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Matrix regular operator space and operator system

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ABSTRACT

We establish a relationship between Schreiner's matrix regular operator space and Werner's (nonunital) operator system. For a matrix ordered operator space V with complete norm, we show that V is completely isomorphic and complete order isomorphic to a matrix regular operator space if and only if both V and its dual space V^* are (nonunital) operator systems.

1. Introduction

The theory of operator spaces has been developed as a noncommutative counterpart of the theory of Banach spaces. The order structures of classical Banach spaces have been studied mostly under the Banach lattice framework. However, the most basic examples of operator algebras such as M_n $(n \geqslant 2)$ do not possess these lattice structures (consider 2×2 matrices 0_2 and $E_{1,2} + E_{2,1}$ and the family $\lambda E_{1,1} + \lambda^{-1} E_{2,2}$ for $\lambda > 0$). Hence, it is natural to consider order structures that will work in a noncommutative setting. In this paper, we focus on such order structures of operator spaces.

Two of the several basics of operator space theory are Ruan's representation theorem and the duality: every operator space can be embedded into $B(H)$ completely isometrically and there is a natural operator space structure on the Banach dual of an operator space. From this standpoint, there are two definitions of the order structures of operator spaces. Werner's (nonunital) operator system corresponds to the representation and Schreiner's matrix regular operator space to the duality. Usually, an operator system means a unital involutive subspace of $B(H)$ or its abstract characterization given by Choi and Effros [3], but here we follow Werner's terminology. In this paper, a (nonunital) operator system means a matrix ordered operator space which is completely isomorphic and complete order isomorphic to an involutive subspace of $B(H)$ or its abstract characterization given by Werner [12]. For a matrix ordered operator space V with complete norm, V is matrix regular if and only if its dual space V^* is matrix regular [11].

The category of (nonunital) operator systems contains the class of C^*-algebras and Haagerup's noncommutative L_p-spaces [6]. The category of matrix regular operator spaces contains the class of C^*-algebras and their duals, preduals of von Neumann algebras, and the Schatten class $\mathcal{S}_p$ [10].

Karn proved that every matrix regular operator space is a (nonunital) operator system [7]. Since the dual space of a matrix regular operator space is matrix regular, the dual space of a matrix regular operator space is also a (nonunital) operator system. Its converse would be reasonable in the completely isomorphic context because Werner's (nonunital) operator system is defined not in a completely isometric sense but in a completely isomorphic sense. The purpose of this paper is to show that a matrix ordered operator space V with complete norm is completely isomorphic and complete order isomorphic to a matrix regular operator space if and only if both V and its dual space V^* are (nonunital) operator systems.

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2. Preliminaries

Recall that a complex vector space V is matrix ordered if

(1) V is a $*$-vector space (hence so is $M_n(V)$ for all $n \geqslant 1$),
(2) each $M_n(V)$, $n \geqslant 1$, is partially ordered by a (not necessarily proper) cone $M_n(V)^+ \subset M_n(V)_{sa}$, and
(3) if $\alpha \in M_{m,n}$, then $\alpha^* M_m(V)^+ \alpha \subset M_n(V)^+$.

Here the positive cone need not be proper, in other words, it may be the case that $M_n(V)^+ \cap -M_n(V)^+ \neq \{0\}$. The reason to exclude the proper condition is due to the fact that the dual cone of a proper cone need not be proper.

An operator space V is called a matrix ordered operator space iff V is a matrix ordered vector space and for every $n \in \mathbb{N}$,

(1) the $*$-operation is an isometry on $M_n(V)$, and
(2) the cones $M_n(V)^+$ are closed.

For a matrix ordered operator space V and its dual space V^*, the positive cone on $M_n(V^*)$ for each $n \in \mathbb{N}$ is defined by

$$M_n\big(V^*\big)^+ = CB(V, M_n) \cap CP(V, M_n).$$

Then the operator space dual V^* with this positive cone is a matrix ordered operator space [11, Corollary 3.2].

For a matrix ordered operator space V with complete norm, we say that V is a matrix regular operator space if for each $n \in \mathbb{N}$ and for all $v \in M_n(V)_{sa}$

(1) $u \in M_n(V)^+$ and $-u \leqslant v \leqslant u$ imply that $\|v\|_n \leqslant \|u\|_n$, and
(2) $\|v\|_n < 1$ implies that there exists $u \in M_n(V)^+$ such that $\|u\|_n < 1$ and $-u \leqslant v \leqslant u$.

Due to condition (1), it is easily seen that the positive cone of a matrix regular operator space is always proper. A matrix regular operator space can be described in another way. A matrix ordered operator space V with complete norm is matrix regular if and only if the following condition holds: for all $x \in M_n(V)$, $\|x\|_n < 1$ if and only if there exist $a, d \in M_n(V)^+$, $\|a\|_n < 1$ and $\|d\|_n < 1$, such that $\left(\begin{smallmatrix} a & x \\ x^* & d \end{smallmatrix}\right) \in M_{2n}(V)^+$ [11, Theorem 3.4].

The class of matrix regular operator spaces has a nice duality property. Let V be a matrix ordered operator space with complete norm. Schreiner showed that V is matrix regular if and only if its dual space V^* is matrix regular [11, Corollary 4.7, Theorem 4.10].

Let X be a matrix ordered operator space with a proper positive cone. For $x \in M_n(X)$, the modified numerical radius is defined by

$$\nu_X(x) = \sup\left\{ \left| \varphi\left(\begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \right) \right| : \varphi \in M_{2n}(X)^*_{1,+} \right\}.$$

We call a matrix ordered operator space with a proper positive cone an operator system iff there is a $k > 0$ such that for all $n \in \mathbb{N}$ and $x \in M_n(X)$,

$$\|x\|_n \leqslant k \nu_X(x).$$

Since we always have the inequality $\nu_X(x) \leqslant \|x\|_n$, we can say that an operator system is a matrix ordered operator space such that the operator space norm and the modified numerical radius are equivalent uniformly for all $n \in \mathbb{N}$.

Werner showed that X is an operator system if and only if there is a complete order isomorphism Φ from X onto an involutive subspace of $B(H)$, which is a complete topological onto-isomorphism [12, Theorem 4.15]. Hence, the operator system is an abstract characterization of the involutive subspace of $B(H)$ in a completely isomorphic and complete order isomorphic sense.

3. Matrix regular operator space and operator system

Karn showed that every matrix regular operator space can be embedded into $B(H)$ 2-completely isomorphically and complete order isomorphically [7]. Here we give another proof of the result that is more fitting to Werner's axiomatic framework. This idea also appears in the recent preprint [8, Theorem 2.4].

Theorem 1. *Every matrix regular operator space is an operator system with a dominating constant* 2.

Proof. Since the dual space of a matrix regular operator space is matrix regular and the canonical inclusion map from a matrix ordered operator space into its bidual is a completely isometric complete order isomorphism [11, Theorem 4.9], it is sufficient to show that the dual space of a matrix regular operator space is an operator system.

Suppose that V is a matrix regular operator space. We choose an element $F=[f_{ij}]$ in $M_n(V^*)$. Its norm can be written as

$$\begin{aligned}
\|F\|_{M_n(V^*)} &= \|F\|_{CB(V,M_n)} \\
&= \sup\bigl\{\bigl\|F_n(x)\bigr\|_{M_{n^2}}\colon x\in M_n(V)_{\|\cdot\|<1}\bigr\} \\
&= \sup\left\{\left\|\begin{pmatrix} 0 & F_n(x) \\ F_n(x)^* & 0\end{pmatrix}\right\|_{M_{2n^2}}\colon x\in M_n(V)_{\|\cdot\|<1}\right\} \\
&= \sup\left\{\left|\left\langle\begin{pmatrix} 0 & F_n(x) \\ F_n(x)^* & 0\end{pmatrix}\xi\,\Big|\,\xi\right\rangle\right|\colon x\in M_n(V)_{\|\cdot\|<1},\ \xi\in\bigl(\ell^2_{2n^2}\bigr)_1\right\}.
\end{aligned}$$

We choose elements x in $M_n(V)_{\|\cdot\|<1}$ and ξ in $(\ell^2_{2n^2})_1$. By [11, Theorem 3.4], there exist a, d in $M_n(V)$ such that $\|a\|_n<1$, $\|d\|_n<1$ and $\begin{pmatrix} a & x \\ x^* & d\end{pmatrix}\in M_{2n}(V)^+$. Then we have $\left\|\begin{pmatrix} a & x \\ x^* & d\end{pmatrix}\right\|_{2n}<2$. We define a linear functional $\varphi_{x,\xi}: M_{2n}(V^*)\to\mathbb{C}$ by

$$\varphi_{x,\xi}(G)=\frac{1}{2}\left\langle\begin{pmatrix} I_{n^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n^2}\end{pmatrix} G_{2n}\left(\begin{pmatrix} a & x \\ x^* & d\end{pmatrix}\right)\begin{pmatrix} I_{n^2} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & I_{n^2}\end{pmatrix}\xi\,\Big|\,\xi\right\rangle,\quad G\in M_{2n}\bigl(V^*\bigr)=CB(V,M_{2n}).$$

Then $\varphi_{x,\xi}$ is a positive contractive functional. Putting $G=\begin{pmatrix} 0 & F \\ F^* & 0\end{pmatrix}$, we get

$$\begin{aligned}
\varphi_{x,\xi}\left(\begin{pmatrix} 0 & F \\ F^* & 0\end{pmatrix}\right) &= \frac{1}{2}\left\langle\begin{pmatrix} I_{n^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n^2}\end{pmatrix}\begin{pmatrix} 0 & 0 & F_n(a) & F_n(x) \\ 0 & 0 & F_n(x^*) & F_n(d) \\ F_n^*(a) & F_n^*(x) & 0 & 0 \\ F_n(x)^* & F_n^*(z) & 0 & 0\end{pmatrix}\begin{pmatrix} I_{n^2} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & I_{n^2}\end{pmatrix}\xi\,\Big|\,\xi\right\rangle \\
&= \frac{1}{2}\left\langle\begin{pmatrix} 0 & F_n(x) \\ F_n(x)^* & 0\end{pmatrix}\xi\,\Big|\,\xi\right\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|F\|_{M_n(V^*)} &= 2\sup\left\{\left|\varphi_{x,\xi}\left(\begin{pmatrix} 0 & F \\ F^* & 0\end{pmatrix}\right)\right|\colon x\in M_n(V)_{\|\cdot\|<1},\ \xi\in\bigl(\ell^2_{2n}\bigr)_1\right\} \\
&\leqslant 2\nu_{V^*}(F),
\end{aligned}$$

where $\nu_{V^*}(F)$ denotes the modified numerical radius of F. $\square$

In general, the embedding cannot be chosen completely isometrically and complete order isomorphically as can be seen from the two-dimensional L^1-space ℓ^1_2 [2, Proposition 1.1]. In the case of the Schatten class $\mathcal{S}_p$, the constant can be chosen to be $2^{\frac{1}{p}}$ [6].

The direct converse of Theorem 1 is false. If we consider the operator system

$$\left\{\begin{pmatrix} 0 & \alpha \\ \beta & 0\end{pmatrix}\in M_{2n}\colon \alpha,\beta\in M_n\right\},$$

then its positive cone is trivial, thus the second condition on matrix regularity cannot be satisfied for any operator system complete order isomorphic to the above one. Because the dual space of a matrix regular operator space is also matrix regular, the matrix regularity of V implies that both V and its dual space V^* are operator systems. Our next goal is to prove the converse in a completely isomorphic and complete order isomorphic sense. Informally, the first and the second conditions on matrix regularity imply that the positive cone is small and large, respectively, in some sense. In other words, we can say that the positive cone of a matrix regular operator space is just the right size. The definition of an operator system means the dual cone is large enough, or equivalently, that the positive cone of an operator system is small enough. As we have just seen, the positive cone of an operator system may be trivial. We see from these informal observations that it is natural to consider the problem set forth as the goal of the present paper.

Lemma 2. *Suppose that V is a matrix ordered operator space with complete norm satisfying the following two conditions*:

(1) *for all $x,y\in M_n(V)_{sa}$, $-y\leqslant x\leqslant y$ implies $\|x\|_n\leqslant\|y\|_n$,*
(2) *for all $x\in M_n(V)_{sa}$ with $\|x\|<1$, there exist $a,d\in M_n(V)^+$ such that $\|a\|_n,\|d\|_n<K$ and $\begin{pmatrix} a & x \\ x^* & d\end{pmatrix}\in M_{2n}(V)^+$.*

Then V is K-completely isomorphic and complete order isomorphic to a matrix regular operator space.

Proof. We first define

$$\|x\|_{reg} = \inf\left\{\max\{\|a\|_n, \|d\|_n\}: \begin{pmatrix} a & x \\ x^* & d \end{pmatrix} \in M_{2n}(V)^+\right\}, \quad x \in M_n(V).$$

Note that this definition is similar to the norm $\|\cdot\|_{dec}$ of a decomposable map [5]. The set which we take an infimum over is not empty and we have $\|x\|_{reg} \leqslant K\|x\|_n$. Multiplying both sides by the scalar matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, we see that

$$\begin{pmatrix} a & x \\ x^* & d \end{pmatrix} \in M_{2n}^+ \quad \text{if and only if} \quad -\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \leqslant \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \leqslant \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

The inequality $\|x\|_n \leqslant \|x\|_{reg}$ follows from condition (1). We choose elements $\left(\begin{smallmatrix} a_1 & x \\ x^* & d_1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a_2 & y \\ y^* & d_2 \end{smallmatrix}\right)$ in $M_{2n}(V)^+$. Since $\left(\begin{smallmatrix} a_1+a_2 & x+y \\ (x+y)^* & d_1+d_2 \end{smallmatrix}\right)$ belongs to $M_{2n}(V)^+$, we have

$$\begin{aligned}
\|x+y\|_{reg} &\leqslant \max\{\|a_1+a_2\|_n, \|d_1+d_2\|_n\} \\
&= \left\| \begin{pmatrix} a_1+a_2 & 0 \\ 0 & d_1+d_2 \end{pmatrix} \right\|_{2n} \\
&\leqslant \left\| \begin{pmatrix} a_1 & 0 \\ 0 & d_1 \end{pmatrix} \right\|_{2n} + \left\| \begin{pmatrix} a_2 & 0 \\ 0 & d_2 \end{pmatrix} \right\|_{2n} \\
&= \max\{\|a_1\|_n, \|d_1\|_n\} + \max\{\|a_2\|_n, \|d_2\|_n\}.
\end{aligned}$$

It follows that $\|x+y\|_{reg} \leqslant \|x\|_{reg} + \|y\|_{reg}$. For $\lambda = e^{it}|\lambda| \in \mathbb{C}$, we have

$$\begin{pmatrix} |\lambda|a & \lambda x \\ (\lambda x)^* & |\lambda|b \end{pmatrix} = \begin{pmatrix} e^{it}|\lambda|^{\frac{1}{2}} & 0 \\ 0 & |\lambda|^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} a & x \\ x^* & d \end{pmatrix} \begin{pmatrix} e^{-it}|\lambda|^{\frac{1}{2}} & 0 \\ 0 & |\lambda|^{\frac{1}{2}} \end{pmatrix} \in M_{2n}(V)^+.$$

It follows that $\|\lambda x\|_{reg} = |\lambda|\|x\|_{reg}$. Because we have $\|x\|_n \leqslant \|x\|_{reg}$ for all $x \in M_n(V)$, $\|x\|_{reg} = 0$ implies $x = 0$. Hence, $\|\cdot\|_{reg}$ is a norm on $M_n(V)$ for each $n \in \mathbb{N}$.

Next let us show that $\|\alpha x\beta\|_{reg} \leqslant \|\alpha\|\|x\|_{reg}\|\beta\|$ for $x \in M_m(V)$, $\alpha \in M_{n,m}$, $\beta \in M_{m,n}$. To this end, we may assume that $\|\alpha\| = \|\beta\|$. For $\left(\begin{smallmatrix} a & x \\ x^* & d \end{smallmatrix}\right) \in M_{2m}(V)^+$, we have

$$\begin{pmatrix} \alpha a\alpha^* & \alpha x\beta \\ (\alpha x\beta)^* & \beta^* d\beta \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta^* \end{pmatrix} \begin{pmatrix} a & x \\ x^* & d \end{pmatrix} \begin{pmatrix} \alpha^* & 0 \\ 0 & \beta \end{pmatrix} \in M_{2n}(V)^+.$$

It follows that

$$\begin{aligned}
\|\alpha x\beta\|_{reg} &\leqslant \max\{\|\alpha a\alpha^*\|_n, \|\beta^* d\beta\|_n\} \\
&\leqslant \|\alpha\|\|\beta\| \max\{\|a\|_m, \|d\|_m\},
\end{aligned}$$

thus $\|\alpha x\beta\|_{reg} \leqslant \|\alpha\|\|\beta\|\|x\|_{reg}$. Suppose that

$$\begin{pmatrix} a_1 & x \\ x^* & d_1 \end{pmatrix} \in M_{2m}(V)^+ \quad \text{and} \quad \begin{pmatrix} a_2 & y \\ y^* & d_2 \end{pmatrix} \in M_{2n}(V)^+$$

with $\|a_1\|_m, \|d_1\|_m < \|x\|_{reg} + \varepsilon$ and $\|a_2\|_n, \|d_2\|_n < \|y\|_{reg} + \varepsilon$. Then we have

$$\begin{pmatrix} a_1 & 0 & x & 0 \\ 0 & a_2 & 0 & y \\ x^* & 0 & d_1 & 0 \\ 0 & y^* & 0 & d_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & x & 0 & 0 \\ x^* & d_1 & 0 & 0 \\ 0 & 0 & a_2 & y \\ 0 & 0 & y^* & d_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in M_{2(m+n)}(V)^+.$$

It follows that

$$\|x \oplus y\|_{reg} \leqslant \max\{\|a_1\|_m, \|a_2\|_n, \|d_1\|_m, \|d_2\|_n\} < \max\{\|x\|_{reg}, \|y\|_{reg}\} + \varepsilon.$$

Hence, $(V, \|\cdot\|_{reg})$ is an operator space.

For $\left(\begin{smallmatrix} a & x \\ x^* & d \end{smallmatrix}\right) \in M_{2n}(V)^+$, we have

$$\begin{pmatrix} d & x^* \\ x & a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & x \\ x^* & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_{2n}(V)^+,$$

thus $\|x\|_{reg} = \|x^*\|_{reg}$. Since the identity map $id: (V, \|\cdot\|) \to (V, \|\cdot\|_{reg})$ is a K-complete isomorphism, the operator space $(V, \|\cdot\|_{reg})$ is complete and the positive cone $M_n(V)^+$ is closed with respect to the norm $\|\cdot\|_{reg}$. For $a \in M_n(V)^+$, we have

$$\begin{pmatrix} a & a \\ a & a \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} a \,(1 \quad 1) \in M_{2n}(V)^+,$$

thus $\|a\|_{reg} = \|a\|_n$. If an element x belongs to $M_n(V)$ with $\|x\|_{reg} < 1$, then there exist elements a and d in $M_n(V)^+$ such that $\|a\|_n$, $\|d\|_n < 1$ and $\left(\begin{smallmatrix} a & x \\ x^* & d \end{smallmatrix}\right) \in M_{2n}(V)^+$. Since a and d are positive, we have $\|a\|_{reg}$, $\|d\|_{reg} < 1$. The converse is obvious. By [11, Theorem 3.4], $(V, \|\cdot\|_{reg}, \{M_n(V)^+\}_{n\in\mathbb{N}})$ is a matrix regular operator space. The identity map $id : (V, \|\cdot\|, \{M_n(V)^+\}_{n\in\mathbb{N}}) \to (V, \|\cdot\|_{reg}, \{M_n(V)^+\}_{n\in\mathbb{N}})$ is a K-completely isomorphic complete order isomorphism. □

Lemma 3. *Suppose that a matrix ordered operator space V and its dual space V^* are operator systems. Then the dual space V^* is completely isomorphic and complete order isomorphic to a matrix regular operator space.*

Proof. By looking at the dual space V^* as a $*$-subspace of $B(K)$, we can say that condition (1) of Lemma 2 is satisfied. Suppose that W is a $*$-subspace of $B(H)$ and $\Phi : W \to V$ is a completely isomorphic complete order isomorphism with $\|\Phi\|_{cb} \leqslant 1$. We put

$$X = \left\{ \begin{pmatrix} \lambda I_H & x \\ y & \mu I_H \end{pmatrix} :\ \lambda, \mu \in \mathbb{C},\ x, y \in W \right\}.$$

Then X is a unital operator system in $B(H^2)$. We take an element F in $M_n(V^*)$ with $\|F\|_{M_n(V^*)} < 1$. By [9, Lemma 8.1], the linear map $\varphi : X \to M_{2n}$ defined by

$$\varphi\left(\begin{pmatrix} \lambda I_H & x \\ y & \mu I_H \end{pmatrix}\right) = \begin{pmatrix} \lambda I_n & F \circ \Phi(x) \\ F^* \circ \Phi(y) & \mu I_n \end{pmatrix}$$

is a unital completely positive map. By Arveson's extension theorem [1], φ has a unital completely positive extension $\psi : M_2(W) + \mathbb{C}I_H \oplus \mathbb{C}I_H \to M_{2n}$. The linear map

$$\theta : x \in W \mapsto \begin{pmatrix} x & x \\ x & x \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} x \,(1 \quad 1) \in M_2(W)$$

is completely positive. We write

$$\psi \circ \theta = \begin{pmatrix} \varphi_1 & F \circ \Phi \\ F^* \circ \Phi & \varphi_2 \end{pmatrix} \in M_{2n}\big(W^*\big)^+, \quad \|\varphi_1\|_{M_n(W^*)} \leqslant 1,\ \|\varphi_2\|_{M_n(W^*)} \leqslant 1.$$

Then we have

$$\begin{pmatrix} \varphi_1 \circ \Phi^{-1} & F \\ F^* & \varphi_2 \circ \Phi^{-1} \end{pmatrix} \in M_{2n}\big(V^*\big)^+ \quad \text{and} \quad \big\|\varphi_1 \circ \Phi^{-1}\big\|_{cb}, \big\|\varphi_2 \circ \Phi^{-1}\big\|_{cb} \leqslant \big\|\Phi^{-1}\big\|_{cb}.$$

By Lemma 2, we conclude that the dual space V^* is completely isomorphic and complete order isomorphic to a matrix regular operator space. □

Theorem 4. *Suppose that both V and its dual space V^* are operator systems with complete norm. Then V is completely isomorphic and complete order isomorphic to a matrix regular operator space.*

Proof. Once again we consider the space V as a $*$-subspace of $B(H)$ and see that condition (1) of Lemma 2 is satisfied. By Lemma 3, there exist a matrix regular operator space W and a completely isomorphic complete order isomorphism $\Phi : V^* \to W$. For $x \in M_n(V)$, we define

$$\|x\|_{M_n(W_*)} = \sup\big\{\big|\varphi(x)\big|:\ \varphi \in M_n(V)^*,\ \big\|\Phi(\varphi)\big\|_{T_n(W)} \leqslant 1\big\}.$$

Endowing V with this matrix norm $\|\cdot\|_{W_*}$, we get an operator space predual of W. Since the two norms $\|\cdot\|_{M_n(W_*)}$ and $\|\cdot\|_{M_n(V)}$ are equivalent, the positive cone $M_n(V)^+$ is closed with respect to the norm $\|\cdot\|_{M_n(W_*)}$ and the operator space $(V, \|\cdot\|_{W_*})$ is complete. Applying the conjugate linear variation of [4, Theorem 4.1.8] to the involution, we get

$$\begin{aligned} \big\|x^*\big\|_{M_n(W_*)} &= \sup\big\{\big|\varphi\big(x^*\big)\big|:\ \big\|\Phi(\varphi)\big\|_{T_n(W)} \leqslant 1\big\} \\ &= \sup\big\{\big|\varphi(x)\big|:\ \big\|\Phi(\varphi)^*\big\|_{T_n(W)} \leqslant 1\big\} \\ &= \|x\|_{M_n(W_*)}. \end{aligned}$$

Hence the space $W_* := (V, \{\|\cdot\|_{M_n(W_*)}\}_{n\in\mathbb{N}}, \{M_n(V)^+\}_{n\in\mathbb{N}})$ is a matrix ordered operator space with a complete norm, and its dual space is W. Since the predual of a matrix regular operator space is also matrix regular, W_* is matrix regular. The predual map $\Phi_* : W_* \to V$ is completely isomorphic and complete order isomorphic. □

Corollary 5. *For a matrix ordered operator space V with complete norm, V is completely isomorphic and complete order isomorphic to a matrix regular operator space if and only if both V and its dual space V^* are operator systems.*

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