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Algebras Related to Matroids Represented in Characteristic Zero†

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Let k be a field of characteristic zero. We consider graded subalgebras A of $k[x_1, \dots, x_m]/(x_1^2, \dots, x_m^2)$ generated by d linearly independent linear forms. Representations of matroids over k provide a natural description of the structure of these algebras. In return, the numerical properties of the Hilbert function of A yield some information about the Tutte polynomial of the corresponding matroid. Isomorphism classes of these algebras correspond to equivalence classes of hyperplane arrangements under the action of the general linear group.

†*Dedicated to the memory of François Jaeger.*

1. INTRODUCTION

We consider the following class of graded algebras over a field k of characteristic zero. Let $B := k[x_1, \dots, x_m]/(x_1^2, \dots, x_m^2)$ with the standard grading (so $B = \bigoplus_{j=0}^m B_j$ and $\dim_k B_j = \binom{m}{j}$), and let $A = \bigoplus_{j=0}^m A_j$ be a subalgebra of B generated by d linearly independent forms of degree one. Two examples motivate the investigation of such algebras.

EXAMPLE 1.1. Let G be a finite undirected graph with m edges, and orient each edge arbitrarily. Fixing a bijection between the edges of G and the indeterminates $\{x_j\}$, we regard a linear form in B_1 as a linear combination of the edges of G. Let A_1 be the 'cycle-space' of G (that is, the subspace of B_1 consisting of linear combinations of the oriented edges satisfying Kirchhoff's First Law: at every vertex the net flux is zero), and let A be the subalgebra of B generated by A_1. In [8] it is shown that this construction may be symmetrized to obtain a graded algebra $\Phi_\cdot(G, k)$ which is independent of the choice of orientation of the edges of G, and which is covariantly functorial with respect to graph morphisms. Formally, $\Phi_\cdot(G, k)$ resembles a cohomology ring for the graph G with coefficients in the field k.

EXAMPLE 1.2. Let G be a connected complex semisimple Lie group, with Borel subgroup B and root system Δ, and consider the homogeneous manifold $X = G/B$ (the 'flag manifold' of type G). Postnikov *et al.* [5] (see also Shapiro *et al.* [6]) identify differential two-forms $\{\phi_\alpha : \alpha \in \Delta\}$ on X such that $\phi_{-\alpha} = -\phi_\alpha$, $\phi_\alpha^2 = 0$, and the ϕ_α pairwise commute. Any weight λ of G determines a holomorphic Hermitian line bundle L_λ on X, and the curvature form $\Theta(L_\lambda)$ of this line bundle is a linear combination of the $\{\phi_\alpha : \alpha \in \Delta\}$. The subalgebra $C(X)$ of the algebra of differential forms on X generated by the curvature forms $\Theta(L_\lambda)$ is of the kind considered here, and the cohomology ring $H^\cdot(X, \mathbf{C})$ is a quotient of $C(X)$.

In the next section we show that an isomorphism class of algebras A as above corresponds to a linear equivalence class of representations of a matroid over the field k. Equivalently, this corresponds to an equivalence class of hyperplane arrangements $\mathcal{H} \subset k^d$ under the action of the general linear group $GL(k^d)$. One direction of this correspondence is immediate (Lemma 2.2) while the other requires substantial preliminaries (Theorem 2.9). We establish a deletion/contraction short exact sequence which proves to be useful (Theorem 2.5). We present A as a quotient of a polynomial ring modulo an explicitly given ideal (Theorem 2.7),

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and prove an analogue of half of the Strong Lefschetz Theorem for these algebras (Theorem 2.11). In Section 3 we discuss inequalities on the Hilbert function of A derived from the algebraic structure of A. The Poincaré polynomial of A is a specialization of the Tutte polynomial of the corresponding matroid, giving the Hilbert function a combinatorial interpretation (Theorem 3.2). Having computed a few hundred random examples, it seems that the Hilbert function of A is logarithmically concave, and we prove this generically and in the case $d = 2$. These results go some way towards addressing Problems 6.8 and 6.10 of [8].

2. Algebraic Structure

For a natural number n we use the notation $[n] := \{1, 2, \ldots, n\}$. For $0 \leq j \leq m$, let Δ_j be the set of square-free monomials $\mathbf{x}^\alpha$ of degree j in $\{x_1, \ldots, x_m\}$, so $\Delta := \bigcup_{j=0}^{m} \Delta_j$ is a k-basis for B. Endomorphisms of B_j are represented by square matrices with rows and columns indexed by Δ_j. A *monomial matrix* has exactly one nonzero entry in each row and each column.

LEMMA 2.1. *The k-algebra automorphisms of B form a group* $\mathrm{Aut}_k(B)$ *which is isomorphic to the group of monomial matrices acting on B_1 with respect to the basis Δ_1.*

PROOF. Note that if $f \in B_1$ is such that $f^2 = 0$, then $f = cx_j$ for some $c \in k$ and $j \in [m]$. Thus, for any automorphism $\phi : B \to B$ there is a permutation $\sigma : [m] \to [m]$ and nonzero scalars $c_j \in k$ such that $\phi(x_j) = c_j x_{\sigma(j)}$ for all $j \in [m]$. Conversely, any such choice of σ and $\{c_j\}$ determines an automorphism of B. □

Let $M = (m_{ij})$ be a d-by-m matrix over k for which the rowspace of M is A_1. (Henceforth we identify row vectors of length m with linear combinations of the indeterminates $\{x_j\}$.) Since M determines A we will often use the notation $A(M)$. The linearly independent sets of columns of M form the independent sets of a matroid $\mathcal{M}$, and M is a *representation of* $\mathcal{M}$ *over* k. (For background information on matroids consult Oxley [4] or Welsh [10].) Two representations M and N of $\mathcal{M}$ are *linearly equivalent* if there is a monomial matrix P and an invertible matrix Q such that $QMP = N$.

LEMMA 2.2. *Let M and N be two d-by-m matrices of rank d over the field k. If M and N are linearly equivalent representations of the same matroid, then $A(M) \simeq A(N)$.*

PROOF. If $QMP = N$ with Q invertible and P a monomial matrix, then by Lemma 2.1, P determines a k-algebra automorphism of B such that $A_1(M) \simeq A_1(MP) = A_1(N)$. Since $A(M)$ and $A(N)$ are generated by linear forms, it follows that $A(M)$ and $A(N)$ are isomorphic k-algebras. □

The converse of Lemma 2.2 also holds but the proof relies on a presentation of $A(M)$ as a quotient of a polynomial ring, which takes some work to derive.

Lemma 2.2 has an interesting geometric interpretation; see Orlik and Terao [3] for background on hyperplane arrangements.

EXAMPLE 2.3. Let $\mathcal{H}$ be a (nonreduced, central, essential) arrangement of m hyperplanes in a d-dimensional k-vectorspace V. Choose an arbitrary basis $\mathcal{B}$ of V^*, an arbitrary enumeration $\mathcal{H} = \{H_1, \ldots, H_m\}$ of $\mathcal{H}$, and arbitrary linear forms $\ell_1, \ldots, \ell_m$ in V^* such that $H_j = \ker(\ell_j)$ for $j \in [m]$. Writing each ℓ_j as a column vector with respect to the basis $\mathcal{B}$

determines a d-by-m matrix M of rank d. If N is another such matrix obtained from $\mathcal{H}$ by different choices of basis, enumeration, and linear forms, then there is an invertible d-by-d matrix Q (for the change of basis) and an m-by-m monomial matrix P (for change of enumeration and rescaling of linear forms) such that $QMP = N$. Therefore, by Lemma 2.2, the algebra $A(\mathcal{H}) := A(M)$ is a well-defined invariant of the hyperplane arrangement. Moreover, if $\mathcal{H}'$ is a hyperplane arrangement which is equivalent to $\mathcal{H}$ under the action of $GL(V)$, then the corresponding matrices M and M' are linearly equivalent representations of the same matroid, and so $A(\mathcal{H}') \simeq A(\mathcal{H})$.

Lemma 2.4 prepares for Theorem 2.5.

LEMMA 2.4. *Consider linear forms $f_i = x_i + \sum_{j=d+1}^{m} c_{ij}x_j$ in B_1 for $i \in [d]$, and a polynomial $p(z_1, \ldots, z_d)$ in $k[z_1, \ldots, z_d]$. If $f_1 p(f_1, \ldots, f_d) = \sum_\alpha s_\alpha \mathbf{x}^\alpha \neq 0$ then there is some $\mathbf{x}^\alpha \in \Delta$ which is divisible by x_1 and such that $s_\alpha \neq 0$.*

PROOF. Since $f_1 p(f_1, \ldots, f_d) \neq 0$, there is some $\mathbf{x}^\beta \in \Delta$ with $s_\beta \neq 0$. Let T be the set of $j \in [m]$ such that x_j divides $\mathbf{x}^\beta$, $c_{1j} \neq 0$, and the coefficient w_j of $\mathbf{x}^\beta x_j^{-1}$ in $p(f_1, \ldots, f_d)$ is nonzero. Thus, $s_\beta = \sum_{j \in T} c_{1j} w_j$. If x_1 divides $\mathbf{x}^\beta$ then the result is proved, so we may assume that x_1 does not divide $\mathbf{x}^\beta$, and hence that $1 \notin T$. Since T is not empty there is some $j \in T$; now consider the monomial $\mathbf{x}^\alpha := x_1 \mathbf{x}^\beta x_j^{-1}$. We claim that this occurs in $f_1 p(f_1, \ldots, f_d)$ with coefficient $s_\alpha = w_j$, which is nonzero. However, this is clear, since in $f_1 p(f_1, \ldots, f_d) = \sum_{a=1}^{b} q_a(f_2, \ldots, f_d) f_1^a$ the terms contributing to $s_\alpha \mathbf{x}^\alpha$ correspond bijectively with the terms contributing to $s_\beta \mathbf{x}^\beta$ which choose x_j from some factor f_1. The correspondence is made simply by replacing x_1 by x_j in each such term, and the ratio of the coefficients of corresponding terms is $1 : c_{1j}$. □

For a d-by-m matrix M and $j \in [m]$, let $M \setminus j$ be the d-by-$(m-1)$ matrix obtained by deleting the jth column from M. If this column is identically zero then $A(M \setminus j) \simeq A(M)$, as is easily seen. As a result, we are free to assume that M has no zero columns in what follows. If column j of M is not zero then let $i \in [d]$ be the greatest index such that $m_{ij} \neq 0$, and produce M' by adding $-m_{ih}/m_{ij}$ times column j to column h of M, for each $h \in [m]$. Finally, M/j is the $(d-1)$-by-$(m-1)$ matrix obtained by deleting the ith row and jth column from M'.

Theorem 2.5 is an analogue of the sequence (3.1) of [8]. (The notation $A(M \setminus j)(-1)$ merely indicates that the grading of $A(M \setminus 1)$ has been shifted up by one degree.)

THEOREM 2.5. *Let M be a d-by-m matrix of rank d over the field k. For each $j \in [m]$ such that column j of M is not zero, there is a short exact sequence of graded k-spaces*

$$0 \longrightarrow A(M \setminus j)(-1) \longrightarrow A(M) \xrightarrow{\pi} A(M/j) \longrightarrow 0$$

in which π is a k-algebra homomorphism.

PROOF. Replacing M, if necessary, by a linearly equivalent representation of the same matroid, we may assume that $j = 1$ and that M has the block structure $M = [I\ N]$ in which I is the d-by-d identity matrix. Let $f_1, \ldots, f_d$ be the rows of M, let $f_1', \ldots, f_d'$ be the rows of $M \setminus 1$, and let $f_2'', \ldots, f_d''$ be the rows of $M/1$. There is certainly an exact sequence

$$0 \longrightarrow (f_1) \longrightarrow A(M) \longrightarrow A(M)/(f_1) \longrightarrow 0$$

for the principal ideal (f_1) of $A(M)$. It remains only to establish isomorphisms $A(M/1) \simeq A(M)/(f_1)$ and $A(M \smallsetminus 1)(-1) \simeq (f_1)$.

Now, since column 1 of M is zero except in row 1, $f_i'' = f_i$ for $2 \le i \le d$; thus, there is a well-defined k-algebra homomorphism from $A(M)$ to $A(M/1)$ given by $f_1 \mapsto 0$ and $f_i \mapsto f_i''$ for $2 \le j \le d$. Clearly this is surjective and has kernel (f_1). For the other isomorphism, notice that $f_1 = x_1 + f_1'$ and $f_i = f_i'$ for $2 \le i \le d$. Thus, $f_1^a = (f_1')^a + ax_1(f_1')^{a-1}$ for every natural number a; it follows that for any polynomial $p(z_1, \ldots, z_d)$,

$$p(f_1, \ldots, f_d) = p(f_1', \ldots, f_d') + x_1 p'(f_1', \ldots, f_d'),$$

in which $p'(\mathbf{z}) := (\partial/\partial z_1)p(\mathbf{z})$. Thus, the rule $p(f_1, \ldots, f_d) \mapsto p'(f_1', \ldots, f_d')$ gives a well-defined k-linear homomorphism $\phi : A(M) \to A(M \smallsetminus 1)(-1)$; this is just the extraction of the coefficient of x_1 from $p(f_1, \ldots, f_d)$. Since for every polynomial $q(\mathbf{z})$ there is a polynomial $p(\mathbf{z})$ such that $(\partial/\partial z_1)z_1 p(\mathbf{z}) = q(\mathbf{z})$, it follows that the restriction of ϕ to (f_1) is surjective onto $A(M \smallsetminus 1)(-1)$. Finally, Lemma 2.4 shows that the restriction of ϕ to (f_1) is injective, establishing the isomorphism $A(M \smallsetminus 1)(-1) \simeq (f_1)$. □

We next present $A(M)$ as a quotient of the polynomial ring $R := k[z_1, \ldots, z_d]$. For any linear form $f = \sum_{j=1}^m c_j x_j$ in B_1, let $\nu(f) := \#\{j : c_j \neq 0\}$. Note that $f^{\nu(f)} \neq 0$ and $f^{1+\nu(f)} = 0$. Identifying a linear form p in R_1 with a row vector of length d, there is a corresponding linear form pM in B_1. Define the ideal $J(M)$ of R by

$$J(M) := (p^{1+\nu(pM)} : p \in R_1).$$

LEMMA 2.6. *Let M and N be two d-by-m matrices of rank d over the field k. If M and N are linearly equivalent representations of the same matroid, then $J(M) \simeq J(N)$.*

PROOF. Let $N = QMP$ with Q invertible and P a monomial matrix. Certainly $\nu(pQMP) = \nu(pQM)$ for every $p \in R_1$. The rule $p \mapsto pQ$ for $p \in R_1$ defininés a k-algebra automorphism $\phi : R \to R$, and

$$J(M) = \{(pQ)^{1+\nu(pQM)} : p \in R_1\}.$$

Therefore, since

$$J(N) = \{p^{1+\nu(pN)} : p \in R_1\},$$

it follows that the restriction of ϕ to $J(N)$ is an isomorphism from $J(N)$ to $J(M)$. □

Theorem 2.7 generalizes Theorem 4.8 of [8] and Proposition 1.1 of Shapiro *et al.* [6].

THEOREM 2.7. *For M a d-by-m matrix of rank d over the field k, $A(M) \simeq R/J(M)$.*

PROOF. We apply Theorem 2.5 for some $j \in [m]$ indexing a nonzero column of M. By Lemmas 2.2 and 2.6 we may replace M, if necessary, by a linearly equivalent representation of the same matroid. Thus we may assume that $j = 1$ and that M has the block structure $M = [I \;\; N]$ in which I is the d-by-d identity matrix. Let $f_1, \ldots, f_d$ be the rows of M, let $f_1', \ldots, f_d'$ be the rows of $M \smallsetminus 1$, and let $f_2'', \ldots, f_d''$ be the rows of $M/1$.

Define a k-algebra homomorphism $\psi : R \to A(M)$ by $\psi(z_i) := f_i$ for $i \in [d]$. Certainly ψ is surjective, as A_1 generates A. We claim that $\ker(\psi) = J(M)$, which we prove by induction on d and m, the bases $d = 1$ and $m = d$ being easily seen. It is clear that $J(M) \subseteq \ker(\psi)$ since for any $p \in R_1$ we have $\psi(p) = pM$ and $(pM)^{1+\nu(pM)} = 0$ in $A(M)$. For the converse, define a k-algebra homomorphism $\psi' : R \to A(M \setminus 1)$ by $\psi'(z_i) := f_i'$ for $i \in [d]$, and define $\psi'' : k[z_2, \ldots, z_d] \to A(M/1)$ by $\psi''(z_i) := f_i''$ for $2 \leq i \leq d$. There is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R(-1) & \xrightarrow{\eta} & R & \xrightarrow{\pi} & k[z_2, \ldots, z_d] & \longrightarrow & 0 \\
 & & \downarrow \psi' & & \downarrow \psi & & \downarrow \psi'' & & \\
0 & \longrightarrow & A(M \setminus 1)(-1) & \longrightarrow & A(M) & \longrightarrow & A(M/1) & \longrightarrow & 0
\end{array}$$

in which the bottom row is the sequence of Theorem 2.5. From the proof of Theorem 2.5 one sees that the homomorphisms in the top row are given by $\pi(p(\mathbf{z})) := p(0, z_2, \ldots, z_d)$ and $\eta(p(\mathbf{z})) := \int p(\mathbf{z})\mathrm{d}z_1$ for all $p(\mathbf{z}) \in R$. Since ψ' is surjective, the kernel-cokernel exact sequence (see, e.g., Lemma II.5.2 of Mac Lane [2]) implies that $0 \to \ker(\psi') \to \ker(\psi) \to \ker(\psi'') \to 0$ is exact. By induction, we deduce that

$$\ker(\psi) = \eta(J(M \setminus 1)(-1)) \oplus \iota(J(M/1)),$$

in which $\iota : k[z_2, \ldots, z_d] \to R$ is the natural inclusion.

To prove that $\ker(\psi) \subseteq J(M)$, it thus suffices to show that $\iota(J(M/1)) \subseteq J(M)$ and $\eta(J(M \setminus 1)(-1)) \subseteq J(M)$. The first of these claims is trivial, since the generators of $\iota(J(M/1))$ are exactly those generators of $J(M)$ which do not involve the indeterminate z_1. For the second claim, by k-linearity it suffices to prove that $\eta(\mathbf{z}^\gamma g(\mathbf{z})) \in J(M)$ for any monomial $\mathbf{z}^\gamma$ and generator $g(\mathbf{z})$ of $J(M \setminus 1)(-1)$. So, let $p(\mathbf{z}) := c_1 z_1 + \cdots + c_d z_d$ and let $\nu := \nu(c_1 f_1' + \cdots + c_d f_d')$, and consider $\eta(\mathbf{z}^\gamma p(\mathbf{z})^{1+\nu})$. If $c_1 = 0$ then $\nu(c_1 f_1 + \cdots + c_d f_d) = \nu$ and $\int \mathbf{z}^\gamma p(\mathbf{z})^{1+\nu}\mathrm{d}z_1 = \mathbf{z}^\gamma z_1 p(\mathbf{z})^{1+\nu}/(\gamma_1 + 1)$ is in $J(M)$. On the other hand, if $c_1 \neq 0$ then $\nu(c_1 f_1 + \cdots + c_d f_d) = \nu + 1$; however, applying integration by parts repeatedly we obtain

$$\begin{aligned}
\int \mathbf{z}^\gamma p(\mathbf{z})^{1+\nu}\mathrm{d}z_1 &= \frac{\mathbf{z}^\gamma p(\mathbf{z})^{2+\nu}}{2+\nu} - \int \left(\frac{\partial \mathbf{z}^\gamma}{\partial z_1}\right) \frac{p(\mathbf{z})^{2+\nu}}{2+\nu}\mathrm{d}z_1 \\
&= \cdots = q(\mathbf{z})p(\mathbf{z})^{2+\nu}
\end{aligned}$$

for some polynomial $q(\mathbf{z}) \in R$. Since $p(\mathbf{z})^{2+\nu}$ is a generator of $J(M)$, the result follows. □

Although Theorem 2.7 gives a good picture of $A(M)$, it would be preferable to have a standard monomial theory for this algebra. Presumably this would rely on matroid-theoretic structure as in the proof of Theorem 3.2 below, but as yet the situation remains unclear.

We can now establish the converse of Lemma 2.2, the proof of which uses the following ‘tomographic’ lemma (valid for any infinite field k).

LEMMA 2.8. *Let $\mathcal{L}$ and $\mathcal{L}'$ be finite multisets of lines in the d-dimensional k-vectorspace V, each line passing through the origin. Assume that for every hyperplane $H \subset V$, the number of lines of $\mathcal{L}$ in H equals the number of lines of $\mathcal{L}'$ in H. Then $\mathcal{L} = \mathcal{L}'$.*

PROOF. Arguing by contradiction, assume that $\mathcal{L} \neq \mathcal{L}'$. Replacing $\mathcal{L}$ by $\mathcal{L} \setminus \mathcal{L}'$ and $\mathcal{L}'$ by $\mathcal{L}' \setminus \mathcal{L}$, we may assume that $\mathcal{L} \cap \mathcal{L}' = \varnothing$. At least one of $\mathcal{L}$ or $\mathcal{L}'$ is nonempty; by symmetry, consider any $\ell \in \mathcal{L}$. Since k is infinite and $\mathcal{L}'$ is finite, there is a hyperplane $H \subset V$ which contains ℓ but does not contain any lines of $\mathcal{L}'$; this contradicts the hypothesis. □

THEOREM 2.9. *Let M and N be two d-by-m matrices of rank d over the field k. Then $A(M)$ and $A(N)$ are isomorphic as k-algebras if and only if M and N are linearly equivalent representations of the same matroid.*

PROOF. Lemma 2.2 establishes one direction. For the converse, assume that $\phi : A(M) \to A(N)$ is a k-algebra isomorphism. By the remarks preceding Theorem 2.5, we may assume that M and N have no zero columns. Let $f_1, \ldots, f_d$ be the rows of M, and let $g_1, \ldots, g_d$ be the rows of N. Replacing N, if necessary, by a linearly equivalent representation of the same matroid, we may assume that $\phi : A_1(M) \to A_1(N)$ is determined by $\phi(f_i) = g_i$ for all $i \in [d]$. Now let $R := k[z_1, \ldots, z_d]$ and define $\psi : R \to A(M)$ and $\psi' : R \to A(N)$ by $\psi(z_i) := f_i$ and $\psi'(z_i) := g_i$ for all $i \in [d]$. From Theorem 2.7 it follows that $J(M) = \ker(\psi) = \ker(\psi') = J(N)$. Let $\mathcal{L}$ be the multiset of lines in k^d consisting of the scalar multiples of the columns of M. Let $\mathcal{L}'$ be the corresponding multiset of lines for N. Since $J(M) = J(N)$ and $\phi : A(M) \to A(N)$ is determined by $\phi(f_i) = g_i$ for $i \in [d]$, it follows that for any linear form $p \in R_1$, $\nu(pM) = \nu(pN)$; that is, the number of lines of $\mathcal{L}$ in $\ker(p)$ equals the number of lines of $\mathcal{L}'$ in $\ker(p)$. By Lemma 2.8 it follows that $\mathcal{L} = \mathcal{L}'$. Thus, there is an m-by-m monomial matrix P such that $MP = N$. This completes the proof. $\square$

COROLLARY 2.10. *Let A and A' be subalgebras of B generated by linear forms. Any k-algebra isomorphism $\phi : A \to A'$ extends to an automorphism of B.*

PROOF. Let $A = A(M)$ and $A' = A(N)$ for d-by-m matrices M and N of rank d. By Theorem 2.9 there is an m-by-m monomial matrix P such that $A_1(MP) = A_1(N)$ and for $f \in A_1(M)$, $\phi(f) = fP$. By Lemma 2.1, P determines an automorphism of B extending $\phi : A \to A'$. $\square$

We close this section with an analogue of half of the Strong Lefschetz Theorem, generalizing Theorem 4.10 of [8].

THEOREM 2.11. *Let M be a d-by-m matrix of rank d over the field k, and assume that M has no zero columns. Let $g = \sum_{j=1}^m c_j x_j \in A_1(M)$ be such that $c_j \neq 0$ for all $j \in [m]$. Then for each $0 \leq j \leq m/2$, the homomorphism $\cdot g^{m-2j} : A_j(M) \to A_{m-j}(M)$ is injective.*

PROOF. Fix any $0 \leq j \leq m/2$. Let W be the matrix with rows indexed by Δ_{m-j} and with columns indexed by Δ_j, with $W_{\mathbf{x}^\alpha, \mathbf{x}^\beta} := 1$ if $\mathbf{x}^\beta$ divides $\mathbf{x}^\alpha$ and $W_{\mathbf{x}^\alpha, \mathbf{x}^\beta} := 0$ otherwise. Multiplication by the element g^{m-2j} of $A(M)$ induces a homomorphism $\cdot g^{m-2j} : B_j \to B_{m-j}$; let G be the matrix representing this homomorphism with repsect to the bases Δ_j and Δ_{m-j}. That is, if $\mathbf{x}^\beta | \mathbf{x}^\alpha$ then $G_{\mathbf{x}^\alpha, \mathbf{x}^\beta} = \prod\{c_j : x_j | \mathbf{x}^{\alpha-\beta}\}$, and $G_{\mathbf{x}^\alpha, \mathbf{x}^\beta} := 0$ otherwise. Let P be the diagonal square matrix indexed by Δ_{m-j}, with $P_{\mathbf{x}^\beta, \mathbf{x}^\beta} := \prod\{c_j : x_j | \mathbf{x}^\beta\}$, and let Q be the diagonal square matrix indexed by Δ_j, with $Q_{\mathbf{x}^\alpha, \mathbf{x}^\alpha} := \prod\{c_j : x_j | \mathbf{x}^\alpha\}$. By the hypothesis on g, both P and Q are invertible. One verifies that the matrix equation $GP = QW$ holds, and hence $\det(G) = \det(Q)\det(W)\det(P)^{-1}$. Wilson [11] proves that

$$\det(W) = \prod_{h=0}^{j} \binom{m-j-h}{j-h}^{\binom{m}{h}-\binom{m}{h-1}},$$

and therefore $\det(G) \neq 0$. Thus, $\cdot g^{m-2j} : B_j(M) \to B_{m-j}(M)$ is an isomorphism, and so $\cdot g^{m-2j} : A_j(M) \to A_{m-j}(M)$ is injective. $\square$

3. Hilbert Functions

The *Poincaré polynomial* of a finite-dimensional graded k-space $A = \bigoplus_{j=0}^{m} A_j$ is $P(A; t) := \sum_{j=0}^{m} (\dim_k A_j) t^j$. The coefficients of $P(A; t)$ form the *Hilbert function* of A. The *Tutte polynomial* $T_{\mathcal{M}}(x, y)$ of a matroid $\mathcal{M}$ is the class of $\mathcal{M}$ in the Grothendieck ring of the category of matroids; see [1, 4, 10].

PROPOSITION 3.1. *Let M be a d-by-m matrix of rank d with no zero columns, representing the matroid $\mathcal{M}$ over the field k. Then the Poincaré polynomial of $A(M)$ depends only on $\mathcal{M}$ and is $P(\mathcal{M}; t) := P(A(M); t) = t^{m-d} T_{\mathcal{M}}(1 + t, t^{-1})$.*

PROOF. From Theorem 2.5 we obtain the recursion

$$P(A(M); t) = t P(A(M \setminus j); t) + P(A(M/j); t) \tag{3.1}$$

for the Poincaré polynomials, with initial conditions $P(A(M); t) = 1$ if $d = 0$, and $P(A(M); t) = 1 + t + \cdots + t^m$ if $d = 1$ and M has m nonzero columns. Defining $\widetilde{P}(A(M); t) := t^{d-m} P(A(M); t)$ we have $\widetilde{P}(A(M); t) = \widetilde{P}(A(M \setminus j); t) + \widetilde{P}(A(M/j); t)$. When $d = 0$ and $m = 1$ we have $\widetilde{P}(A(M); t) = t^{-1}$, and when $d = 1$ and $m = 1$ we have $\widetilde{P}(A(M); t) = 1 + t$. Since $T_{\mathcal{M}}(x, y)$ is the universal Tutte–Grothendieck invariant of the category of matroids (see Brylawski and Oxley [1]), it follows by induction on d and m that $\widetilde{P}(A(M); t) = T_{\mathcal{M}}(1 + t, t^{-1})$. □

For the next result we need some operations on sets of columns of the matrix M. For $S \subseteq [m]$, let $\operatorname{span}_k(S)$ be the k-space spanned by the columns of M in S, and let $\overline{S}$ be the set of columns of M contained in $\operatorname{span}_k(S)$. The *rank* of S is $r(S) := \dim_k \operatorname{span}_k(S)$. Let $I(S)$ be the lexicographically earliest basis of $\operatorname{span}_k(S)$ contained in $\overline{S}$. A column j is *externally active* for S if and only if $j \in \overline{S} \setminus S$ and $I(S \cup \{j\}) = I(S)$. Let $EA(S)$ be the set of columns externally active for S, and let $\operatorname{ea}(S)$ be the cardinality of this set.

What follows is a new proof of Theorem 2 of Postnikov *et al.* [5].

THEOREM 3.2. *Let M be a d-by-m matrix of rank d with no zero columns, representing the matroid $\mathcal{M}$ over the field k. For $0 \le j \le m$, $\dim_k A_j(M)$ is the number of independent sets of $\mathcal{M}$ such that $m - \#S - \operatorname{ea}(S) = j$.*

PROOF. The rank-polynomial expansion (see (6.12) of Brylawski and Oxley [1]) of $T_{\mathcal{M}}(x, y)$ is

$$T_{\mathcal{M}}(x, y) = \sum_{S \subseteq [m]} (x - 1)^{d - r(S)} (y - 1)^{\#S - r(S)}.$$

Making the substitution of Proposition 3.1 leads to

$$\begin{aligned} P(\mathcal{M}; t) &= \sum_{S \subseteq [m]} t^{m - \#S} (1 - t)^{\#S - r(S)} = \sum_{S \subseteq [m]} t^{m - \#S} \sum_{T \subseteq S \setminus I(S)} (-1)^{\#T} t^{\#T} \\ &= \sum_{R \subseteq [m]} t^{m - \#R} \sum_{T \subseteq EA(R)} (-1)^{\#T} = \sum_{R \subseteq [m]:\ EA(R) = \varnothing} t^{m - \#R}. \end{aligned}$$

For the third equality, notice that $S \setminus I(S) \subseteq EA(I(S))$ for every $S \subseteq [m]$. Thus, if $T \subseteq S \setminus I(S)$ and $R := S \setminus T$ then $I(R) = I(S)$ and $T \subseteq EA(R)$. Conversely, if $T \subseteq EA(R)$ then $T \subseteq (R \cup T) \setminus I(R \cup T)$.

Since $EA(R) = EA(I(R)) \smallsetminus R$, it follows that $EA(R) = \varnothing$ if and only if $R = I(R) \cup EA(I(R))$. Conversely, if S is independent then $I(S \cup EA(S)) = S$. Thus, the functions $R \mapsto I(R)$ and $S \mapsto S \cup EA(S)$ are mutually inverse bijections between the sets $\{R \subseteq [m] : EA(R) = \varnothing\}$ and $\{S \subseteq [m] : S \text{ is independent in } \mathcal{M}\}$. Therefore,

$$P(A(M); t) = \sum_{R \subseteq [m]:\ EA(R) = \varnothing} t^{m - \#R} = \sum_S t^{m - \#S - \mathrm{ea}(S)}, \tag{3.2}$$

with the last sum over the independent sets of $\mathcal{M}$. □

The notation $d_j(M) := \dim_k A_j(M)$ will be convenient. For positive integers a and j there is a unique expression

$$a = \binom{a_j}{j} + \binom{a_{j-1}}{j-1} + \cdots + \binom{a_i}{i}$$

such that $a_j > a_{j-1} > \cdots > a_i \geq i > 0$. The *$j$th pseudopower of a* is

$$\psi_j(a) := \binom{a_j + 1}{j+1} + \binom{a_{j-1} + 1}{j} + \cdots + \binom{a_i + 1}{i+1}.$$

PROPOSITION 3.3. *Let M be a d-by-m matrix of rank d with no zero columns. Then $d_0(M) = 1$, $d_1(M) = d$, $d_m(M) = 1$, and for $j \in [m-1]$, we have $0 < d_{j+1}(M) \leq \psi_j(d_j(M))$.*

PROOF. Since $A_0(M) = k$ and $\dim_k A_1(M) = d$, the first two statements are clear. Since M has no zero columns and k is infinite, there is a linear form $g \in A_1$ with $\nu(g) = m$. Thus, g^m is a nonzero multiple of $x_1 \cdots x_m$; since B_m is one-dimensional it follows that $d_m(M) = 1$. The remaining inequalities are a direct application of Macaulay's Theorem (see Theorems II.2.2 and II.2.3 of Stanley [7]), since A is generated by linear forms. □

PROPOSITION 3.4. *Let M be a d-by-m matrix of rank d with no zero columns. Then $d_0(M) \leq d_1(M) \leq \cdots \leq d_{\lfloor m/2 \rfloor}(M)$, and if $0 \leq j \leq m/2$, then $d_j(M) \leq d_{m-j}(M)$.*

PROOF. Since M has no zero columns and k is infinite, there is a linear form $g \in A_1$ with $\nu(g) = m$. The monomorphisms $\cdot g^{m-2j} : A_j(M) \to A_{m-j}(M)$ of Theorem 2.11 show that $d_j(M) \leq d_{m-j}(M)$ for all $0 \leq j \leq m/2$. Since each of these maps is injective, each of the maps $\cdot g : A_j(M) \to A_{j+1}(M)$ for $0 \leq j < m/2$ must also be injective, implying the remaining inequalities. □

Note that $d_j(M) \leq \binom{d+j-1}{j}$ for all $j \geq 0$ since A is generated by d linear forms, and that $d_j(M) \leq \binom{m}{j}$ for all $j \geq 0$ since A is a subalgebra of B. Next, we see that generically these bounds are attained. Recall that the uniform matroid $\mathcal{U}_m^d$ has for its independent sets all subsets of $[m]$ of size at most d.

PROPOSITION 3.5. *Let M be a d-by-m matrix over the field k representing the uniform matroid $\mathcal{U}_m^d$ of rank d on m elements. Then for $0 \leq j \leq m$, $d_j(M) = \min\left\{\binom{d+j-1}{j}, \binom{m}{j}\right\}$.*

PROOF. When $d \geq 2$, $M \setminus 1$ represents $\mathcal{U}^d_{m-1}$ and $M/1$ represents $\mathcal{U}^{d-1}_{m-1}$. Thus, from (3.1) we obtain

$$P(\mathcal{U}^d_m; t) = tP(\mathcal{U}^d_{m-1}; t) + P(\mathcal{U}^{d-1}_{m-1}; t)$$

for $d \geq 2$, with initial conditions $P(\mathcal{U}^1_m; t) = 1 + t + \cdots + t^m$. The result follows by induction on d and m, using familiar recurrences for binomial coefficients. □

Let $\mathcal{A}(m, d)$ be the collection of all graded subalgebras $A \subseteq B$ generated by A_1 and with $\dim_k A_1 = d$. This $\mathcal{A}(m, d)$ is a sub-bundle of the trivial vector-bundle $\mathbf{G}(B_1, d) \times B$ over the Grassmann variety $\mathbf{G}(B_1, d)$ of d-dimensional subspaces of B_1; for a given d-plane $A_1 \subseteq B_1$, the fibre over A_1 is the subalgebra of B generated by A_1. As the Poincaré polynomial $P(A; t)$ varies with A_1, the rank of $\mathcal{A}(m, d)$ is not constant, so $\mathcal{A}(m, d)$ is not complete. By upper semicontinuity, the rank of the fibre of $\mathcal{A}(m, d)$ over $A_1(M)$ attains its generic value if the matroid represented by M is uniform. We next prove the converse, giving equations in local coordinates for the degeneracy locus of $\mathcal{A}(m, d)$. Consider the affine open chart $\mathcal{C} \subset \mathbf{G}(B_1, d)$ of d-planes $A_1 \subseteq B_1$ of the form $A_1(M)$ for a d-by-m matrix $M = [I\ N]$ with I the d-by-d identity matrix. The entries of $N = (n_{ij})$ are local coordinates on $\mathcal{C}$. Since $\mathbf{G}(B_1, d)$ is covered by affine opens which are in the orbit of $\mathcal{C}$ under $\mathrm{Aut}_k(B)$, it suffices to consider just this one chart $\mathcal{C}$.

PROPOSITION 3.6. *Let $M = [I\ N]$ be a d-by-m matrix with I the d-by-d identity matrix, representing the matroid $\mathcal{M}$ over the field k. The following are equivalent:*

(a) *$\mathcal{M}$ is not the uniform matroid $\mathcal{U}^d_m$.*
(b) *For some $j \in [m]$, $d_j(M) < \min\{\binom{d+j-1}{j}, \binom{m}{j}\}$.*
(c) *For some $h \in [\min\{d, m-d\}]$ and some h-by-h submatrix N' of N, $\det(N') = 0$.*

PROOF. Proposition 3.5 shows that (b) implies (a). To see that (a) implies (c), if $\mathcal{M}$ is not uniform then there is a d-by-d submatrix M' of M which is singular. Deleting the rows and columns of M' which contain a 1 from the I block of M produces a singular square submatrix of N, proving (c). This argument may be reversed to show that (c) also implies (a). Finally, assuming (a), if $\mathcal{M}$ is not $\mathcal{U}^d_m$ then from (3.2), with S ranging over the independent sets of $\mathcal{M}$,

$$P(A(M); 1) = \sum_S 1 < \binom{m}{0} + \binom{m}{1} + \cdots + \binom{m}{d} = \sum_{j=0}^{m} \min\left\{\binom{d+j-1}{j}, \binom{m}{j}\right\},$$

and (b) follows. □

A sequence $(d_0, \ldots, d_m)$ of positive integers is *logarithmically concave* if $d_j^2 \geq d_{j-1}d_{j+1}$ for all $2 \leq j \leq m-1$. From Proposition 3.5 it follows that if M represents the uniform matroid $\mathcal{U}^d_m$ then the Hilbert function of $A(M)$ is logarithmically concave. (The argument is easy: each of the sequences $\binom{d+j-1}{j}$ and $\binom{m}{j}$ for $0 \leq j \leq m$ is logarithmically concave, and the coefficientwise minimum of two logarithmically concave sequences is also logarithmically concave.) Thus, generically, the sequence of ranks of the graded pieces of $\mathcal{A}(m, d)$ is logarithmically concave. Whether or not this remains true over the degeneracy locus of $\mathcal{A}(m, d)$ is an interesting question; one possible approach is as follows.

As observed in [9], the Hilbert function of a graded $\mathbf{C}$-space $A = \bigoplus_{j=0}^m A_j$ is logarithmically concave if and only if there is a representation of $\mathfrak{sl}_2(\mathbf{C})$ on $A \otimes A$ for which the standard basis elements $\{\mathsf{X}, \mathsf{Y}, \mathsf{H}\}$ of $\mathfrak{sl}_2(\mathbf{C})$ act such that $\mathsf{X} : A_i \otimes A_j \to A_{i-1} \otimes A_{j+1}$ and $\mathsf{Y} : A_i \otimes A_j \to A_{i+1} \otimes A_{j-1}$ for all i and j. Hence, such a representation exists on the generic fibre of $\mathcal{A}(m, d) \otimes \mathcal{A}(m, d)$. The difficulty lies in degenerating this generic representation over Spec $\mathbf{C}[u]$ so that at $u = 0$ a representation on the fibre above an arbitrary point of $\mathbf{G}(B_1, d)$ is obtained. It is not clear how (or whether!) this can be done, but the following degenerations of the irreducible representations of $\mathfrak{sl}_2(\mathbf{C})$ seem relevant. For a proposition P, let $\langle P \rangle$ be 1 if P is true and 0 if P is false. For integers $1 \le r \le n$ of the same parity, let $X_{n,r}(u)$ be the n-by-n matrix with entries

$$X_{n,r}(u)_{ij} := \begin{cases} i u^{\langle i \le r \rangle - \langle j > n-r \rangle} & \text{if } i = j - 1, \\ 0 & \text{otherwise,} \end{cases}$$

let $Y_{n,r}(u)$ be the n-by-n matrix with entries

$$Y_{n,r}(u)_{ij} := \begin{cases} (n - j) u^{\langle i > n-r \rangle - \langle j \le r \rangle} & \text{if } i = j + 1, \\ 0 & \text{otherwise,} \end{cases}$$

and let $H_{n,r}(u) = X_{n,r}(u) Y_{n,r}(u) - Y_{n,r}(u) X_{n,r}(u)$. For example, with $n = 5$ and $r = 3$,

$$X_{5,3}(z) := \begin{bmatrix} 0 & u & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4u^{-1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Y_{5,3}(z) := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 4u^{-1} & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & u & 0 \end{bmatrix}.$$

For $0 \neq u \in \mathbf{C}$ these matrices define an irreducible representation of $\mathfrak{sl}_2(\mathbf{C})$ on $\mathbf{C}^n$. As $u \to 0$ these linear transformations cease to be defined on all of $\mathbf{C}^n$. At $u = 0$ they remain defined on an r-dimensional subspace, on which they still provide an irreducible representation of $\mathfrak{sl}_2(\mathbf{C})$.

In the special case of $d = 2$ we can establish a property stronger than logarithmic concavity by other means.

THEOREM 3.7. *Let M be a 2-by-m matrix of rank 2 over the field k. Then the sequence $d_j(M) - d_{j-1}(M)$ is nonincreasing as j goes from 1 to m. Consequently, the Hilbert function of $A(M)$ is logarithmically concave.*

PROOF. We may assume that $M = [I \ N]$ with I the 2-by-2 identity matrix, and denote the rows of M by f_1 and f_2. By Theorem 2.7, $A(M) \simeq R/J(M)$ in which $R = k[z_1, z_2]$ and

$$J(M) := ((c_1 z_1 + c_2 z_2)^{1+\nu(c_1 f_1 + c_2 f_2)} : c_1, c_2 \in k).$$

The columns of M are partitioned uniquely into subsets $E_1, \ldots, E_s$ such that columns j and j' belong to the same part E_h if and only if they are proportional; for each $h \in [s]$, let $e_h := \#E_h$. For each $h \in [s]$ there is a particular ratio $c_1 : c_2$ such that the jth entry of $c_1 f_1 + c_2 f_2$ is zero if and only if $j \in E_h$. Thus, for each $h \in [s]$ there is a linear form $p_h \in R$ such that $\nu(p_h M) = m - e_h$, and in fact $J(M)$ is generated by $\{p_1^{1+m-e_1}, \ldots, p_s^{1+m-e_s}\}$. For each $0 \le j \le m$ let $w_j := \#\{h \in [s] : 1 + m - e_h = j\}$.

Now $\dim_k R_j = j + 1$ for all $j \ge 0$, and $\dim_k A_m(M) = 1$ by Proposition 3.3, so $\dim_k J_m(M) = m$. However

$$\dim_k J_m(M) \le \sum_{h=1}^{s} \dim_k (p_h^{1+m-e_h})_m = \sum_{h=1}^{s} [m - (1 + m - e_h) + 1] = m,$$

and since equality holds, the forms $p_h^{1+m-e_h}$ for $h \in [s]$ impose independent conditions on homogeneous j-forms in R, for all $0 \leq j \leq m$. Thus, for each $0 \leq j \leq m$,

$$\dim_k A_j(M) = j + 1 - \sum_{i=0}^{j} w_i(j - i + 1).$$

From this the inequalities $d_j(M) - d_{j-1}(M) \geq d_{j+1}(M) - d_j(M)$ for $j \in [m-1]$ follow. By the inequality of arithmetic and geometric means it follows that $d_j(M) \geq (d_{j-1}(M) + d_{j+1}(M))/2 \geq (d_{j-1}(M)d_{j+1}(M))^{1/2}$, completing the proof. □

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