above PVP has transformed itself into a macro-scale one having the typical format as for a second strain gradient material model, known from the literature (Mindlin, 1965; Germain, 1973).

- The exploitation of the latter (macro-scale) PVP has led to the well-known field and boundary equilibrium equations of second strain gradient solids with edge line and corner points (Mindlin, 1965; Germain, 1973; Gurtin, 2001). A suitable re-interpretation of the latter equations has permitted us to specify the (mechanical) surface effects as in the following:

1. A second strain gradient elastic material can be conceived as the combination of two subsystems, that is, the bulk material operating as a Cauchy continuum, and the boundary surface operating as a membrane-like boundary layer, each subsystem being in global and local equilibrium.

2. The (ordinary) boundary traction decomposes (in unknown proportions) into the sum of two parts, i.e. the Cauchy traction supported by the bulk material, and the Gurtin–Murdoch traction which, together with all other external boundary actions, is supported by the boundary layer.

3. The boundary layer incorporates three levels of mutually interacting surface microstructures. Local equilibrium equations hold for these surface microstructures, which obey the principles of surface mechanics (Gurtin and Murdoch, 1975, 1978) and in addition take into account the surface curvature effects. The boundary layer constitutes a structured two-dimensional manifold that substitutes the classical purely geometrical concept of boundary surface.

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Appendix A. Global equilibrium equations of the boundary layer

In this Appendix, the global equilibrium Eqs. (56) and (57) are proved. This task can be easily fulfilled by considering the internal and external virtual powers (17) and (20) and by defining the following quantities:

\[
L_{\text{tot}}(\mathbf{\hat{v}}, \mathbf{B}) := L_{\text{int}}(\mathbf{\hat{v}}, \mathbf{B}) - \int_B \mathbf{b} \cdot \mathbf{\hat{v}} \, d\mathbf{v} - \int_{\partial B} \mathbf{t}_c \cdot \mathbf{\hat{v}} \, d\mathbf{a} \quad (A.1)
\]

\[
L_{\text{tot}}(\mathbf{\hat{v}}, \mathbf{S}_B) := \int_{\partial B} \left( \mathbf{b}_S \cdot \mathbf{\hat{v}} \right) + t^{(1)} \cdot \partial_n \mathbf{\hat{v}} + t^{(2)} \cdot \partial_{nn} \mathbf{\hat{v}} \right) \, d\mathbf{s} + \sum_{c=1}^{N_{\text{bc}}} F_c \cdot \mathbf{\hat{v}}_c \quad (A.2)
\]

Here above, \( \mathbf{\hat{v}} \) denotes virtual velocities over \( \partial B \) and the equality \( \mathbf{t} = \mathbf{t}_c + \mathbf{b}_S \) has been used. Then, the PVP (24) can be enforced by the equality

\[
L_{\text{tot}}(\mathbf{\hat{v}}, \mathbf{B}) = L_{\text{tot}}(\mathbf{\hat{v}}, \mathbf{S}_B) \quad (A.3)
\]

which has to be satisfied identically for any choice of \( B \), as well as of the fields \( \mathbf{\hat{v}} \) and \( \mathbf{\hat{v}} \) complying with the following compatibility conditions:

\[
\varepsilon = \nabla \mathbf{\hat{v}}, \quad \varepsilon^{(1)} = \nabla \varepsilon(\mathbf{\hat{v}}), \quad \varepsilon^{(2)} = \nabla \nabla \varepsilon(\mathbf{\hat{v}}) \quad \text{in } B \quad (A.4)
\]

\[
\mathbf{\hat{v}} = \mathbf{\hat{v}} \quad \text{on } \partial B \quad (A.5)
\]

If the variational procedure is fully pursued, at the end the same local equilibrium Eqs. (50)–(53) would be obtained. Here instead we
are interested in the invariance conditions for (A.3), enforced separately for the two members. For this purpose, let us relax (A.5) and choose $\mathbf{v}$ and $\mathbf{v}'$ as follows:

$$
\begin{align*}
\mathbf{v}(x, t) &= \mathbf{c}'(t) + \mathbf{x} \times \mathbf{\phi}'(t) \ \forall \mathbf{x} \in B \\
\mathbf{v}(x, t) &= \mathbf{c}'(t) + \mathbf{x} \times \mathbf{\phi}'(t) \ \forall \mathbf{x} \in \partial B
\end{align*}
$$

(A.6)

where the simply and doubly starred vectors denote free independent rigid-body translations and rotations of $B$ and $\Sigma$, respectively. Then, skipping the mathematical details for brevity, by (A.6), we can obtain Eqs. (58) and (59), whereas by (A.6) we get the equilibrium equations (56) and (57). QED

Appendix B. Double and triple stresses and related issues

Double and triple stresses are, together with the related higher order strains, indispensable analytical tools necessary to describe the intrinsic mechanisms of higher order strain gradient materials. Although explanations on this issue can be found in the literature, (see e.g. Mindlin, 1964, 1965; Love, 1927; Jaunzemis, 1967; Lazar et al., 2006; Gronwald and Hehl, 1993), we believe that some further explanations, particularly regarding the extra indexes indicating the direction of the lever arm(s) of double and triple stresses would be most useful.

B.1. Higher order deformation states

Within a continuum being in a state of deformation, a distortion component, say $h_{ij} := \partial u_i / \partial x_j$, can be thought of to specify a homogeneous deformation state of a material element, or particle, whereby the linear fibers of direction $i$ undergo relative displacements in direction $j$ (consisting in a rotation $\partial u_i / \partial x_j$ if $i \neq j$, but a constant dilatation $\Phi$ (no sum on $i$) if $i = j$). Similarly, a first distortion gradient component, say $h_{i0j}^{(1)} := \partial^2 u_i / \partial x_j \partial x_k$, specifies a linearly varying deformation state of the particle, whereby the extra index $k$ indicates the gradient direction, i.e. the direction along which the variation occurs. However, due to the symmetry of the index pair $(k, l)$, the roles of the latter indices are interchangeable. This implies that the distortion gradients $h_{i0j}^{(1)}$ and $h_{i0j}^{(1)}$ are equal to each other, i.e. $h_{i0j}^{(1)} = h_{i0j}^{(1)}$. Physically, this means that the distortion gradient components $h_{i0j}^{(1)}$ and $h_{i0j}^{(1)}$ specify a same deformation state of the material, which can thus be considered as the result of either a distortion $h_{i0j}$ linearly variable in the direction $k$, or a distortion $h_{i0j}$ linearly variable in the direction $i$. For instance, in Fig. B.1(a) and (b) a deformation state described by $h_{i0j}^{(2)} = h_{i0j}^{(1)}$ is considered as the result of: in Fig. B.1(a), distortion $h_{i0j}^{(2)}$ linearly variable with $x_k$; in Fig. B.1(b), a strain $h_{i0j}^{(2)}$ linearly variable with $x_i$.

Analogously, a second distortion gradient component, say $h_{i0j}^{(2)} = \partial^2 u_i / \partial x_j \partial x_k$, specifies a bi-linearly varying deformation state of the particle, whereby the extra index pair $(k, l)$ denotes the gradient direction, i.e. the two directions along which the deformation variations occur (but the variation will be quadratic if $k = i$). However, for $h_{i0j}^{(1)}$, the index $i$ can be interchanged, as for its specific role, either with $k$, or with $l$. This means that the three distortion gradient components $h_{i0j}^{(2)}$, $h_{i0j}^{(2)}$, and $h_{i0j}^{(2)}$ specify a same deformation state.

The first and second strain gradients are obtained combining the first and second distortion gradients, respectively. For instance, $h_{i0j}^{(1)} = (h_{i0j}^{(2)} + h_{i0j}^{(2)})/2$ can be obtained combining $h_{i0j}^{(2)}$ and $h_{i0j}^{(2)}$, as shown in Fig. B.2(a)-(c).

B.2. Dipole and quadrupole forces and higher order stress states

A double force can be obtained by considering two equal, but opposite, ordinary forces, say $\mathbf{F} = m\mathbf{f}$ of direction $\mathbf{f}$ and amplitude $m > 0$, applied at two points located at a distance $c$ from each other along a straight line of direction, say $\lambda$, Fig. B.3(a). The (finite) limit of the product $cm \to m^{(1)}$ for $c \to 0$ and $m \to \infty$ (while both $\mathbf{f}$ and $\lambda$ remain fixed) specifies a double (or dipole) force with direction $\mathbf{f}$ and lever arm in the direction $\lambda$ (Love, 1927). The double force possesses a moment whenever the lever arm is not parallel to the force, otherwise it is without moment. Couples are double forces with moments.

Analogously, a triple force can be generated by considering four ordinary equal forces, say $\mathbf{F} = mf$, every two of them being opposite to the others. The forces are respectively applied at the corners of a parallelogram of sides $c_1$, $c_2$ and directions $\lambda_1$, $\lambda_2$, as shown in Fig. B.3(b). The (finite) limit of the product $c_1c_2m \to m^{(2)}$ as $(c_1, c_2) \to 0$ and $m \to \infty$, (while $\mathbf{f}$, $\lambda_1$ and $\lambda_2$ remain fixed), constitutes a triple (or quadrupole) force of direction $\mathbf{f}$ and lever arms in the directions $\lambda_1$, $\lambda_2$. Every such force is self-equilibrated.

Double forces are like double forces, but the force is distributed over a unit plane area. With reference to Cartesian orthogonal axes $x_i$, (i = 1, 2, 3), a typical notation for a double stress is, say, $\sigma_{ij}^{(1)}$, which is here assumed to represent a double force of direction $i$, acting on a plane of normal $\mathbf{f}$ and having one lever arm in the direction $k$. The double stress $\sigma_{ij}^{(1)}$ is defined as the (finite) limit of the product $c_1\sigma_{ij}$, (where $c_2$ is the length of a line segment of direction $k$), when $c_1 \to 0$ and $\sigma_{ij} \to \infty$, while the directions $k, i, j$ remain fixed.

In Fig. B.4(a), the double stresses $\sigma_{112}^{(1)}$ and $\sigma_{121}^{(1)}$ are schematically shown aside the faces of the material element on which they act. In this concern one has to be aware about the fact that the boundary surface of a strain gradient material element has to be conceived not as a sharply polished surface, but instead as a thin layer composed of a few particle surface networks. In this way, the lever arm of a double stress can find itself either lying, say, on the outer network, or to cross all networks orthogonally, as shown in Fig. B.4(b). The stresses $\sigma_{112}^{(1)}$ and $\sigma_{121}^{(1)}$ turn out to be different physically, but equal numerically, since $\lim (c_1\sigma_{121}) = \lim (c_1\sigma_{121})$ as $c_1 \to 0$ and $\sigma_{121} = \sigma_{121} \to \infty$.

The above reminds the concept of primary and secondary planes introduced by Askes and Metrikine (2005), by which these authors interpret physically the relation between the higher order stresses and their divergence-form contributions to the total stress as balance equations at the microscale. In the present paper, instead, the higher order stresses themselves with the inherent lever arms are physically interpreted as force systems which at the microscale are distributed within a thin boundary layer.

As mentioned before in Section 4.1, an important feature of the double stress $\sigma_{ij}^{(1)}$ is that the first two indexes, $k$ and $i$, can be indifferently interpreted either with our own rule (according to which $k$ gives the lever arm direction, $i$ the normal to the plane on which the force of direction $j$ acts), or with the Mindlin rule (whereby the roles of these two indexes interchange). For more clarity, the
reads $\lambda$), but are however numerically equal $\lambda$, and $\lambda_h$. The distortion gradient $h_{12}$, varying linearly with $x_2$, generates $h_{12}^{(1)}$. (b) The distortion gradient $h_{12}$, varying linearly with $x_2$, generates $h_{12}^{(1)}$. (c) The shear strain $\varepsilon_{12} = h_{12} - (h_{12} + h_{12})/2$ varying linearly with $x_2$, generates $\varepsilon_{12}^{(2)} = h_{12}^{(2)} - h_{12}^{(2)}$.

For this purpose, let us consider the stress power

$$\mathcal{W}^{(i)} = \sigma_{ij}^{(i)} \varepsilon_{ij}^{(i)} = \sigma_{ij}^{(i)} h_{ij}^{(i)} = \sigma_{ij}^{(i)} \partial_i \partial_j u_j$$

(\text{B.1})

where the symmetry of $\sigma_{ij}^{(i)}$ with respect to $(i,j)$ has been exploited to replace $\varepsilon_{ij}^{(i)}$ (symmetric with respect to these indexes) with the non-symmetric $h_{ij}^{(i)}$. With a change of the index names, we can rewrite (B.1) as

$$\mathcal{W}^{(i)} = \sigma_{ij}^{(i)} \varepsilon_{ij}^{(i)} = \sigma_{ij}^{(i)} h_{ij}^{(i)} = \sigma_{ij}^{(i)} \partial_i \partial_j u_j$$

(\text{B.2})

On subtracting (B.1) and (B.2) from each other, we have

$$\sigma_{ij}^{(i)} - \sigma_{ij}^{(i)} \partial_i \partial_j u_j = 0$$

(\text{B.3})

which, having to hold for arbitrary second partial derivatives of $u_j$, gives

$$\sigma_{ij}^{(i)} = \sigma_{ij}^{(i)}, \quad \forall (i,j,k) \in \{1,2,3\}$$

(\text{B.4})

Since $\sigma_{ij}^{(i)} = \sigma_{ij}^{(i)}$, (B.4) is equivalent to

$$\sigma_{ij}^{(i)} = \sigma_{ij}^{(i)}, \quad \forall (i,j,k) \in \{1,2,3\}$$

(\text{B.5})

The latter result is illustrated in Fig. B.5(a) and (b), where a material element exhibits strain gradient deformations $\varepsilon_{12}^{(2)}$ and $\varepsilon_{21}^{(2)}$, respectively, and is subjected to the respective work-conjugate double stresses, i.e. $\sigma_{12}^{(2)}$ and $\sigma_{21}^{(2)}$. Although the element suffers, in the two deformation cases, different microstructural deformation mechanisms—either shear stain $\varepsilon_{12}$ varying linearly along the $x_1$ direction in Fig. B.5(a), or normal strain $\varepsilon_{11}$ varying linearly along $x_2$ in Fig. B.5(a)—there results ultimately a unique deformation pattern measured either by $\varepsilon_{12}^{(2)}$, or $\varepsilon_{21}^{(2)}$, which are accompanied by the work-conjugate double stresses $\sigma_{12}^{(2)} = \sigma_{21}^{(2)}$. This result was previously derived by Askes and Metrikine (2005) by means of a discrete model.

Extensions to triple stresses are straightforward. Indeed, the stress components $\sigma_{ij}^{(i)}$ have each two possible interpretations, one is based on our own rule (whereby the first two indexes $(k,l)$ denote the extra index pair specifying the directions of the relevant lever arms, whereas $i$ denotes the normal to the plane on which the force of direction $j$ acts), another is based on Mindlin rule (whereby the second and third indexes $(l,i)$ denote lever arms, whereas $k$ denotes the normal to the plane on which the mentioned force acts). An equality like (B.5) holds, i.e.

$$\sigma_{ij}^{(i)} = \sigma_{ij}^{(i)}$$

(\text{B.6})

This implies that $\sigma_{ij}^{(i)}$ is fully symmetric with respect to the three indexes $(k,l,i)$ and that the number of its independent components reduces from $6 \times 6 = 36$ to $10 \times 3 = 30$.

In conclusion of the latter discussion we can state the following statement on the complementary components of the higher order stresses, (that is, components which differ from each other by
interchanging the one index denoting a lever arm direction with the index denoting the normal to the plane on which the stress acts), namely: *Every two complementary components of the higher order stresses are numerically equal to each other.*

**References**


