JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **216,** 581–592 (1997)
ARTICLE NO. AY975690

On Internal Characterizations of Completely L-Regular Spaces

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Received October 11, 1996

Complete L-regularity is internally characterized in terms of separating chains of open L-sets. A possible characterization in terms of normal and separating families of closed L-sets is discussed and it is shown that spaces admitting such families are completely L-regular. The question of whether the converse holds true remains open. Some partial solutions are however given, e.g. in the class of countably compact spaces.

1. INTRODUCTION

Completely L-regular spaces, introduced by Hutton [8], constitute one of the best established categories of L-topological spaces. These spaces have satisfactory theories of uniform and metric structures, provided L is completely distributive [3, 8, 20]. There are a number of characterizations in terms of continuous L-real functions under various assumptions about L (including completeness). For L-Tychonoff spaces ($=$ completely L-regular spaces in which open L-sets separate points) there is the Tychonoff embedding theorem, holding for any complete L. The complete L-regularity and the L-Tychonoff behave well with respect to the functors ι_L and ω_L, where L is a hypercontinuous lattice, in particular, a completely distributive one. For these results see [17]. There are also further results,

0022-247X/97 $25.00

among them those related to compactness in completely L-regular spaces [1] and to probabilistic metric spaces [6]. Also cf. [11, 19, 24].

Internal characterizations of complete regularity (i.e., complete $\{0,1\}$-regularity) have been given by several authors. The paper [2] has an almost complete bibliography. (We recall that an internal characterization is one that depends only on the lattice of all open (or closed) subsets and does not involve the real numbers, even as an index set.) Among the T_1-free characterizations are those of Kerstan [13], Steiner [25], and Johnson and Mandelker [10].

In this paper we discuss the possibility of an internal characterization of complete regularity in an L-topological setting. We shall show that the characterization of [10] in terms of separating chains continues to hold for completely L-regular spaces with an arbitrary complete L. This provides a positive answer to the (first named author's) question of [23, p. 349]. Since a ring structure is not available in $C(X, \mathbb{R}(L))$ (with $|L| > 2$), there are difficulties in implementing the characterization in terms of normal and separating families of closed sets due to Steiner [25]. Spaces that admit normal and separating families of closed L-sets will be shown to be completely L-regular for any complete L. We have not yet been able to prove or disprove the converse for any L with $|L| > 2$. Our conjecture is that this may be the case for any complete Boolean algebra (see Question 5.10 and Conjecture 5.11 for details). However, a partial solution is given for those completely L-regular spaces that are countably compact in the sense of [9] (with L is a complete lattice). Among the less interesting special cases are second countable completely L-regular spaces (L meet-continuous) and topologically generated completely L-regular spaces (where $L = [0,1]$). The paper also includes a number of open questions.

2. PRELIMINARY NOTIONS AND RESULTS

We recall a bit of standard terminology, mainly to fix notation.

Let $L = (L, ')$ be a complete lattice with an order-reversing involution (0 and 1 are the bounds). For a set X, L^X is the complete lattice (under pointwise defined ordering and involution) of all maps from X to L. These maps are called L-*sets*. Bounds of L^X are denoted $1_\varnothing$ and 1_X (in general, the characteristic function of $A \subset X$ is denoted 1_A). A subfamily T of L^X, which is closed under arbitrary sups and finite infs (both formed in L^X), is called an L-*topology* on X, and (X, T) is then an L-*topological space* (L-ts for short). The space will often be denoted just by X if T is understood from the context. Members of T are called open, and $k \in L^X$ is closed iff k' is open. If $a \in L^X$, we denote by $\bar{a}$ the closure of a, and by Int a the interior of a. If $A \subset X$, then the set of all restrictions $\{u|A: u \in T\}$ is the *subspace L-topology* on A. We say that $S \subset L^X$ *generates* T if $T = \bigcap\{T_1 \supset S: T_1$ is an L-topology on $X\}$. A map f between two L-topological spaces X and Y is *continuous* if uf is open in X for every open u in Y (where uf is the composition of f and u).

The following lemma is important because it enables us to deal with continuous L-real functions in a complete lattice setting. For a proof and remarks about it, see [23, p. 282].

SUBBASE LEMMA 2.1. *Let L be a complete lattice. Let X and Y be two L-topological spaces and let $S \subset L^Y$ generate the L-topology of Y. Then $f: X \to Y$ is continuous if and only if uf is open in X for every $u \in S$.*

We now recall the concepts of the L-line and the unit L-interval [4, 7]. Let L be complete. Let $\mathbb{R}_L$ be the set of all order-reversing members $\lambda \in L^{\mathbb{R}}$ such that $\bigvee \lambda(\mathbb{R}) = 1$ and $\bigwedge \lambda(\mathbb{R}) = 0$. For every $t \in \mathbb{R}$ we let $\lambda^+(t) = \bigvee \lambda(t, \infty)$ and $\lambda^-(t) = \bigwedge \lambda(-\infty, t)$. Define $\lambda \sim \mu$ iff $\lambda^+ = \mu^+$ (this is equivalent to the statement that $\lambda^- = \mu^-$ [17, Remark 1.3.1]). The quotient set $\mathbb{R}(L) = \mathbb{R}_L/\sim$ is called the *L-real line*. It becomes a poset with the ordering $[\lambda] \le [\mu]$ iff $\lambda^+ \le \mu^+$(iff $\lambda^- \le \mu^-$). For every $t \in \mathbb{R}$, $L_t, R_t \in L^{\mathbb{R}(L)}$ are defined by $L_t[\lambda] = \lambda^-(t)'$ and $R_t[\lambda] = \lambda^+(t)$. The *natural L-topology* on $\mathbb{R}(L)$ is generated by the L-sets L_t and R_t $(t \in \mathbb{R})$. The set $I(L) = \{[\lambda] \in \mathbb{R}(L): \lambda^-(0)' = \lambda^+(1) = 0\}$ is called the (*unit*) *L-interval*. It has the subspace L-topology and the ordering induced from $\mathbb{R}(L)$. We write R_t rather than $R_t|I(L)$ (and similarly for L_t) to denote the open L-sets of $I(L)$. We write $C(X)$ (resp., $C(X, I(L))$) for the collection of all continuous functions from X to $\mathbb{R}(L)$ (resp., $I(L)$).

For a fuller account about the lattice-theoretic properties which follow, we refer to [17, Sect. 1.3]. For any complete $L, (I(L), \le)$ is a (complete) lattice in which $[\lambda] \vee [\mu] = [\lambda \vee \mu]$ and the same for $\wedge$. Furthermore, $I(L)$ has a continuous order-reversing involution $(\cdot)^\bullet$ defined by $[\lambda]^\bullet = [\lambda^\bullet]$, where $\lambda^\bullet(t) = \lambda(1-t)'$, $t \in \mathbb{R}$ (see [14, 17]). Given an $f \in C(X, I(L))$, we write $f^\bullet$ for the function in $C(X, I(L))$, obtained by composing f and $(\cdot)^\bullet$.

Let $\mathbf{t}$ denote the constant map of X into $\mathbb{R}(L)$ with the value $[1_{(-\infty, t)}]$. For any complete L and any $f \in C(X)$, we have $\mathbf{t} \wedge f, \mathbf{t} \vee f \in C(X)$. However, if L is meet-continuous, i.e., for every $\alpha \in L$ and every directed subset $D \subset L$:

$$\text{(MC)} \quad \alpha \wedge \bigvee D = \bigvee \{\alpha \wedge \delta : \delta \in D\},$$

then $C(X)$ is closed under arbitrary finite (pointwisely defined) $\vee$ and $\wedge$. We do not need this information; yet it may be useful when dealing with the open questions of Section 5.

Notation. Given an $f: X \to \mathbb{R}(L)$, we write f_x for a representative of $f(x)$, i.e., $f(x) = [f_x]$.

For $f \in C(X)$ and $s \in \mathbb{R}$, we define $f + \mathbf{s}$ by $(f + \mathbf{s})_x(t) = f_x(t - s)$ for every $x \in X$ and $t \in \mathbb{R}$. We have $R_t(f + \mathbf{s}) = R_{t-s}f$ and $L_t(f + \mathbf{s}) = L_{t-s}f$; hence $f + \mathbf{s} \in C(X)$ by 2.1. We write $f - \mathbf{s}$ for $f + (-\mathbf{s})$.

Remark. In point-set topology a subset A of a topological space X is called a zero-set if $A = f^{-1}\{0\}$ for some continuous real-valued function f of X. Since $f^{-1}\{0\} = |f|^{-1}(-\infty, 0]$ and $g^{-1}(-\infty, 0] = (g \vee 0)^{-1}\{0\}$, A is a zero-set iff $A = h^{-1}(-\infty, 0]$ for some continuous h. This motivates the following definition.

DEFINITION 2.2. Let X be an L-topological space, where L is a complete lattice. An $a \in L^X$ is called an *L-zero-set* if $a = R'_0 f$ for some $f \in C(X)$.

The obvious proof of what follows is included just to make it clear that it is a complete lattice proof.

LEMMA 2.3. *Let X be an L-topological space, where L is a complete lattice. For $a \in L^X$, the following are equivalent*:

(1) *a is an L-zero-set.*

(2) *There exists $f \in C(X, I(L))$ such that $a = R'_0 f$.*

(3) *There exists $f \in C(X, I(L))$ and $0 \le t \le 1$ such that $a = R'_t f$.*

(4) *There exists $f \in C(X, I(L))$ and $0 \le t \le 1$ such that $a = L'_t f$.*

Proof. (1) $\Rightarrow$ (2): Let $a = R'_0 g$ with $g \in C(X)$. Put $f = (g \vee \mathbf{0}) \wedge \mathbf{1}$. Then $f \in C(X, I(L))$, and for every $x \in X$ we have

$$\begin{aligned} R_0 f(x) &= R_0\big[\big(g_x \vee 1_{(-\infty,0)}\big) \wedge 1_{(-\infty,1)}\big] \\ &= \bigvee_{s>0} \big(g_x(s) \vee 1_{(-\infty,0)}(s)\big) \wedge 1_{(-\infty,1)}(s) \\ &= \bigvee_{s>0} g_x(s) = R_0 g(x). \end{aligned}$$

(2) $\Rightarrow$ (3): Obvious

(3) $\Rightarrow$ (4): We have $a = R'_t f = L'_{1-t} f^{\bullet}$.

(4) $\Rightarrow$ (3): Similarly, $a = L'_t f = R'_{1-t} f^{\bullet}$.

(3) $\Rightarrow$ (1): $a = R'_t f = R'_0(f - \mathbf{t})$. ∎

Remark. If L satisfies (MC), then the family of all L-zero-sets is a ring, i.e., it is closed under finite $\wedge$ and $\vee$. This is not needed for us, but, note again, it may be useful for solving the questions of Section 5.

Remark 2.4. Recall [14] that the characteristic function of $a \in L^X$ is the map $\chi_a: X \to I(L)$, where $\chi_a(x)$ is defined by

$$(\chi_a)_x(t) = \begin{cases} a(x) & \text{if } 0 \leq t \leq 1, \\ 1 & \text{if } t < 0, \\ 0 & \text{if } t > 1. \end{cases}$$

If a is a closed-and-open L-set in an L-ts X, then $\chi_a \in C(X, I(L))$ and vice versa (see [14, Proposition 3.2]).

3. GENERATING CONTINUOUS L-REAL FUNCTIONS

Let X be a set. A nondecreasing family $\mathbf{C} = \{c_r: r \in \mathbb{Q}\} \subset L^X$ is called a *scale* on X if $\bigwedge \mathbf{C} = 1_\varnothing$ and $\bigvee \mathbf{C} = 1_X$. For every $x \in X$ and $t \in \mathbb{R}$ let

$$f_x(t) = \bigwedge_{r<t} c'_r(x).$$

It is easy to see that $f_x \in \mathbb{R}_L$ and $f_x = f_x^-$. The function $f: X \to \mathbb{R}(L)$, defined by $f(x) = [f_x]$, is said to be *generated* by $\mathbf{C}$ (cf. [7, 12, 15]). In what follows p, q, r stand for rationals.

LEMMA 3.1. *Let L be complete and let f: $X \to \mathbb{R}(L)$ be generated by the scale $\{c_r$: $r \in \mathbb{Q}\} \subset L^X$. Then*:

(1) $L_t f = \bigvee_{r<t} c_r$ *for every* $t \in \mathbb{R}$;

(2) $R'_t f = \bigwedge_{r>t} c_r$ *for every* $t \in \mathbb{R}$.

For X an L-topological space we have

(3) $f \in C(X)$ *if and only if* $\overline{c_r} \leq \operatorname{Int} c_q$ *whenever* $r < q$;

(4) $f \in C(X, I(L))$ *if and only if* $f \in C(X)$, *and* $c_r = 1_\varnothing$ *if* $r < 0$, *and* $c_r = 1_X$ *if* $r > 1$.

Proof. (1) For each $x \in X$ we have $L'_t f(x) = f_x^-(t) = f_x(t) = \bigwedge_{r<t} c'_r(x)$. Hence $L_t f = \bigvee_{r<t} c_r$.

(2) Since $R'_t f = \bigwedge_{s>t} L_s f$, by (1) we get

$$R'_t f = \bigwedge_{s>t} \bigvee_{r<s} c_r \leq \bigwedge_{s>t} \bigwedge_{r>s} c_r = \bigwedge_{r>t} c_r.$$

For the reverse inequality observe that $\bigwedge_{r>t} c_r \leq \bigvee_{r<s} c_r$ for each $s > t$, and therefore $\bigwedge_{r>t} c_r \leq \bigwedge_{s>t} \bigvee_{r<s} c_r$.

(3) $\Rightarrow$: For $r < q$ we have, by (1) and (2):

$$c_r \leq \bigwedge_{p>r} c_p = R'_r f \leq L_q f = \bigvee_{p<q} c_p \leq c_q.$$

Since $R'_r f$ is closed and $L_q f$ is open, we obtain $\overline{c_r} \leq \operatorname{Int} c_q$.

$\Leftarrow$: It is clear that $\bigvee_{r<t} c_r = \bigvee_{r<t} \operatorname{Int} c_r$; hence $L_t f$ is open by (1). Similarly, since $\bigwedge_{r>t} c_r = \bigwedge_{r>t} \overline{c_r}$, $R'_t f$ is closed by (2). By the Subbase Lemma, $f \in C(X)$.

(4) $\Rightarrow$: $f \in C(X, I(L))$ if and only if $L_0 f = R_1 f = 1_\phi$. But, by (1) and (2), we have $L_0 f = \bigvee_{r<0} c_r$ and $R_1 f = \bigvee_{r>1} c'_r$; hence $c_r = 1_\varnothing$ if $r < 0$, and $c_r = 1_X$ iif $r > 1$.

$\Leftarrow$: Reverse the steps of the forward implication. ∎

LEMMA 3.2. *Let X be an L-topological space, where L is a complete lattice. For $a \leq b$ in L^X, the following are equivalent*:

(1) $a \leq L'_t f \leq R_s f \leq b$ *for some* $f \in C(X)$ *and* $s < t$ *in* $\mathbb{R}$;

(2) $a \leq L'_1 f \leq R_0 f \leq b$ *for some* $f \in C(X, I(L))$;

(3) $a \leq R'_0 f \leq L_1 f \leq b$ *for some* $f \in C(X, I(L))$;

(4) *There exists a family* $\{u_r : r \in \mathbb{Q} \cap (0, 1)\}$ *of open L-sets of X such that* $a \leq u_r \leq b$ *for every* $r \in \mathbb{Q} \cap (0, 1)$, *and* $\overline{u_r} \leq u_q$ *whenever* $r < q$.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3): [17, Remark 1.4.1].

(3) $\Rightarrow$ (4): For every $r \in \mathbb{Q} \cap (0, 1)$ we have $R'_0 f \leq L_r f \leq L_1 f$. Furthermore, $L_r f \leq R'_p f \leq L_q f$ if $0 < r < p < q < 1$; hence $\overline{L_r f} \leq L_q f$ if $r < q$. Therefore $\{L_r f : r \in \mathbb{Q} \cap (0, 1)\}$ has the required properties.

(4) $\Rightarrow$ (3): Let $\{u_r : r \in \mathbb{Q} \cap (0, 1)\}$ be given. Put $u_r = 1_\varnothing$ for $r \leq 0$ and $u_r = 1_X$ for $r \geq 1$ Then $\boldsymbol{U} = \{u_r : r \in \mathbb{Q}\}$ is a scale of open L-sets such that $\overline{u_r} \leq u_q$ if $r < q$ in $\mathbb{Q}$. Let f be the function generated by $\boldsymbol{U}$. By Lemma 3.1 (4), $f \in C(X, I(L))$. Moreover, since $a \leq u_r \leq b$ if $0 < r < 1$, we obtain by (1) and (2) of Lemma 3.1

$$a \leq \bigwedge_{r>0} u_r = R'_0 f \leq L_1 f = \bigvee_{r<1} u_r \leq b. \qquad ∎$$

Notation. If $a, b \in L^X$, we write $a \prec b$ if a and b satisfy condition (2) of Lemma 3.2 (and thus all of them). If necessary, we shall write $a \prec_f b$ to indicate the function involved in (2).

DEFINITION 3.3. Two L-sets c and d are *completely separated* in an L-ts X if $c \prec d'$. We then also have $d \prec c'$, because of (2) $\Leftrightarrow$ (3) in Lemma 3.2.

4. AN INTERNAL CHARACTERIZATION OF COMPLETELY L-REGULAR SPACES.

DEFINITION 4.1 (Hutton [8]). Let L be a complete lattice. An L-topological space X is *completely L-regular* if, whenever u is an open L-set, there exists a family $\mathcal{A} \subset L^X$ such that $\bigvee \mathcal{A} = u$ and $a \prec u$ for every $a \in \mathcal{A}$.

PROPOSITION 4.2 [17, Section 2.1]. *Let L be complete. For X an L-ts, the following are equivalent*:

(1) *X is completely L-regular.*

(2) *For every open u there exists a family $\mathcal{V}$ of open L-sets such that $\bigvee \mathcal{V} = u$ and $v \prec u$ for every $v \in \mathcal{V}$.*

We are now going to show that the internal characterization of topological complete regularity due to Johnson and Mandelker [10] generalizes to completely L-regular spaces, where L is a complete lattice.

DEFINITION 4.3 (cf. [10] or [21]). Let L be a complete lattice. A *separating chain* in an L-ts X is a family $\mathcal{U}$ of open L-sets of X such that

(1) $\mathcal{U}$ is a countable chain in L^X (i.e., totally ordered under $\leq$);

(2) $\bigwedge \mathcal{U} = 1_\phi$ and $\mathcal{U} = 1_X$;

(3) If $u, v \in \mathcal{U}, u \leq v$, and $u \neq v$, then there exists $w \in \mathcal{U}$ such that $\bar{u} \leq w \leq \overline{w} \leq v$.

A $\mathbb{Q}$-*indexing* of $\mathcal{U}$ is a function $f \colon \mathbb{Q} \to \mathcal{U}$ that is onto and such that $\overline{c(p)} \leq c(r)$ whenever $p < r$.

It is well known that if $L = \{0, 1\}$, then every separating chain can be $\mathbb{Q}$-indexed ([10], [21]). In the proof of this fact (as in [21]), the argument is seen on examination to be both point-free and distributivity-free, and does not use the Boolean complements. Thus, it goes unchanged to the case that follows, and we therefore omit the proof.

LEMMA 4.4. *Let L be a complete lattice. Any separating chain in an L-topological space can be $\mathbb{Q}$-indexed.*

Remark. A separating chain, when $\mathbb{Q}$-indexed, becomes a scale.

THEOREM 4.5 (internal characterization). *Let L be a complete lattice. For X an L-topological space, the following are equivalent*:

(1) *X is completely L-regular.*

(2) *For every open L-set u there exists a family $\mathcal{V}$ of open L-sets such that $\bigvee \mathcal{V} = u$, and for every $v \in \mathcal{V}$ there exists a separating chain $\mathcal{G}$ such that $v \leq g \leq \bar{g} \leq h \leq u$ for some $g, h \in \mathcal{G}$.*

Proof. (1) ⇒ (2): If u is open, there exists by Proposition 4.2 a family $\mathcal{V}$ of open L-sets with $\bigvee \mathcal{V} = u$ and $v \prec u$ for all $v \in \mathcal{V}$. Let $\{g_r : r \in \mathbb{Q} \cap (0,1)\}$ be the associated scale of open L-sets that exists on account of Lemma 3.2 (4), i.e., $v \le g_r \le u$ and $\overline{g_r} \le g_q$ if $r < q$. Put $g_r = 1_\varnothing$ if $r \le 0$, and $g_r = 1_X$ if $r \ge 1$. Then $\mathcal{G} = \{g_r : r \in \mathbb{Q}\}$ is the required separating chain.

(2) ⇒ (1): Let $\mathcal{V}$ be as in (2). Fix $v \in \mathcal{V}$ and assume that $v \le g \le \bar{g} \le h \le u$ for some g and h of a separating chain $\mathcal{G}$. Let c be a $\mathbb{Q}$-indexing of $\mathcal{G}$ (cf. Lemma 4.4). Let $g = c(p)$ and $h = c(r)$ for some $p, r \in \mathbb{Q}$.

If $g = h$, then $g = \bar{g}$ is a closed-and-open L-set; hence $\chi_g \in C(X, I(L))$ (cf. Remark 2.4). We have $v \le g = L_1' \chi_g = R_0 \chi_g = g \le u$, i.e., $v \prec u$.

If $g \ne h$, then $p < r$. Thus $v \le c(p) \le \bigwedge_{q>p} c(q) = R_p' f$, where f is the continuous function generated by the scale $\{c(r) : r \in \mathbb{Q}\}$. Similarly, $L_r f = \bigvee_{q<r} c(q) \le c(r) \le u$. Thus $u' \le L_r' f \le R_p f \le v'$, *i.e.*, $u' \prec v'$. Hence $v \prec u$ (cf. Lemma 3.2 and Definition 3.3). ∎

5. NORMAL AND SEPARATING FAMILIES OF CLOSED L-SETS

The definition below is a point-free variant of the concept of a normal and separating family of closed subsets of a topological space (cf. Steiner [25]).

DEFINITION 5.1. Let L be complete. A family $\mathcal{K}$ of closed L-sets of an L-ts is called:

(1) *separating*, if for every open L-set u there exist two families $\{a_\gamma : \gamma \in \Gamma\}, \{b_\gamma : \gamma \in \Gamma\} \subset \mathcal{K}$ such that $u = \bigvee_{\gamma \in \Gamma} a_\gamma$ and $a_\gamma \le b_\gamma' \le u$ for every $\gamma \in \Gamma$ (therefore $u = \bigvee_{\gamma\in\Gamma} b_\gamma'$ and hence $\mathcal{K}' = \{a' : a \in \mathcal{K}\}$ is a base of the L-topology of X);

(2) *normal*, if for every $a, b \in \mathcal{K}$ with $a \le b'$ there exist $c, d \in \mathcal{K}$ such that $a \le c' \le d \le b'$.

LEMMA 5.2. *Let L be a complete lattice and let $\mathcal{K}$ be a normal family of closed L-sets of an L-topological space X. Then any $a, b \in \mathcal{K}$ such that $a \le b'$ are completely separated.*

Proof. This is essentially the proof of Urysohn's lemma. Let $\{q_n\}$ be an enumeration of $\mathbb{Q} \cap [0,1]$, where $q_1 = 0$ and $q_2 = 1$. For every $n \ge 2$ we shall inductively define families $\{u_{q_i} : i < n\} \subset \mathcal{K}'$ and $\{k_{q_i} : i < n\} \subset \mathcal{K}$ such that

$$(I_n) \qquad a \le u_{q_i} \le k_{q_i} \le u_{q_j} \le b' \quad \text{if } q_i < q_j \quad (i, j < n).$$

There are u_{q_1} and k_{q_1} where $a \le u_{q_1} \le k_{q_1} \le b' =: u_{q_2}$. This is (I_2). Assume that u_{q_i} and k_{q_i} are already defined for $i < n$ and satisfy (I_n). Let $q_l = \max\{q_i < q_n : i < n\}$ and $q_r = \min\{q_i > q_n : i < n\}$. Then $q_l < q_r$ and thus there are $c, d \in \mathsf{K}$ such that $k_{q_l} \le c' \le d \le u_{q_r}$. Put $u_{q_n} = c'$ and $k_{q_n} = d$. Then $u_{q_i}, k_{q_i} (i < n+1)$ satisfy (I_{n+1}). We thus have a family $\{u_r : r \in \mathbb{Q} \cap (0,1)\}$ of open L-sets such that $a \le u_r \le b'$ and $\overline{u_r} \le u_q$ if $r < q$. By Lemma 3.2 we conclude that $a \prec b'$. ∎

We recall that X is *L-normal* [7] if, given a closed k and an open u with $k \le u$, there exists an open v with $k \le v \le \bar{v} \le u$. Clearly, X is L-normal iff its family of closed L-sets is normal. Thus we get by Lemma 5.2 the following.

COROLLARY 5.3 (Hutton-Urysohn lemma [7]). *Let L be a complete lattice. An X is L-normal if and only if, whenever k is closed, u is open and $k \le u$, there holds $k \prec u$.*

PROPOSITION 5.4. *Let L be a complete lattice. Every L-topological space with a normal and separating family of closed L-sets is completely L-regular.*

Proof. Let X be an L-ts with a normal and separating family K of closed L-sets. For an open u, let $u = \bigvee_{\gamma \in \Gamma} a_\gamma$ and $a_\gamma \le b'_\gamma \le u$ for $a_\gamma, b_\gamma \in \mathsf{K}, \gamma \in \Gamma$. By Lemma 5.2, $a_\gamma \le b'_\gamma$ implies $a_\gamma \prec b'_\gamma$, and thus $a_\gamma \prec u$. This shows complete L-regularity of X. ∎

The above generalizes half of Steiner's [25] internal characterization: *a topological space is completely regular if and only if it has a normal and separating family of closed subsets*. In a completely regular X the family $Z(X)$ of all zero-sets is clearly separating (cf. Remark 5.5 below), while $Z(X)$ is normal for any X. The usual proof of the latter statement depends heavily upon the ring structure of $C(X, \mathbb{R})$ [viz., if $f^{-1}\{0\} \cap g^{-1}\{0\} = \emptyset$ with $f, g \in C(X, \mathbb{R})$, then $h = f/(f+g)$ completely separates these disjoint zero-sets, and this is equivalent to the normality of $Z(X)$; [(see Proposition 5.6 below)]. In an L-topological setting, $C(X)$ is merely a poset (or a lattice if L is meet-continuous [17]).

Notation. In what follows, $Z_L(X)$ stands for the family of all L-zero-sets of an L-topological space X.

Remark 5.5. $Z_L(X)$ is a separating family for every completely L-regular space X with L a complete lattice.

Indeed, if $\bigvee \mathsf{A} = u$ and $a \prec_{fa} u$ for all $a \in \mathsf{A}$, then $u = \bigvee_{a \in \mathsf{A}} L'_1 f_a$ and $L'_1 f_a \le R_0 f_a \le u$.

PROPOSITION 5.6. *Let L be complete and let X be an L-ts. Then $Z_L(X)$ is a normal family if and only if $a \prec b'$ for every $a, b \in Z_L(X)$ with $a \le b'$.*

Proof. The necessity follows from Lemma 5.2. For the sufficiency: if $a \leq L'_1 f \leq R_0 f \leq b'$, then $a \leq c' \leq d \leq b'$ with $c = R'_{1/2} f$ and $d = L'_{1/2} f$. ∎

The following provides a number of open questions suggested by the above discussion.

Question 5.7. Does there exist a lattice $L \neq \{0, 1\}$ such that $Z_L(X)$ is a normal family for every L-ts X?

By Proposition 5.6, this is the same question as that of [17, Open question 10.4].

Question 5.8 (cf. [23, p. 350] and 17, Open question 10.5]). Does every completely L-regular space have a normal and separating family of closed L-sets for some $L \neq \{0, 1\}$?

Question 5.9. Is the property of having a normal and separating family of closed L-sets hereditary if $L \neq \{0, 1\}$?

Question 5.10. Let L be a complete Boolean algebra. Is there a bijection from the L-topology of the L-cube $I(L)^\Gamma$ onto the topology of the cube $[0, 1]^\Gamma$ that preserves arbitrary suprema and finite infima? (We recall that this is the case when $|\Gamma| = 1$; cf. Hutton [7].) If the answers to Questions 5.9 and 5.10 were "yes," then the Tychonoff embedding theorem for L-topological spaces [17] would yield the following, which we state as a conjecture. (We note that when L is a complete Boolean algebra, then the concept of a normal and separating family of closed L-sets can be formulated in terms of open L-sets involving arbitrary suprema and finite infima.)

Conjecture 5.11. Let L be a complete Boolean algebra. An L-T_0 space (= open L-sets separate points of the space) is completely L-regular if and only if it has a normal and separating family of closed L-sets.

An L-topological space X is *countably compact* [9] if, given a closed k and a countable family $\mathcal{U}$ of open L-sets of X such that $k \leq \bigvee \mathcal{U}$, there exists a finite subfamily $\mathcal{U}_0 \subset \mathcal{U}$ with $k \leq \bigvee \mathcal{U}_0$.

Since a countably compact completely L-regular space need not be L-normal (as the case of $L = \{0, 1\}$ shows; cf. [5, 3L]), the following is of some interest.

PROPOSITION 5.15. *Let L be a complete lattice. A countably compact L-ts X is completely L-regular if and only if it has a normal and separating family of closed L-sets.*

Proof. After Proposition 5.4 and Remark 5.5 it suffices to observe that $Z_L(X)$ is a normal family. Indeed, let $a, b \in Z_L(X)$ and $a \leq b'$. By Lemma 2.3, we can assume that $b = R'_0 f$ for some $f \in C(X, I(L))$. Since a is closed and $R_0 f = \bigvee\{R_r f : r > 0, r \in \mathbb{Q}\}$, there are $r_1, \dots, r_n > 0$ such that $a \le R_{r_1} f \vee \dots \vee R_{r_n} f = R_r f$, where $r = \min\{r_i : i \le n\}$. But then $a \le R_r f \le L'_r f \le R_0 f = b'$; hence $Z_L(X)$ is normal by Lemma 2.3. ∎

Remark 5.13. (1) If L is meet-continuous, then every second countable completely L-regular space X is L-normal [17, Theorem 9.11]; hence $Z_L(X)$ is normal by Corollary 5.3.

(2) There is also an obvious argument for the solution to Question 5.8 in the class of topologically generated spaces with $L = I = [0,1]$. Recall [18] that ωX is said to be topologically generated from a topological space X if ωX is the set X endowed with the I-topology consisting of all lower semicontinuous functions from the space X to I. It is proved in [16, Corollary 3.4 and Remark 4.4] that X is a completely regular space if and only if ωX is completely I-regular if and only if ωX is zero-dimensional ($=$ there is a base consisting of closed-and-open I-sets). Thus, clearly, the family of all closed-and-open I-sets ($= C(X, I)$) is then both normal and separating in ωX.

ACKNOWLEDGMENTS

This work was done while the first named author was visiting the University of the Basque Country, in summer 1996, supported by the Government of the Basque Country.

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