Nonlinear boundary value problem for first order impulsive integro-differential equations of mixed type

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Received 1 January 2006
Available online 20 March 2006
Submitted by William F. Ames

Abstract

This paper discusses nonlinear boundary value problem for first order impulsive integro-differential equations of mixed type. The lower and upper solutions and monotone iterative techniques are used to achieve the existence results. Two examples are provided to illustrate the efficiency of the obtained results.
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Keywords: Impulsive differential equations; Nonlinear boundary; Upper and lower solution; Monotone iterative technique; Extremal solutions

1. Introduction

In this paper, we study the following impulsive integro-differential problem:

\[ \begin{align*}
  x'(t) &= f(t, x(t), x(\theta(t)), (Wx)(t), (Sx)(x)), \quad t \in J = [0, T], \; t \neq t_k, \; k = 1, 2, \ldots, p, \\
  \Delta x(t_k) &= I_k(x(t_k)), \quad k = 1, 2, \ldots, p, \\
  g(x(0), x(T)) &= 0, \end{align*} \]

(1)

where \( f \in C(J \times R^4, R), \; I_k \in C(R, R), \; g \in C(R \times R, R), \; \theta \in C(J, J). \) \( \Delta x(t_k) = x(t_k^+) - x(t_k^-), \) in which \( x(t_k^+), x(t_k^-) \) denote the right and left limits of \( x(t) \) at \( t_k, \; k = 1, 2, \ldots, p, \) which

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doi:10.1016/j.jmaa.2006.01.084 \]
are fixed points such that $0 < t_1 < t_2 < \cdots < t_p < T$, $J_0 = J \setminus \{t_1, t_2, \ldots, t_p\}$. The integral part in Eq. (1) is defined by

$$\gamma(t) \int_0^T K(t, s)x(\delta(s)) \, ds, \quad (Sx)(t) = \int_0^T H(t, s)x(\sigma(s)) \, ds,$$

$\gamma, \delta, \sigma \in C(J, J)$, $K \in C(D, R_+)$, $D = \{(t, s) \in J \times J: 0 \leq s \leq \gamma(t)\}$, $H \in C(J \times J, R_+)$, $K^* = \max\{K(t, s): (t, s) \in D\}$, $H^* = \max\{H(t, s): (t, s) \in J \times J\}$. Let $PC(J) = \{u: J \to R: u \text{ is continuous for any } t \in J_0; u(t^+_k), u(t^-_k) \text{ exist, } k = 1, 2, \ldots, p \}$ and $PC^1(J) = \{u \in PC(J): u \text{ is continuously differentiable for any } t \in J_0; u'(0^+), u'(T^-), u'(t^+_k), u'(t^-_k) \text{ exist, } k = 1, 2, \ldots, p\}$. It is clear that $PC(J)$ and $PC^1(J)$ are Banach spaces with the respective norms

$$\|u\|_{PC(J)} = \sup \{|u(t)|: t \in J\}$$

and

$$\|u\|_{PC^1(J)} = \|u\|_{PC(J)} + \|u'\|_{PC(J)}.$$ 

The study of boundary value problems for first order differential equations with impulse or without impulse [1,2,6–24] are often seen in the literature. This type of equations has become an important aspect of impulsive differential equations [1–5] for its deep physical background and realistic mathematical model. It is well known that the monotone iterative technique is quite useful. In our recent work in [8], this method, combining with the upper and lower solutions, has been successfully applied to obtain the existence of the extremal solutions for a class of nonlinear first order impulsive functional differential equations. In [9–21], the method also has been used to solve the problems for nonlinear differential equations, integro-differential equations, functional differential equations with either initial value or boundary value conditions. Recently, it has been considered in [22] when the functional dependence is not necessarily a Lipschitzian function. In this paper, we will apply the monotone iterative method to achieve existence results for (1), which has a very general form. Note that as special instances resulting from (1), we can have impulsive differential equations with deviating arguments and impulsive differential equations with the Volterra or Fredholm operators.

This paper is organized as follows. First, we will establish some comparison principles. Next the existence and uniqueness of the solutions for linear boundary value problems will be discussed. Then, using the definition of upper and lower solutions $\alpha(t), \beta(t)$ and monotone iterative technique, we will obtain the existence of the extremal solutions for (1), with $\alpha(t) \geq \beta(t)$ or in the reverse order. Finally, some examples are provided to verify the required assumptions.

A function $x \in PC^1(J)$ is called a solution of (1), if it satisfies (1).

2. Lemmas

**Lemma 1.** If $u \in PC^1(J)$,

$$u'(t) \leq -M_1u(t) - M_2u(\theta(t)) - M_3(Wu)(t) - M_4(Su)(t), \quad t \neq t_k, \ t \in J,$$

$$\Delta u(t_k) \leq L_k u(t_k), \quad k = 1, 2, \ldots, p,$$

$$u(0) \leq 0,$$

$$u(t) \leq M_3(Wu)(t) + M_4(Su)(t), \quad t \in J_0.$$
where $M_i \geq 0$, $i \in \{2, 3, 4\}$, $-1 < L_k \leq 0$ for $k = 1, 2, \ldots, p$. Assume that $0 \leq \theta, \delta, \sigma \leq t$ and

$$\left(1 + \prod_{j=1}^{p}(1 + L_j)^{-1}\right) \int_{0}^{T} \left[ M_2 e^{M_1(t-\theta(t))} + M_3 \int_{0}^{T} K_1(t, s) \, ds + M_4 \int_{0}^{T} H_1(t, s) \, ds \right] \, dt \leq 1,$$

where $K_1(t, s) = e^{M_1(t-\delta(s))} K(t, s)$, $H_1(t, s) = e^{M_1(t-\sigma(s))} H(t, s)$, then $u(t) \leq 0$ on $J$.

**Proof.** Let $v(t) = e^{M_1 t} u(t)$, then $v \in PC^1(J)$ and

$$v'(t) \leq -M_2 e^{M_1(t-\theta(t))} v(\theta(t)) - M_3 (W_1 v)(t) - M_4 (S_1 v)(t), \quad t \neq t_k, \ t \in J,$$

$$\Delta v(t_k) \leq L_k v(t_k), \quad k = 1, 2, \ldots, p,$$

$$v(0) \leq 0,$$

$$(W_1 v)(t) = \int_{0}^{\gamma(t)} K_1(t, s) (\delta(s)) \, ds, \quad (S_1 v)(t) = \int_{0}^{T} H_1(t, s) v(\sigma(s)) \, ds. \quad (4)$$

Obviously, $v(t) \leq 0$ implies $u(t) \leq 0$. To show $v(t) \leq 0$, suppose on the contrary, that $v(t^*) > 0$ for some $t^* \in J$. Then there are two cases:

(a) $v(t) \geq 0$ for all $t \in [0, t^*]$;

(b) there exists $t_\ast \in [0, t^*)$, such that $v(t_\ast) < 0$.

In case (a), (4) implies that $v'(t) \leq 0$ for $t \neq t_k$ and $\Delta v(t_k) \leq 0$, here $t, t_k \in [0, t^*]$, hence $v(t)$ is nonincreasing in $[0, t^*]$. Then $v(t^*) \leq v(0) \leq 0$. It contradicts to $v(t^*) > 0$.

In case (b), let

$$\inf_{t \in J} v(t) = -\gamma_1.$$

Then $\gamma_1 > 0$, and for some $i \in \{1, 2, \ldots, p\}$, exists $t_\ast \in (t_i, t_{i+1}]$, such that $v(t_\ast) = -\gamma_1$ or $v(t_\ast^+) = -\gamma_1$. We only consider $v(t_\ast) = -\gamma_1$, as the proof is similar for the case $v(t_\ast^+) = -\gamma_1$.

Now for some $j$ satisfying $t^* \in (t_j, t_{j+1}]$, it is obviously $t_\ast < t^*$, then $j \geq i$. Now, we integrate (4) between $t_\ast$ and $t_{i+1},$

$$\int_{t_\ast}^{t_{i+1}} v'(t) \, dt \leq \int_{t_\ast}^{t_{i+1}} (-M_2 e^{M_1(t-\theta(t))} v(\theta(t)) - M_3 (W_1 v)(t) - M_4 (S_1 v)(t)) \, dt,$$

$$v(t_{i+1}) - v(t_\ast) \leq \gamma_1 \int_{t_i}^{t_{i+1}} \left[ M_2 e^{M_1(t-\theta(t))} + M_3 \int_{0}^{T} K_1(t, s) \, ds + M_4 \int_{0}^{T} H_1(t, s) \, ds \right] \, dt,$$

similarly we get

$$v(t_{i+2}) - (1 + L_{i+1}) v(t_{i+1}) \leq \gamma_1 \int_{t_{i+1}}^{t_{i+2}} \left[ M_2 e^{M_1(t-\theta(t))} + M_3 \int_{0}^{T} K_1(t, s) \, ds + M_4 \int_{0}^{T} H_1(t, s) \, ds \right] \, dt,$$
\begin{align*}
v(t_j) - (1 + L_{j-1})v(t_{j-1}) & \leq \gamma_1 \int_{t_{j-1}}^{t_j} \left[ M_2 e^{M_1(t - \theta(t))} + M_3 \int_0^{t_j} K_1(t, s) \, ds + M_4 \int_0^T H_1(t, s) \, ds \right] \, dt, \\
v(t^*) - (1 + L_j)v(t_j) & \leq \gamma_1 \int_{t_j}^{t_j+1} \left[ M_2 e^{M_1(t - \theta(t))} + M_3 \int_0^{t_j} K_1(t, s) \, ds + M_4 \int_0^T H_1(t, s) \, ds \right] \, dt.
\end{align*}

By adding together and observing \( v(t^*) = -\gamma_1 \), we have

\[
1 < \left( 1 + \prod_{j=1}^p (1 + L_j)^{-1} \right) \int_0^T \left[ M_2 e^{M_1(t - \theta(t))} + M_3 \int_0^{t_j} K_2(t, s) \, ds + M_4 \int_0^T H_2(t, s) \, ds \right] \, dt \leq 1,
\]

which contradicts to (3). So we have \( v(t) \leq 0, \quad t \in J \). This completes the proof. \( \square \)

**Lemma 2.** If \( u \in PC^1(J) \),

\[
u'(t) \geq M_1 u(t) + M_2 u(\theta(t)) + M_3 (Wu)(t) + M_4 (Su)(t), \quad t \neq t_k, \quad t \in J,
\]

\[
\Delta u(t_k) \geq L_k u(t_k), \quad k = 1, 2, \ldots, p,
\]

\[
u(T) \leq 0,
\]

where \( M_i \geq 0, \quad i \in \{2, 3, 4\} \), \( L_k \geq 0 \) for \( k = 1, 2, \ldots, p \). Assume that \( t \leq \theta, \delta, \sigma \leq T \) and

\[
\left( 1 + \prod_{j=1}^p (1 + L_j)^{-1} \right) \int_0^T \left[ M_2 e^{M_1(\theta(t) - t)} + M_3 \int_0^{t_j} K_2(t, s) \, ds + M_4 \int_0^T H_2(t, s) \, ds \right] \, dt \leq 1,
\]

where \( K_2(t, s) = e^{M_1(\delta(s) - t)} K(t, s) \), \( H_2(t, s) = e^{M_1(\sigma(s) - t)} H(t, s) \), then \( u(t) \leq 0 \) on \( J \).

The proof is similar to that of Lemma 1.

**Lemma 3.** If \( u \in PC^1(J) \),

\[
u'(t) \leq -M_1 u(t) - M_2 u(\theta(t)) - M_3 (Wu)(t) - M_4 (Su)(t), \quad t \neq t_k, \quad t \in J,
\]

\[
\Delta u(t_k) \leq L_k u(t_k), \quad k = 1, 2, \ldots, p,
\]

\[
u(0) \leq \lambda u(T),
\]

where \( 0 < \lambda e^{-M_1 T} \leq 1 \), with any of \( M_2, M_3, M_4 > 0 \) and the other two constants be nonnegative, \( -1 < L_k \leq 0 \) for \( k = 1, 2, \ldots, p \). Assume that

\[
\left( 1 + \prod_{j=1}^p (1 + L_j)^{-1} \right) \int_0^T \left[ M_2 e^{M_1(t - \theta(t))} + M_3 \int_0^{t_j} K_1(t, s) \, ds + M_4 \int_0^T H_1(t, s) \, ds \right] \, dt \leq \lambda e^{-M_1 T},
\]

(8)
where $K_1(t,s), H_1(t,s)$ are defined in the same way as in Lemma 1, then $u(t) \leq 0$ on $J$.

**Proof.** The proof is similar to the proof of Lemma 1 [8], therefore we omit it. \qed

Similarly we have

**Lemma 4.** If $u \in PC^1(J)$,

\[
\begin{align*}
&u'(t) \geq M_1 u(t) + M_2 u(\theta(t)) + M_3 (Wu)(t) + M_4 (Su)(t), \quad t \neq t_k, \ t \in J, \\
&\Delta u(t_k) \geq L_k u(t_k), \quad k = 1, 2, \ldots, p, \\
u(T) \leq \lambda u(0),
\end{align*}
\]

where $0 < \lambda e^{-M_1 T} \leq 1$, with any of $M_2, M_3, M_4 > 0$ and the other two constants be nonnegative, $L_k \geq 0$ for $k = 1, 2, \ldots, p$. Assume that

\[
(1 + \prod_{j=1}^p (1 + L_j)) \int_0^T \left[ M_2 e^{M_1 (\theta(t) - t)} + M_3 \int_0^t K_2(t,s) ds + M_4 \int_0^T H_2(t,s) ds \right] dt \\
\leq \lambda e^{-M_1 T},
\]

where $K_2(t,s), H_2(t,s)$ are defined in the same way as in Lemma 2, then $u(t) \leq 0$ on $J$.

**Remark 1.** Let $M_3 = M_4 = 0, M_2 > 0$, then Lemmas 3, 4 in this paper are identical with Lemmas 1, 2 in [8].

Consider the problem

\[
\begin{align*}
y'(t) &= -M_1 y(t) - M_2 y(\theta(t)) - M_3 (W y)(t) - M_4 (S y)(t) + \varphi(t), \\
&\quad t \in J, \ t \neq t_k, \ k = 1, 2, \ldots, p, \\
&\Delta y(t_k) = L_k y(t_k) + I_k (u(t_k)) - L_k u(t_k), \quad k = 1, 2, \ldots, p, \\
g(u(0), u(T)) + N_1 (y(0) - u(0)) - N_2 (y(T) - u(T)) = 0
\end{align*}
\]

and

\[
\begin{align*}
y'(t) &= M_1 y(t) + M_2 y(\theta(t)) + M_3 (W y)(t) + M_4 (S y)(t) + \varphi(t), \\
&\quad t \in J, \ t \neq t_k, \ k = 1, 2, \ldots, p, \\
&\Delta y(t_k) = L_k y(t_k) + I_k (u(t_k)) - L_k u(t_k), \quad k = 1, 2, \ldots, p, \\
g(u(0), u(T)) + N_1 (y(0) - u(0)) - N_2 (y(T) - u(T)) = 0.
\end{align*}
\]

By direct computation, we have the following results.

**Lemma 5.** $y \in PC^1(J)$ is a solution of (11) if and only if $y \in PC(J)$ is a solution of the impulsive integral equation

\[
y(t) = C e^{-M_1 t} B u + \int_0^T G(t,s) \left[-M_2 y(\theta(s)) - M_3 (W y)(s) - M_4 (S y)(s) + \varphi(s) \right] ds \\
+ \sum_{0 < t_k < T} G(t,t_k) (L_k y(t_k) + I_k (u(t_k)) - L_k u(t_k)),
\]

where $K_1(t,s), H_1(t,s)$ are defined in the same way as in Lemma 1, then $u(t) \leq 0$ on $J$. 

**Proof.** The proof is similar to the proof of Lemma 1 [8], therefore we omit it. \qed
where \( Bu = -g(u(0), u(T)) + N_1u(0) - N_2u(T) \), \( C = (N_1 - N_2e^{-M_1T})^{-1} \), \( M_i, N_j, i = 1, 2, 3, 4, j = 1, 2 \), are constants with \( M_i \geq 0, i = 1, 2, 3, 4, N_1 \neq N_2e^{-M_1T} \) and

\[
G(t, s) = \begin{cases} 
CN_2e^{-M_1(T+t-s)} + e^{-M_1(t-s)} & 0 \leq s < t \leq T, \\
CN_2e^{-M_1(T+t-s)} & 0 \leq t \leq s \leq T.
\end{cases}
\]

Lemma 6. \( y \in PC^1(J) \) is a solution of (12) if and only if \( y \in PC(J) \) is a solution of the impulsive integral equation

\[
y(t) = Ce^{M_1}Bu + \int_0^T G(t, s)\left[M_2y(\theta(s)) + M_3(Wy)(s) + M_4(Sy)(s) + \varphi(s)\right]ds \\
+ \sum_{0 < t_k < T} G(t, t_k)\left(L_ky(t_k) + I_k(u(t_k)) - L_ku(t_k)\right), \tag{14}
\]

where \( Bu = -g(u(0), u(T)) + N_1u(0) - N_2u(T) \), \( C = (N_1 - N_2e^{M_1T})^{-1} \), \( M_i, N_j, i = 1, 2, 3, 4, j = 1, 2 \), are constants with \( M_i \geq 0, N_1 \neq N_2e^{M_1T} \) and

\[
G(t, s) = \begin{cases} 
CN_2e^{M_1(T+t-s)} + e^{M_1(t-s)} & 0 \leq s < t \leq T, \\
CN_2e^{M_1(T+t-s)} & 0 \leq t \leq s \leq T.
\end{cases}
\]

Lemma 7. If \( N_1 \neq N_2e^{-M_1T} \), with any of \( M_2, M_3, M_4 > 0 \) and the other two constants be non-negative, when

\[
\left(M_2T + M_3K^*T^2 + M_4H^*T^2 + \sum_{k=1}^p |L_k|\right)r < 1, \quad r = \max\{|CN_1|, |CN_2|\},
\]

\[
C = (N_1 - N_2e^{-M_1T})^{-1}
\]

then Eq. (11) has a unique solution \( y \in PC^1(J) \).

Proof. By Lemma 5 and Banach fixed point theorem applied to the right hand of (13), the proof is apparent. \( \square \)

Remark 2. If \( N_1 \geq N_2 > 0 \), with any of \( M_2, M_3, M_4 > 0 \) and the other two constants be non-negative, when

\[
CN_1\left(M_2T + M_3K^*T^2 + M_4H^*T^2 + \sum_{k=1}^p |L_k|\right) < 1, \quad C = (N_1 - N_2e^{-M_1T})^{-1}
\]

from Lemma 7, Eq. (11) has a unique solution \( y \in PC^1(J) \).

Remark 3. Let \( N_2 = 0 \) in (11), then (11) is an initial value problem. \( y \in PC^1(J) \) is a solution of (11) if and only if \( y \in PC(J) \) is a solution of the impulsive integral equation

\[
y(t) = e^{-M_1t}Bu + \int_0^t e^{-M_1(t-s)}\left(-M_2y(\theta(s)) - M_3(Wy)(s) - M_4(Sy)(s) + \varphi(s)\right)ds \\
+ \sum_{0 < t_k < t} e^{-M_1(t-t_k)}\left(L_ky(t_k) + I_k(u(t_k)) - L_ku(t_k)\right), \tag{16}
\]

where \( Bu = -g(u(0), u(T)) + N_1u(0) - N_2u(T) \), \( C = (N_1 - N_2e^{-M_1T})^{-1} \), \( M_i, N_j, i = 1, 2, 3, 4, j = 1, 2 \), are constants with \( M_i \geq 0, i = 1, 2, 3, 4, N_1 \neq N_2e^{-M_1T} \) and
where $Bu = -\frac{1}{N_1} g(u(0), u(T)) + u(0)$, $N_1 \neq 0$. Furthermore, assume that

$$M_2 T + M_3 K^* T^2 + M_4 H^* T^2 + \sum_{k=1}^{p} |L_k| < 1, \quad (17)$$

where $M_i \geq 0$, $i = 2, 3, 4$, then (11) has a unique solution.

Similarly we have

**Lemma 8.** If $N_1 \neq N_2 e^{M_1 T}$, with any of $M_2, M_3, M_4 > 0$ and the other two constants be non-negative

$$
\left( M_2 T + M_3 K^* T^2 + M_4 H^* T^2 + \sum_{k=1}^{p} |L_k| \right) r < 1, \quad r = \max \{ |CN_1 e^{M_1 T}|, |CN_2 e^{M_1 T}| \},
$$

$$C = (N_1 - N_2 e^{M_1 T})^{-1}, \quad (18)$$

then Eq. (12) has a unique solution $y \in PC^1(J)$.

**Remark 4.** If $N_2 \geq N_1 > 0$, with any of $M_2, M_3, M_4 > 0$ and the other two constants be non-negative, when

$$(1 - CN_1) \left( M_2 T + M_3 K^* T^2 + M_4 H^* T^2 + \sum_{k=1}^{p} |L_k| \right) < 1, \quad C = (N_1 - N_2 e^{M_1 T})^{-1}$$

from Lemma 8, Eq. (12) has a unique solution $y \in PC^1(J)$.

**Remark 5.** Let $N_1 = 0$ in (12), then (12) is a terminal value problem. $y \in PC^1(J)$ is a solution of (12) if and only if $y \in PC(J)$ is a solution of the impulsive integral equation

$$
y(t) = e^{-M_1 (T-t)} Bu - \int_t^T e^{M_1 (t-s)} \left( M_2 y(\theta(s)) + M_3 (Wy)(s) + M_4 (Sy)(s) + \varphi(s) \right) ds$$

$$- \sum_{t<t_k<T} e^{M_1 (t-t_k)} (L_k y(t_k) + I_k (u(t_k)) - L_k u(t_k)),
$$

where $Bu = \frac{1}{N_2} g(u(0), u(T)) + u(T)$, $N_2 \neq 0$. Furthermore, assume that

$$
M_2 T + M_3 K^* T^2 + M_4 H^* T^2 + \sum_{k=1}^{p} |L_k| < 1, \quad (19)
$$

where $M_i \geq 0$, $i = 2, 3, 4$, then (12) has a unique solution.

3. Main results

Functions $\alpha, \beta \in PC^1(J)$ are called upper and lower solutions of problem (1) if
\[ \alpha'(t) \geq f(t, \alpha(t), \alpha(\theta(t)), (W\alpha)(t), (S\alpha)(t)), \quad t \neq t_k, \ t \in J, \]
\[ \Delta \alpha(t_k) \geq I_k(\alpha(t_k)), \quad k = 1, 2, \ldots, p, \]
\[ g(\alpha(0), \alpha(T)) \geq 0 \]

and

\[ \beta'(t) \leq f(t, \beta(t), \beta(\theta(t)), (W\beta)(t), (S\beta)(t)), \quad t \neq t_k, \ t \in J, \]
\[ \Delta \beta(t_k) \leq I_k(\beta(t_k)), \quad k = 1, 2, \ldots, p, \]
\[ g(\beta(0), \beta(T)) \leq 0. \]

**Theorem 1.** Let \( 0 \leq \theta, \delta, \sigma \leq t, \) (3), (17) hold, let \( \alpha(t), \beta(t) \in PC^1(J) \) be upper and lower solutions of (1) and \( \beta(t) \leq \alpha(t) \) and assume that

- \( A_1 \) The function \( f \in C(J \times R^4, R) \) satisfies
  \[ f(t, x_1, x_2, x_3, x_4) - f(t, y_1, y_2, y_3, y_4) \]
  \[ \geq -M_1(x_1 - y_1) - M_2(x_2 - y_2) - M_3(x_3 - y_3) - M_4(x_4 - y_4), \]
  \[ \beta \leq y_1 \leq x_1 \leq \alpha, \quad \beta(\theta(t)) \leq y_2 \leq x_2 \leq \alpha(\theta(t)), \quad W\beta \leq y_3 \leq x_3 \leq W\alpha, \]
  \[ S\beta \leq y_4 \leq x_4 \leq S\alpha, \quad t \in J, \]
where \( M_i \geq 0, i = 1, 2, 3, 4. \)

- \( A_2 \) The function \( I_k \in C(R, R) \) satisfies
  \[ I_k(x) - I_k(y) \leq L_k(x - y), \]
wherever \( \beta(t_k) \leq y \leq x \leq \alpha(t_k) \), where \( -1 < L_k \leq 0 \) and \( k = 1, 2, \ldots, p. \)

- \( A_3 \) The function \( g(x, y) \in C(R, R) \) is nonincreasing in the second variable and satisfies
  \[ g(x_1, y) - g(x_2, y) \leq N_1(x_1 - x_2), \]
wherever \( \beta(0) \leq x_2 \leq x_1 \leq \alpha(0), N_1 > 0. \)

Then there exist monotone sequences \( \{\alpha_n(t)\}, \{\beta_n(t)\} \) with \( \alpha_0(t) = \alpha(t), \beta_0(t) = \beta(t) \), which converge in \( PC^1(J) \) to the extremal solutions of (1) in \( \beta, \alpha \), \( \beta, \alpha = \{x \in PC^1(J): \beta(t) \leq x(t) \leq \alpha(t), \ t \in J\}. \)

**Proof.** For any \( u \in [\beta, \alpha] \), consider (16) with

\[ \varphi(t) = f(t, u(t), u(\theta(t)), (Wu)(t), (Su)(t)) + M_1u(t) + M_2u(\theta(t)) + M_3(Wu)(t) \]
\[ + M_4(Su)(t). \]

By Remark 2, (16) has a unique solution \( y \in PC^1(J) \). Denote an operator \( A: PC(J) \rightarrow PC(J) \) by \( y = Au \), then the operator \( A \) has the following properties:

- (i) \( \beta_0 \leq A\beta_0, A\alpha_0 \leq \alpha_0 \).
- Set \( m = \beta_0 - \beta_1 \), where \( \beta_1 = A\beta_0 \).
\[ m'(t) = \beta'_0(t) - \beta'_1(t) \leq -M_1 m(t) - M_2 m(\theta(t)) - M_3 (Wm)(t) - M_4 (Sm)(t), \quad t \neq t_k, \ t \in J, \]
\[ \Delta m(t_k) = \Delta \beta_0(t_k) - \Delta \beta_1(t_k) \leq I_k(\beta_0(t_k)) - (L_k \beta_1(t_k) + I_k(\beta_0(t_k)) - L_k \beta_0(t_k)) = L_k m(t_k), \quad k = 1, 2, \ldots, p, \]
\[ m(0) = \beta_0(0) - \beta_1(0) = \beta_0(0) - \left[ -\frac{1}{N_1} g(\beta_0(0), \beta_0(T)) + \beta_0(0) \right] \leq 0. \]

By Lemma 1, we get \( m(t) \leq 0 \) when \( t \in J \), i.e., \( \beta_0 \leq A \beta_0 \). Similar arguments show that \( A \alpha_0 \leq \alpha_0 \).

(ii) \( A \eta_1 \leq A \eta_2 \), if \( \beta \leq \eta_1 \leq \eta_2 \leq \alpha \).

Let \( u_1 = A \eta_1, u_2 = A \eta_2 \), set \( m = u_1 - u_2 \). Using \( A_1, A_2 \) and \( A_3 \), we get
\[ m'(t) = u'_1(t) - u'_2(t) \leq -M_1 m(t) - M_2 m(\theta(t)) - M_3 (Wm)(t) - M_4 (Sm)(t), \quad t \neq t_k, \ t \in J, \]
\[ \Delta m(t_k) = \Delta u_1(t_k) - \Delta u_2(t_k) = L_k u_1(t_k) + I_k(\eta_1(t_k)) - L_k \eta_1(t_k) - \left[ L_k u_2(t_k) + I_k(\eta_2(t_k)) - L_k \eta_2(t_k) \right] \leq L_k m(t_k), \quad k = 1, 2, \ldots, p, \]
\[ m(0) = u_1(0) - u_2(0) = -\frac{1}{N_1} g(\eta_1(0), \eta_1(T)) + \eta_1(0) + \frac{1}{N_1} g(\eta_2(0), \eta_2(T)) - \eta_2(0) \leq 0 \]
from Lemma 1, we have \( m(t) \leq 0 \) when \( t \in J \), i.e., \( A \eta_1 \leq A \eta_2 \). From (i) and (ii), we get \( \beta_0 \leq A \beta_0 \leq A \alpha_0 \leq \alpha_0 \), and it is apparent that \( A \beta_0, A \alpha_0 \) are lower and upper solutions of (1), respectively.

Now let \( \alpha_n = A \alpha_{n-1}, \beta_n = A \beta_{n-1}, n = 1, 2, \ldots \). Following (i) and (ii), we have
\[ \beta_0 \leq \beta_1 \leq \cdots \leq \beta_n \leq \cdots \leq \alpha_n \leq \cdots \leq \alpha_1 \leq \alpha_0. \]

Obviously, each \( \alpha_i, \beta_i \ (i = 1, 2, \ldots) \) satisfies
\[ \left\{ \begin{array}{l}
\alpha'_n(t) = F(\alpha_{n-1}(t), \alpha_n(t)), \\
\Delta \alpha_n(t_k) = L_k \alpha_n(t_k) + I_k(\alpha_{n-1}(t_k)) - L_k \alpha_{n-1}(t_k), \\
g(\alpha_{n-1}(0), \alpha_n(T)) + N_1(\alpha_n(0) - \alpha_{n-1}(0)) = 0
\end{array} \right. \]
and
\[ \left\{ \begin{array}{l}
\beta'_n(t) = F(\beta_{n-1}(t), \beta_n(t)), \\
\Delta \beta_n(t_k) = L_k \beta_n(t_k) + I_k(\beta_{n-1}(t_k)) - L_k \beta_{n-1}(t_k), \\
g(\beta_{n-1}(0), \beta_n(T)) + N_1(\beta_n(0) - \beta_{n-1}(0)) = 0
\end{array} \right. \]
with \( F \) defined by
\[ F(x(t), y(t)) = f(t, x(t), x(\theta(t)), (Wx)(t), (Sx)(t)) + M_1(x(t) - y(t)) + M_2(\theta(t) - x(\theta(t))) + M_3((Wx)(t) - (Wy)(t)) + M_4((Sx)(t) - (Sy)(t)) \]
Theorem 2. Therefore there exist \( x^*, x_* \), such that \( \lim_{n \to \infty} \alpha_n(t) = x^* \), \( \lim_{n \to \infty} \beta_n(t) = x_* \), uniformly on \( J \). Clearly, \( x^* \) and \( x_* \) satisfy (1).

To prove that \( x^*, x_* \) are extremal solutions of (1), let \( x(t) \) be any solution of (1) such that \( \beta \leq x(t) \leq \alpha \).

Suppose there exists a positive integer \( n \) such that \( \beta_n(t) \leq x(t) \leq \alpha_n(t) \), for \( t \in J \).

Setting \( m(t) = \beta_{n+1}(t) - x(t) \), then for \( t \in J \)

\[
m'(t) = \beta'_{n+1}(t) - x'(t) 
\]

\[
\leq -M_1 m(t) - M_2 m(\theta(t)) - M_3(Wm)(t) - M_4(Sm)(t), \quad t \neq t_k, \ t \in J,
\]

\[
\Delta m(t_k) = \Delta \beta_{n+1}(t_k) - \Delta x(t_k) 
\]

\[
= L_k \beta_{n+1}(t_k) + I_k(\beta_n(t_k)) - L_k \beta_n(t_k) - I_k(x(t_k)) \leq L_k m(t_k), \quad k = 1, 2, \ldots, p,
\]

\[
m(0) = -\frac{1}{N_1} g(\beta_n(0), \beta_n(T)) + \beta_n(0) - x(0) \leq 0.
\]

By Lemma 1, \( m(t) \leq 0 \), when \( t \in J \), i.e., \( \beta_{n+1}(t) \leq x(t) \). Similarly, we get \( x(t) \leq \alpha_{n+1}(t) \) on \( t \in J \). By induction we acquire \( \beta_{n+1}(t) \leq x(t) \leq \alpha_{n+1}(t) \), for all \( t \in J \) and any \( n \), which implies \( x_n(t) \leq x(t) \leq x^*(t) \). This completes the proof of the theorem. \( \square \)

Similarly we have

Theorem 2. \( t \leq \theta, \delta, \sigma \leq T, (6), (19) \) hold, let \( \alpha(t), \beta(t) \in PC^1(J) \) be upper and lower solutions of (1) and \( \alpha(t) \leq \beta(t) \) and assume that

\( (A'_1) \) The function \( f \in C(J \times R^2, R) \) satisfies

\[
f(t, y_1, y_2, y_3, y_4) - f(t, x_1, x_2, x_3, x_4) 
\]

\[
\geq -M_1(x_1 - y_1) - M_2(x_2 - y_2) - M_3(x_3 - y_3) - M_4(x_4 - y_4),
\]

\( \beta \leq y_1 \leq x_1 \leq \alpha, \quad \beta(\theta(t)) \leq y_2 \leq x_2 \leq \alpha(\theta(t)), \quad W\beta \leq y_3 \leq x_3 \leq W\alpha,
\]

\( S\beta \leq y_4 \leq x_4 \leq S\alpha, \quad t \in J,
\]

where \( M_i \geq 0, i = 1, 2, 3, 4. \)

\( (A'_2) \) The function \( I_k \in C(R, R) \) satisfies

\[
I_k(x) - I_k(y) \geq -L_k(y - x),
\]

wherever \( \alpha(t_k) \leq x \leq y \leq \beta(t_k) \), where \( L_k \geq 0 \) and \( k = 1, 2, \ldots, p. \)

\( (A'_3) \) The function \( g(x, y) \in C(R, R) \) is nondecreasing in the first variable and satisfies

\[
g(x, y_1) - g(x, y_2) \leq N_2(y_2 - y_1),
\]

wherever \( \alpha(T) \leq y_1 \leq y_2 \leq \beta(T), N_2 > 0. \)

Then there exist monotone sequences \( \{\alpha_n(t)\}, \{\beta_n(t)\} \) with \( \alpha_0(t) = \alpha(t), \beta_0(t) = \beta(t) \), which converge in \( PC^1(J) \) to the extremal solutions of (1) in \( [\alpha, \beta] \), \( [\alpha, \beta] = \{x \in PC^1(J) : \alpha(t) \leq x(t) \leq \beta(t), t \in J\}. \)

Theorem 3. Assume that conditions \( (A_1)-(A_2), (8), (15) \) hold, \( M_1 \geq 0 \), with any of \( M_2, M_3, M_4 > 0 \) and the other two constants be nonnegative, \( \lambda = \frac{N_2}{N_1} \neq e^{M_1T} \) and
Let $\alpha(t), \beta(t) \in PC^1(J)$ be upper and lower solutions of (1) and $\beta(t) \leq \alpha(t)$. Then there exist monotone sequences $\{\alpha_n(t)\}, \{\beta_n(t)\}$ with $\alpha_0(t) = \alpha(t), \beta_0(t) = \beta(t)$, which converge in $PC^1(J)$ to the extremal solutions of (1) in $[\beta, \alpha], [\beta, \alpha] = \{x \in PC^1(J): \beta(t) \leq x(t) \leq \alpha(t), t \in J\}$.

**Theorem 4.** Assume that conditions $(A_1')-(A_2'), (10), (18)$ hold, $M_1 \geq 0$, with any of $M_2, M_3, M_4 > 0$ and the other two constants be nonnegative, $\lambda = \frac{N_1}{N_2} \neq e^{M_1 T}$ and

$(A_4')$ The function $g(x, y) \in C(R, R)$ satisfies

$$g(x_1, y_1) - g(x_2, y_2) \leq N_1 (x_1 - x_2) + N_2 (y_2 - y_1),$$

wherever $\beta(0) \leq x_2 \leq x_1 \leq \alpha(0), \alpha(T) \leq y_2 \leq y_1 \leq \alpha(T), \text{ where } N_2 \geq N_1 > 0$.

Let $\alpha(t), \beta(t) \in PC^1(J)$ be upper and lower solutions of (1) and $\alpha(t) \leq \beta(t)$ then there exist monotone sequences $\{\alpha_n(t)\}, \{\beta_n(t)\}$ with $\alpha_0(t) = \alpha(t), \beta_0(t) = \beta(t)$, which converge in $PC^1(J)$ to the extremal solutions of (1) in $[\alpha, \beta], [\alpha, \beta] = \{x \in PC^1(J): \alpha(t) \leq x(t) \leq \beta(t), t \in J\}$.

4. Examples

**Example 1.** Consider the problem of

$$x'(t) = 2x(t) + e^{\frac{1}{2} (t-5)} e^{x(2t)} + e^{\frac{1}{2} t} \int_0^T \frac{\gamma(t)}{s} e^{-2\sqrt{s}} x(\sqrt{s}) ds$$

$$- \int_0^t \left( 1 - \sqrt{s} \right) e^{-2\sqrt{s}} x(\sqrt{s}) ds - e^{\frac{1}{2} (t-5)} , \quad t \in [0, T], \ t \neq t_k,$$

$$\Delta x(t_k) = \frac{1}{3} x(t_k), \quad t = t_k,$$

$$x(0) = \frac{6}{7} x(T) - \frac{1}{7},$$

(20)

here $T = \frac{1}{3}, k = 1, \theta(t) = 2t, \delta(t) = \sigma(t) = \sqrt{t}, \gamma(t) \in C(J, J)$. Set

$$\alpha(t) = 0, \quad \beta(t) = \begin{cases} 1 + t, & t \in [0, \frac{T}{2}], \\ 1 + 3t, & t \in (\frac{T}{2}, T]. \end{cases}$$

we can easily verify that $\alpha(t)$ is an upper solution, $\beta(t)$ is a lower solution with $\alpha(t) \leq \beta(t)$.

Set $M_1 = 2, M_2 = e^{-\frac{1}{4}}, M_3 = 0, M_4 = e^{\frac{1}{2}}, L_1 = \frac{1}{3}, N_2 = \frac{6}{7}$, the conditions of Theorem 2 are all satisfied. So problem (20) has extremal solutions in the segment $[\alpha(t), \beta(t)]$.

On the other hand, we cannot use Theorem 4 to obtain the result, for some conditions of Theorem 4 such as (18) are not satisfied.
Example 2. Consider the problem of

\[ x'(t) = 2x(t) + e^{\frac{1}{2}(t-10)}e^{x(t/2)} + e^{\frac{1}{2}t} \int_0^T \left( 1 - \frac{s}{2} \right) e^{-s} x\left( \frac{s}{2} \right) ds \]

\[ - \int_0^t \left( 1 - \frac{s}{2} \right) e^{-s} x\left( \frac{s}{2} \right) ds - e^{\frac{1}{2}(t-10)}, \quad t \in [0, T], \ t \neq t_k, \]

\[ \Delta x(t_k) = \frac{1}{20} x(t_k), \quad t = t_k, \]

\[ x(0) = \frac{5}{7} x(T) - \frac{1}{4}, \] (21)

Here \( T = \frac{1}{3}, \ k = 1, \ \theta(t) = \delta(t) = \sigma(t) = \frac{t}{2}, \ \gamma(t) \in C(J, J) \).

Set

\[ \alpha(t) = 0, \quad \beta(t) = \begin{cases} 1 + t, & t \in [0, \frac{T}{2}], \\ 1 + \frac{t}{2}, & t \in (\frac{T}{2}, T], \end{cases} \]

we can easily verify that \( \alpha(t) \) is an upper solution, \( \beta(t) \) is a lower solution with \( \alpha(t) \leq \beta(t) \).

Set \( M_1 = 2, \ M_2 = e^{-\frac{11}{3}}, \ M_3 = 0, \ M_4 = e^{\frac{1}{6}}, \ L_1 = \frac{1}{20}, \ N_1 = 1, \ N_2 = \frac{5}{7} \), the conditions of Theorem 4 are all satisfied. So problem (21) has extremal solutions in the segment \([\alpha(t), \beta(t)]\).

On the other hand, we cannot use Theorem 2 to obtain the result, for \( 0 \leq \theta(t), \ \delta(t), \ \sigma(t) \leq t \).

References