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Ground state solutions for the nonlinear Schrödinger–Maxwell equations

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ABSTRACT

In this paper we study the nonlinear Schrödinger–Maxwell equations

$$\begin{cases} -\Delta u + V(x)u + \phi u = |u|^{p-1}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3. \end{cases}$$

If V is a positive constant, we prove the existence of a ground state solution (u,ϕ) for $2 < p < 5$. The non-constant potential case is treated for $3 < p < 5$, and V possibly unbounded below. Existence and nonexistence results are proved also when the nonlinearity exhibits a critical growth.

Contents

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1. Introduction

In this paper we consider the following system

$$\begin{cases} -\Delta u + V(x)u + \phi u = f'(u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{$\mathcal{SM}$}$$

where $V : \mathbb{R}^3 \to \mathbb{R}$ and $f \in C^1(\mathbb{R}, \mathbb{R})$. Such a system, also known as the nonlinear Schrödinger–Poisson, arises in an interesting physical context. In fact, according to a classical model, the interaction of a charge particle with an electromagnetic field can be described by coupling the nonlinear Schrödinger's and the Maxwell's equations (we refer to [3] for more details on the physical aspects). In particular, if we are looking for electrostatic-type solutions, we just have to solve $(\mathcal{SM})$. In [3], the potential V has been supposed constant, and the linear version of the problem (i.e. $f \equiv 0$) has been studied as an eigenvalue problem for a bounded domain. The linear Schrödinger–Maxwell equations have been treated also in [10,12], where the potential V has been supposed radial.

The nonlinear case has been considered in [1,11,14,16,17,23], where existence and multiplicity results have been stated when V is a positive constant. By means of the Pohozaev's fibering method, a multiplicity result has been proved in [24] also in the non-homogeneous case, that is when a non-homogeneous term $g(x) \in L^2(\mathbb{R}^3)$ is added on the right-hand side of the first equation of $(\mathcal{SM})$ (see also [7]). On the other hand, nonexistence results for $(\mathcal{SM})$ can be found in [15,23]. For a related problem see [21].

Up to our knowledge, the literature does not contain any result on the existence of *ground state solutions* to the problem $(\mathcal{SM})$, namely couples (u, ϕ) which solve $(\mathcal{SM})$ and minimize the action functional associated to $(\mathcal{SM})$ among all possible solutions: this is the aim of our paper. The problem of finding such a type of solutions is a very classical problem: it has been introduced by Coleman, Glazer and Martin in [13], and reconsidered by Berestycki and Lions in [5] for a class of nonlinear equations including the Schrödinger's one. Later on the existence and the profile of ground state solutions have been studied for a plethora of problems by many authors; of course we cannot mention all these results.

In the first part of the paper, we are interested in considering pure power type nonlinearities so that the problem we will deal with becomes

$$\begin{cases} -\Delta u + V(x)u + \phi u = |u|^{p-1}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1}$$

where $2 < p < 5$. The solutions $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ of (1) are the critical points of the action functional $\mathcal{E} : H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3) \to \mathbb{R}$, defined as

$$\mathcal{E}(u, \phi) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 - \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \phi|^2 + \frac{1}{2} \int_{\mathbb{R}^3} \phi u^2 - \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1}.$$

We are interested in finding a ground state solution of (1), that is a solution (u_0, ϕ_0) of (1) such that $\mathcal{E}(u_0, \phi_0) \leqslant \mathcal{E}(u, \phi)$, for any solution (u, ϕ) of (1).

The action functional $\mathcal{E}$ exhibits a strong indefiniteness, namely it is unbounded both from below and from above on infinite dimensional subspaces. This indefiniteness can be removed using the reduction method described in [4], by which we are led to study a one variable functional that does not present such a strongly indefinite nature.

The main difficulty related with the problem of finding the critical points of the new functional consists in the lack of compactness of the Sobolev spaces embeddings in the unbounded domain $\mathbb{R}^3$. Usually, at least when V is radially symmetric, such a difficulty is overcome by restricting the functional to the natural constraint of the radial functions where compact embeddings hold. In particular, in [14] a radial solution having minimal energy among all the radial solutions has been found. However we are not able to say if that solution actually is a ground state for our equation. This is the reason why we will use an alternative method, based on a concentration-compactness argument on suitable measures, to recover compactness. Such an approach, very standard in studying the compactness in problems involving the Schrödinger equation, seems to be quite new for the nonlinear Schrödinger–Maxwell equations and presents several difficulties due to the coupling.

We analyze two different situations. First we assume that V is a positive constant and we look for a minimizer of the reduced functional restricted to a suitable manifold $\mathcal{M}$ introduced by Ruiz in [23]. Such a manifold has two interesting features: it is a natural constraint for the reduced functional and it contains, in a sense that we will explain later (see Remark 2.2), every solution of the problem (1). The main result we get is the following

Theorem 1.1. *If V is a positive constant, then the problem* (1) *has a ground state solution for any $p \in \,]2, 5[$.*

Remark 1.2. By using the strong maximum principle and quite standard arguments, it is easy to see that such a ground state solution does not change sign, so we can assume it positive.

Afterwards we study (1) assuming the following hypotheses on V:

(V1) $V:\mathbb{R}^3 \to \mathbb{R}$ is a measurable function;
(V2) $V_\infty := \lim_{|y|\to\infty} V(y) \geqslant V(x)$, for almost every $x \in \mathbb{R}^3$, and the inequality is strict in a non-zero measure domain;
(V3) there exists $\overline{C} > 0$ such that, for any $u \in H^1(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 \geqslant \overline{C}\|u\|^2.$$

Remark 1.3. These hypotheses on V, which have been introduced to study singular nonlinear Schrödinger equations in [18], are satisfied by a large class of potentials including those most meaningful by a physical point of view. Here we give some examples of admissible potentials $V:\mathbb{R}^3 \to \mathbb{R}$:

1. $V(x) = V_1 - \lambda|x|^{-\alpha}$, where V_1 is a positive constant, $\alpha = 1, 2$ and λ is a positive constant small enough;
2. $V(x) = V_1(x) - \lambda|x|^{-\alpha}$, where V_1 is a potential bounded below by a positive constant and satisfying (V2), $\alpha = 1, 2$ and λ is a sufficiently small positive constant;
3. $V(x) = V_1(x) - \lambda V_2(x)$, where V_1 is a potential bounded below by a positive constant and satisfying (V2), λ is a sufficiently small positive constant and V_2 is a positive function such that

$$\exists \alpha_1 > 0,\ \alpha_2 \geqslant 0\colon\quad \int_{\mathbb{R}^3} V_2(x)u^2 \leqslant \int_{\mathbb{R}^3} \alpha_1|\nabla u|^2 + \alpha_2 u^2, \quad \text{for any } u \in H^1(\mathbb{R}^3),$$

and

$$\lim_{|x|\to+\infty} V_2(x) = 0.$$

Because of technical difficulties related with the presence of the potential, we are not allowed to use the same device as in the previous case. In particular the use of the Ruiz' constraint appears quite involved, and minimizing the functional on the Nehari manifold turns out to be a more natural approach. However this causes that only the case $3 < p < 5$ can be considered.

Another difficulty consists in the fact that we are not allowed to repeat the same concentration and compactness argument on positive measures as in the constant potential case. The reason is that, since V may have some singularities, we have no way to affirm that the integral

$$\int_{\Omega} |\nabla u|^2 + V(x)|u|^2$$

is nonnegative for any $u \in H^1(\mathbb{R}^3)$ and $\Omega \subset \mathbb{R}^3$, and consequently the measures could be not positive. We get the following

Theorem 1.4. *If V satisfies* (V1)–(V3)*, then the problem* (1) *has a ground state solution for any $p \in]3, 5[$.*

Theorems 1.1 and 1.4 will be proved in Section 2.

It is remarkable that, up to our knowledge, this latter theorem is the first existence result obtained for (1) when V is non-radial, and the nonlinearity is superlinear. Actually, in [26], existence and nonexistence results have been proved when the nonlinearity is asymptotically linear. However, the device used in [26] seems that does not work for nonlinearities such as $|u|^{p-1}u$, with $1 < p < 5$.

In the second part of the paper we consider the critical case, namely the case when the nonlinearity presents at infinity the same behavior of the power t^{2^*-1}, where $2^* = 6$ is the critical exponent for the Sobolev embeddings in dimension 3. Here a further obstacle to compactness arises: in fact, it is well known that the embedding of the space $H^1(\Omega)$ into the Lebesgue space $L^{2^*}(\Omega)$ is not compact, even if Ω is a bounded set in $\mathbb{R}^3$.

The problem becomes

$$\begin{cases} -\Delta u + V(x)u + \phi u = u^5 & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{2}$$

By [15], we have the following

Theorem 1.5 *(D'Aprile and Mugnai [15]). Suppose that V is a positive constant. Let $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ be a solution of the problem* (2)*, then $u = \phi = 0$.*

We extend this nonexistence result to the case of a non-constant potential V. We prove the following nonexistence theorem, based on a Pohozaev-type identity.

Theorem 1.6. *Suppose that V satisfies*

(V4) $V \in C^1(\mathbb{R}^3, \mathbb{R})$;
(V5) $0 < C_3 \leqslant V(x) \leqslant C_4$, *for all* $x \in \mathbb{R}^3$;
(V6) $0 \leqslant 2V(x) + (\nabla V(x) \mid x)$, *for all* $x \in \mathbb{R}^3$.

Let $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ *be a solution of the problem* (2), *then* $u = \phi = 0$.

Then, in the same spirit of [6] (see also [8] for the Klein–Gordon–Maxwell equation), we add a lower order perturbation to the first equation of (2), namely we look for solutions to the system

$$\begin{cases} -\Delta u + V(x)u + \phi u = |u|^{q-1}u + u^5 & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{3}$$

where $q \in]3, 5[$. The solutions $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ of (3) are the critical points of the action functional $\mathcal{E}^* : H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3) \to \mathbb{R}$, defined as

$$\mathcal{E}^*(u, \phi) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 - \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \phi|^2 + \frac{1}{2} \int_{\mathbb{R}^3} \phi u^2 - \frac{1}{q+1} \int_{\mathbb{R}^3} |u|^{q+1} - \frac{1}{6} \int_{\mathbb{R}^3} u^6.$$

The effect of the additive perturbation is to lower the energy. This causes that the ground state level of the functional falls into an interval where compactness holds. As a consequence we get the following two results, respectively for the constant and the non-constant potential case:

Theorem 1.7. *Let V be a positive constant. Then the problem* (3) *has a ground state solution.*

Theorem 1.8. *Let V satisfy* (V1)–(V3). *Then the problem* (3) *has a ground state solution.*

We will prove these three last theorems in Section 3.

Notation.

- For any $1 \leqslant s < +\infty$, $L^s(\mathbb{R}^3)$ is the usual Lebesgue space endowed with the norm

$$\|u\|_s^s := \int_{\mathbb{R}^3} |u|^s;$$

- $H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed with the norm

$$\|u\|^2 := \int_{\mathbb{R}^3} |\nabla u|^2 + u^2;$$

- $\mathcal{D}^{1,2}(\mathbb{R}^3)$ is completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)}^2 := \int_{\mathbb{R}^3} |\nabla u|^2;$$

- for any $r > 0$, $x \in \mathbb{R}^3$ and $A \subset \mathbb{R}^3$

$$B_r(x) := \{ y \in \mathbb{R}^3 \mid |y - x| \leqslant r \},$$
$$B_r := \{ y \in \mathbb{R}^3 \mid |y| \leqslant r \},$$
$$A^c := \mathbb{R}^3 \setminus A;$$

- C, C', C_i are positive constants which can change from line to line;
- $o_n(1)$ is a quantity which goes to zero as $n \to +\infty$.

2. The subcritical case

2.1. Some preliminary results

We first recall some well-known facts (see, for instance [3,10–12,14,23]). For every $u \in L^{12/5}(\mathbb{R}^3)$, there exists a unique $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ solution of

$$-\Delta\phi = u^2 \quad \text{in } \mathbb{R}^3.$$

It can be proved that $(u,\phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ is a solution of (1) if and only if $u \in H^1(\mathbb{R}^3)$ is a critical point of the functional $I : H^1(\mathbb{R}^3) \to \mathbb{R}$ defined as

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}, \tag{4}$$

and $\phi = \phi_u$.

The functions ϕ_u possess the following properties (see [14] and [23]).

Lemma 2.1. *For any $u \in H^1(\mathbb{R}^3)$, we have*:

(i) $\|\phi_u\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)} \leqslant C\|u\|^2$, *where C does not depend from u. As a consequence there exists $C' > 0$ such that*

$$\int_{\mathbb{R}^3} \phi_u u^2 \leqslant C'\|u\|^4;$$

(ii) $\phi_u \geqslant 0$;
(iii) *for any $t > 0$*: $\phi_{tu} = t^2\phi_u$;
(iv) *for any $\theta > 0$*: $\phi_{u_\theta}(x) = \theta^2\phi_u(\theta x)$, *where* $u_\theta(x) = \theta^2 u(\theta x)$;
(v) *for any $\Omega \subset \mathbb{R}^3$ measurable,*

$$\int_{\Omega} \phi_u u^2 = \int_{\Omega}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}\,dx\,dy.$$

2.2. The constant potential case

In this section we will assume that V is a positive constant. Without lost of generality, we suppose $V \equiv 1$. It can be proved (see [15,23]) that if $(u,\phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ is a solution of (1), then it satisfies the following Pohozaev type identity

$$\int_{\mathbb{R}^3} \frac{1}{2}|\nabla u|^2 + \frac{3}{2}u^2 + \frac{5}{4}\phi u^2 - \frac{3}{p+1}|u|^{p+1} = 0. \tag{5}$$

As in [23], we introduce the following manifold

$$\mathcal{M} := \left\{u \in H^1\left(\mathbb{R}^3\right) \setminus \{0\} \;\middle|\; G(u) = 0\right\},$$

where

$$G(u) := \int_{\mathbb{R}^3} \frac{3}{2}|\nabla u|^2 + \frac{1}{2}u^2 + \frac{3}{4}\phi_u u^2 - \frac{2p-1}{p+1}|u|^{p+1}.$$

Remark 2.2. Observe that if $u \in H^1(\mathbb{R}^3)$ is a nontrivial critical point of I, then $u \in \mathcal{M}$, since $G(u) = 0$ can be obtained by a linear combination of $\langle I'(u), u\rangle = 0$ and (5), with $\phi = \phi_u$. As a consequence if $(u,\phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ is a solution of (1), then $u \in \mathcal{M}$.

The next lemma describes some properties of the manifold $\mathcal{M}$:

Lemma 2.3.

1. *For any $u \in H^1(\mathbb{R}^3)$, $u \neq 0$, there exists a unique number $\bar\theta > 0$ such that $u_{\bar\theta} \in \mathcal{M}$ (where $u_{\bar\theta}$ is defined in Lemma* 2.1*). Moreover*

$$I(u_{\bar\theta}) = \max_{\theta \geqslant 0} I(u_\theta);$$

2. *there exists a positive constant C, such that for all $u \in \mathcal{M}$, $\|u\|_{p+1} \geqslant C$;*
3. *$\mathcal{M}$ is a natural constraint of I, namely every critical point of $I|_{\mathcal{M}}$ is a critical point for I.*

Proof. We refer to [23]. In particular, as regards point 3, we have to point out that Ruiz in [23] has just proved that the minimum of $I|_{\mathcal{M}}$ is in fact a critical point of I: the same arguments can be adapted to prove that $\mathcal{M}$ is a natural constraint of I. □

By 3 of Lemma 2.3 we are allowed to look for critical points of I restricted to $\mathcal{M}$.

With an abuse of notations, we denote by $\theta : H^1(\mathbb{R}^3)\setminus\{0\} \to \mathbb{R}_+$ also the map such that for any $u \in H^1(\mathbb{R}^3)$, $u \neq 0$:

$$I(u_{\theta(u)}) = \max_{\theta \geqslant 0} I(u_\theta).$$

By 1 of Lemma 2.3, it is well defined.

Set

$$c_1 = \inf_{g\in\Gamma}\max_{\theta\in[0,1]} I\big(g(\theta)\big), \qquad c_2 = \inf_{u\neq 0}\max_{\theta\geqslant 0} I(u_\theta), \qquad c_3 = \inf_{u\in\mathcal{M}} I(u),$$

where

$$\Gamma = \big\{g \in C\big([0,1], H^1\big(\mathbb{R}^3\big)\big) \;\big|\; g(0)=0,\ I\big(g(1)\big) \leqslant 0,\ g(1)\neq 0\big\}. \tag{6}$$

Lemma 2.4. *The following equalities hold*

$$c := c_1 = c_2 = c_3.$$

Proof. Taking into account 1 of Lemma 2.3 and the fact that for small $\|u\|$ we have (see [23, Theorem 3.2, Step 1])

$$\int_{\mathbb{R}^3} \frac{3}{2}|\nabla u|^2 + \frac{1}{2}u^2 + \frac{3}{4}\phi_u u^2 > \int_{\mathbb{R}^3} \frac{2p-1}{p+1}|u|^{p+1},$$

the conclusion follows using the same arguments of [22, Proposition 3.11]. □

Remark 2.5. By point 3 of Lemma 2.3 and Remark 2.2, we argue that if $u \in \mathcal{M}$ is such that $I(u)=c$, then (u,ϕ_u) is a ground state solution of (1).

2.2.1. Proof of Theorem 1.1

Let $(u_n)_n \subset \mathcal{M}$ such that

$$\lim_n I(u_n) = c. \tag{7}$$

We define the functional $J : H^1(\mathbb{R}^3) \to \mathbb{R}$ as:

$$J(u) = \int_{\mathbb{R}^3} \frac{p-2}{2p-1}|\nabla u|^2 + \frac{p-1}{2p-1}u^2 + \frac{p-2}{2(2p-1)}\phi_u u^2.$$

Observe that for any $u \in \mathcal{M}$, by (ii) of Lemma 2.1 we have $I(u) = J(u) \geqslant 0$.

By (7), we deduce that $(u_n)_n$ is bounded in $H^1(\mathbb{R}^3)$, so there exists $\bar{u} \in H^1(\mathbb{R}^3)$ such that, up to a subsequence,

$$\begin{aligned} &u_n \rightharpoonup \bar{u} \quad \text{weakly in } H^1\big(\mathbb{R}^3\big), \\ &u_n \to \bar{u} \quad \text{in } L^s(B),\ \text{with } B \subset \mathbb{R}^3,\ \text{bounded, and } 1 \leqslant s < 6. \end{aligned} \tag{8}$$

To prove Theorem 1.1, we need some compactness on the sequence $(u_n)_n$. To this end, we use a concentration-compactness argument on the positive measures so defined: for every $u_n \in H^1(\mathbb{R}^3)$,

$$\nu_n(\Omega) = \int_{\Omega} \frac{p-2}{2p-1}|\nabla u_n|^2 + \frac{p-1}{2p-1}u_n^2 + \frac{p-2}{2(2p-1)}\phi_{u_n} u_n^2. \tag{9}$$

By (7) we have

$$\nu_n\big(\mathbb{R}^3\big) = J(u_n) \to c$$

and then, by P.L. Lions [19], there are three possibilities:

vanishing: for all $r > 0$

$$\lim_n \sup_{\xi \in \mathbb{R}^3} \int_{B_r(\xi)} d\nu_n = 0;$$

dichotomy: there exist a constant $\tilde{c} \in (0, c)$, two sequences $(\xi_n)_n$ and $(r_n)_n$, with $r_n \to +\infty$ and two nonnegative measures ν_n^1 and ν_n^2 such that

$$0 \leqslant \nu_n^1 + \nu_n^2 \leqslant \nu_n, \quad \nu_n^1(\mathbb{R}^3) \to \tilde{c}, \ \nu_n^2(\mathbb{R}^3) \to c - \tilde{c},$$
$$\operatorname{supp}(\nu_n^1) \subset B_{r_n}(\xi_n), \qquad \operatorname{supp}(\nu_n^2) \subset \mathbb{R}^3 \setminus B_{2r_n}(\xi_n);$$

compactness: there exists a sequence $(\xi_n)_n$ in $\mathbb{R}^3$ with the following property: for any $\delta > 0$, there exists $r = r(\delta) > 0$ such that

$$\int_{B_r(\xi_n)} d\nu_n \geqslant c - \delta.$$

Arguing as in [27], we prove the following

Lemma 2.6. *Compactness holds for the sequence of measures $(\nu_n)_n$, defined in* (9).

Proof. *Vanishing does not occur.*

Suppose by contradiction, that for all $r > 0$

$$\lim_n \sup_{\xi \in \mathbb{R}^3} \int_{B_r(\xi)} d\nu_n = 0.$$

In particular, we deduce that there exists $\bar{r} > 0$ such that

$$\lim_n \sup_{\xi \in \mathbb{R}^3} \int_{B_{\bar{r}}(\xi)} u_n^2 = 0.$$

By [20, Lemma I.1], we have that $u_n \to 0$ in $L^s(\mathbb{R}^3)$, for $2 < s < 6$. As a consequence, since $(u_n)_n \subset \mathcal{M}$ and by Lemma 2.1, we get

$$0 \leqslant I(u_n) \leqslant \int_{\mathbb{R}^3} \frac{3}{2}|\nabla u_n|^2 + \frac{1}{2}u_n^2 + \frac{1}{4}\phi_{u_n} u_n^2 - \frac{1}{p+1}|u_n|^{p+1} = -\frac{1}{2}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 + \frac{2p-2}{p+1}\int_{\mathbb{R}^3} |u_n|^{p+1} \to 0$$

which contradicts (7).

Dichotomy does not occur.

Suppose by contradiction that there exist a constant $\tilde{c} \in (0, c)$, two sequences $(\xi_n)_n$ and $(r_n)_n$, with $r_n \to +\infty$ and two nonnegative measures ν_n^1 and ν_n^2 such that

$$0 \leqslant \nu_n^1 + \nu_n^2 \leqslant \nu_n, \qquad \nu_n^1(\mathbb{R}^3) \to \tilde{c}, \qquad \nu_n^2(\mathbb{R}^3) \to c - \tilde{c},$$
$$\operatorname{supp}(\nu_n^1) \subset B_{r_n}(\xi_n), \qquad \operatorname{supp}(\nu_n^2) \subset \mathbb{R}^3 \setminus B_{2r_n}(\xi_n).$$

Let $\rho_n \in C^1(\mathbb{R}^3)$ be such that $\rho_n \equiv 1$ in $B_{r_n}(\xi_n)$, $\rho_n \equiv 0$ in $\mathbb{R}^3 \setminus B_{2r_n}(\xi_n)$, $0 \leqslant \rho_n \leqslant 1$ and $|\nabla \rho_n| \leqslant 2/r_n$.

We set

$$v_n := \rho_n u_n, \qquad w_n := (1 - \rho_n) u_n.$$

It is easy to see that

$$\liminf_n J(v_n) \geqslant \tilde{c}, \qquad \liminf_n J(w_n) \geqslant c - \tilde{c}.$$

Moreover, denoting $\Omega_n := B_{2r_n}(\xi_n) \setminus B_{r_n}(\xi_n)$, we have

$$\nu_n(\Omega_n) \to 0, \quad \text{as } n \to \infty,$$

namely

$$\int_{\Omega_n} |\nabla u_n|^2 + u_n^2 \to 0, \quad \text{as } n \to \infty,$$

$$\int_{\Omega_n} \phi_{u_n} u_n^2 \to 0, \quad \text{as } n \to \infty. \tag{10}$$

By simple computations, we infer also

$$\int_{\Omega_n} |\nabla v_n|^2 + v_n^2 \to 0, \quad \text{as } n \to \infty,$$

$$\int_{\Omega_n} |\nabla w_n|^2 + w_n^2 \to 0, \quad \text{as } n \to \infty.$$

Hence, we deduce that

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 + u_n^2 = \int_{\mathbb{R}^3} |\nabla v_n|^2 + v_n^2 \quad + \int_{\mathbb{R}^3} |\nabla w_n|^2 + w_n^2 + o_n(1), \tag{11}$$

$$\int_{\mathbb{R}^3} |u_n|^{p+1} = \int_{\mathbb{R}^3} |v_n|^{p+1} + \int_{\mathbb{R}^3} |w_n|^{p+1} + o_n(1). \tag{12}$$

Moreover, by point v of Lemma 2.1 and (10), we have

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 = \int_{\mathbb{R}^3} \phi_{v_n} v_n^2 + \int_{\mathbb{R}^3} \phi_{w_n} w_n^2 + 2 \int_{B_{r_n}} \int_{B_{2r_n}^c} \frac{u_n^2(x) u_n^2(y)}{|x-y|} \, dx \, dy + o_n(1) \geqslant \int_{\mathbb{R}^3} \phi_{v_n} v_n^2 + \int_{\mathbb{R}^3} \phi_{w_n} w_n^2 + o_n(1). \tag{13}$$

Hence, by (11) and (13), we get

$$J(u_n) \geqslant J(v_n) + J(w_n) + o_n(1).$$

Then

$$c = \lim_n J(u_n) \geqslant \liminf_n J(v_n) + \liminf_n J(w_n) \geqslant \tilde{c} + (c - \tilde{c}) = c,$$

hence

$$\lim_n J(v_n) = \tilde{c}, \qquad \lim_n J(w_n) = c - \tilde{c}. \tag{14}$$

We recall the definition of the functional $G : H^1(\mathbb{R}^3) \to \mathbb{R}$

$$G(u) = \int_{\mathbb{R}^3} \frac{3}{2} |\nabla u|^2 + \frac{1}{2} u^2 + \frac{3}{4} \phi_u u^2 - \frac{2p-1}{p+1} |u|^{p+1}$$

and that if $u \in \mathcal{M}$, then $G(u) = 0$. By (11)–(13), we have

$$0 = G(u_n) \geqslant G(v_n) + G(w_n) + o_n(1). \tag{15}$$

By Lemma 2.3, for any $n \geqslant 1$, there exists $\theta_n > 0$ such that $(v_n)_{\theta_n} \in \mathcal{M}$, and then

$$\int_{\mathbb{R}^3} \frac{3}{2} \theta_n^2 |\nabla v_n|^2 + \frac{1}{2} v_n^2 + \frac{3}{4} \theta_n^2 \phi_{v_n} v_n^2 = \int_{\mathbb{R}^3} \frac{2p-1}{p+1} \theta_n^{2p-2} |v_n|^{p+1}. \tag{16}$$

We have to distinguish three cases.

Case 1: Up to a subsequence, $G(v_n) \leqslant 0$.
By (16) we have

$$\int_{\mathbb{R}^3} \frac{3}{2} \big(\theta_n^{2p-2} - \theta_n^2\big) |\nabla v_n|^2 + \frac{1}{2} \big(\theta_n^{2p-2} - 1\big) v_n^2 + \frac{3}{4} \big(\theta_n^{2p-2} - \theta_n^2\big) \phi_{v_n} v_n^2 \leqslant 0,$$

which implies that $\theta_n \leqslant 1$. Therefore, for all $n \geqslant 1$

$$c \leqslant I\big((v_n)_{\theta_n}\big) = J\big((v_n)_{\theta_n}\big) \leqslant J(v_n) \to \tilde{c} < c,$$

which is a contradiction.

Case 2: Up to a subsequence, $G(w_n) \leqslant 0$.
We can argue as in the previous case.

Case 3: Up to a subsequence, $G(v_n) > 0$ and $G(w_n) > 0$.
By (15), we infer that $G(v_n) = o_n(1)$ and $G(w_n) = o_n(1)$. If $\theta_n \leqslant 1 + o_n(1)$, we can repeat the arguments of Case 1. Suppose that

$$\lim_n \theta_n = \theta_0 > 1.$$

We have

$$\begin{aligned} o_n(1) = G(v_n) &= \int_{\mathbb{R}^3} \frac{3}{2}|\nabla v_n|^2 + \frac{1}{2}v_n^2 + \frac{3}{4}\phi_{v_n} v_n^2 - \frac{2p-1}{p+1}|v_n|^{p+1} \\ &= \int_{\mathbb{R}^3} \frac{3}{2}\left(1 - \frac{1}{\theta_n^{2p-4}}\right)|\nabla v_n|^2 + \frac{1}{2}\left(1 - \frac{1}{\theta_n^{2p-2}}\right)v_n^2 + \int_{\mathbb{R}^3} \frac{3}{4}\left(1 - \frac{1}{\theta_n^{2p-4}}\right)\phi_{v_n} v_n^2 \end{aligned}$$

and so $v_n \to 0$ in $H^1(\mathbb{R}^3)$, but we get a contradiction with (14).
Hence we conclude that dichotomy cannot occur. □

Now we are able to yield the following

Proof of Theorem 1.1. Let $(u_n)_n$ be a sequence in $\mathcal{M}$ such that (7) holds. We define the measures $(\nu_n)_n$ as in (9); by Lemma 2.6 there exists a sequence $(\xi_n)_n$ in $\mathbb{R}^N$ with the following property: for any $\delta > 0$, there exists $r = r(\delta) > 0$ such that

$$\int_{B_r^c(\xi_n)} \frac{p-2}{2p-1}|\nabla u_n|^2 + \frac{p-1}{2p-1}u_n^2 + \frac{p-2}{2(2p-1)}\phi_{u_n} u_n^2 < \delta. \tag{17}$$

We define the new sequence of functions $v_n := u_n(\cdot - \xi_n) \in H^1(\mathbb{R}^3)$. It is easy to see that $\phi_{v_n} = \phi_{u_n}(\cdot - \xi_n)$, and hence $v_n \in \mathcal{M}$. Moreover, by (17), we have that for any $\delta > 0$, there exists $r = r(\delta) > 0$ such that

$$\|v_n\|_{H^1(B_r^c)} < \delta \quad \text{uniformly for } n \geqslant 1. \tag{18}$$

Since, by (8), $(v_n)_n$ is bounded in $H^1(\mathbb{R}^3)$, certainly there exist a subsequence (likewise labelled) and $\bar{v} \in H^1(\mathbb{R}^3)$ such that

$$v_n \rightharpoonup \bar{v} \quad \text{weakly in } H^1\big(\mathbb{R}^3\big), \tag{19}$$

$$v_n \to \bar{v} \quad \text{in } L^s(B),\ \text{with } B \subset \mathbb{R}^3,\ \text{bounded, and } 1 \leqslant s < 6. \tag{20}$$

By (18), (19) and (20), we have that, taken $s \in [2, 6[$, for any $\delta > 0$ there exists $r > 0$ such that, for any $n \geqslant 1$ large enough

$$\|v_n - \bar{v}\|_{L^s(\mathbb{R}^3)} \leqslant \|v_n - \bar{v}\|_{L^s(B_r)} + \|v_n - \bar{v}\|_{L^s(B_r^c)} \leqslant \delta + C\big(\|v_n\|_{H^1(B_r^c)} + \|\bar{v}\|_{H^1(B_r^c)}\big) \leqslant (1 + 2C)\delta,$$

where $C > 0$ is the constant of the embedding $H^1(B_r^c) \hookrightarrow L^s(B_r^c)$. We deduce that

$$v_n \to \bar{v} \quad \text{in } L^s\big(\mathbb{R}^3\big),\ \text{for any } s \in [2, 6[. \tag{21}$$

Since ϕ is continuous from $L^{12/5}(\mathbb{R}^3)$ to $\mathcal{D}^{1,2}(\mathbb{R}^3)$, from (21) we deduce that

$$\begin{aligned} &\phi_{v_n} \to \phi_{\bar{v}} \quad \text{in } \mathcal{D}^{1,2}\big(\mathbb{R}^3\big),\ \text{as } n \to \infty, \\ &\int_{\mathbb{R}^3} \phi_{v_n} v_n^2 \to \int_{\mathbb{R}^3} \phi_{\bar{v}} \bar{v}^2, \quad \text{as } n \to \infty. \end{aligned} \tag{22}$$

Since $(v_n)_n$ is in $\mathcal{M}$, by 2 of Lemma 2.3 $(\|v_n\|_{p+1})_n$ is bounded below by a positive constant. As a consequence, (21) implies that $\bar{v} \neq 0$. Proceeding as in [23, Theorem 3.2, Step 4], by (21) and (22) we can show that $v_n \to \bar{v}$ in $H^1(\mathbb{R}^3)$ so that $\bar{v} \in \mathcal{M}$ and $I(\bar{v}) = c$. By Remark 2.5, we have that $(\bar{v}, \phi_{\bar{v}})$ is a ground state solution of (1). □

2.3. The non-constant potential case

In this section we suppose that the potential V satisfies (V1)–(V3) and that $p \in]3,5[$.

In order to get our result, we will use a very standard device: we will look for a minimizer of the functional (4) restricted to the Nehari manifold

$$\mathcal{N} = \{u \in H^1(\mathbb{R}^3) \setminus \{0\} \mid \widetilde{G}(u) = 0\},$$

where

$$\widetilde{G}(u) := \int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 + \phi_u u^2 - |u|^{p+1}.$$

The following lemma describes some properties of the Nehari manifold $\mathcal{N}$:

Lemma 2.7.

1. *For any $u \neq 0$ there exists a unique number $\bar{t} > 0$ such that $\bar{t}u \in \mathcal{N}$ and*

$$I(\bar{t}u) = \max_{t \geqslant 0} I(tu);$$

2. *there exists a positive constant C, such that for all $u \in \mathcal{N}$, $\|u\|_{p+1} \geqslant C$;*
3. *$\mathcal{N}$ is a C^1 manifold.*

Proof. Points 1 and 2 can be proved using standard arguments (see, for example, [22]).

3. Observe that for any $u \in H^1(\mathbb{R}^3)$ we have

$$\widetilde{G}(u) = 4I(u) - \int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) - \frac{p-3}{p+1} \int_{\mathbb{R}^3} |u|^{p+1},$$

and then, by point 2, for any $u \in \mathcal{N}$ we have

$$\langle \widetilde{G}'(u), u \rangle = -2 \int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) - (p-3) \int_{\mathbb{R}^3} |u|^{p+1} \leqslant -C < 0. \qquad \square$$

The Nehari manifold $\mathcal{N}$ is a natural constraint for the functional I, therefore we are allowed to look for critical points of I restricted to $\mathcal{N}$.

In view of this, we assume the following definition

$$c_V := \inf_{u \in \mathcal{N}} I(u),$$

so that our goal is to find $\bar{u} \in \mathcal{N}$ such that $I(\bar{u}) = c_V$, from which we would deduce that $(\bar{u}, \phi_{\bar{u}})$ is a ground state solution of (1).

First we recall some preliminary lemmas which can be obtained by using the same arguments as in [22] (see also [2]).

As a consequence of Lemma 2.7, we are allowed to define the map $t : H^1(\mathbb{R}^3) \setminus \{0\} \to \mathbb{R}_+$ such that for any $u \in H^1(\mathbb{R}^3)$, $u \neq 0$:

$$I(t(u)u) = \max_{t \geqslant 0} I(tu).$$

Lemma 2.8. *The following equalities hold*

$$c_V = \inf_{g \in \Gamma} \max_{t \in [0,1]} I(g(t)) = \inf_{u \neq 0} \max_{t \geqslant 0} I(tu),$$

where Γ is the same set defined in (6).

Lemma 2.9. *Let $u_n \in H^1(\mathbb{R}^3)$, $n \geqslant 1$, such that $\|u_n\| \geqslant C > 0$ and*

$$\max_{t \geqslant 0} I(tu_n) \leqslant c_V + \delta_n,$$

with $\delta_n \to 0^+$. Then, there exist a sequence $(y_n)_n \subset \mathbb{R}^N$ and two positive numbers $R, \mu > 0$ such that

$$\liminf_n \int_{B_R(y_n)} |u_n|^2 \, dx > \mu.$$

Lemma 2.10. *Let $(u_n)_n \subset H^1(\mathbb{R}^3)$ such that $\|u_n\| = 1$ and*

$$I\big(t(u_n)u_n\big) = \max_{t \geqslant 0} I(tu_n) \to c_V, \quad \text{as } n \to \infty.$$

Then the sequence $(t(u_n))_n \subset \mathbb{R}_+$ possesses a bounded subsequence in $\mathbb{R}$.

Proof. We have

$$C \geqslant \int_{\mathbb{R}^3} |\nabla u_n|^2 + V(x)u_n^2 = t_n^2\bigg(t_n^{p-3}\int_{\mathbb{R}^3} |u_n|^{p+1} - \int_{\mathbb{R}^3} \phi_{u_n} u_n^2\bigg).$$

The conclusion follows from (i) of Lemma 2.1 and Lemma 2.9. □

Lemma 2.11. *Suppose that V, $V_n \in L^\infty$, for all $n \geqslant 1$. If $V_n \to V$ in $L^\infty(\mathbb{R}^N)$, then $c_{V_n} \to c_V$.*

Now define

$$I_\infty(u) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + V_\infty u^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1},$$
$$c_\infty := c_{V_\infty}.$$

As in [22], we have

Lemma 2.12. *If V satisfies* (V1)–(V3)*, we get $c_V < c_\infty$.*

Proof. By Theorem 1.1, there exists $(w, \phi_w) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ a ground state solution of the problem

$$\begin{cases} -\Delta u + V_\infty u + \phi u = |u|^{p-1}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3. \end{cases}$$

Let $t(w) > 0$ be such that $t(w)w \in \mathcal{N}$. By (V2), we have

$$c_\infty = I_\infty(w) \geqslant I_\infty\big(t(w)w\big) = I\big(t(w)w\big) + \int_{\mathbb{R}^N} \big(V_\infty - V(x)\big)\big|t(w)w\big|^2 > c_V,$$

and then we conclude. □

2.3.1. Proof of Theorem 1.4

Let $(u_n)_n \subset \mathcal{N}$ such that

$$\lim_n I(u_n) = c_V. \tag{23}$$

We define the functional $J : H^1(\mathbb{R}^3) \to \mathbb{R}$ as:

$$J(u) = \bigg(\frac{1}{2} - \frac{1}{p+1}\bigg)\int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 + \bigg(\frac{1}{4} - \frac{1}{p+1}\bigg)\int_{\mathbb{R}^3} \phi_u u^2.$$

Observe that for any $u \in \mathcal{N}$, we have $I(u) = J(u)$.

By (V3) and (23), we deduce that $(u_n)_n$ is bounded in $H^1(\mathbb{R}^3)$, so there exists $\bar{u} \in H^1(\mathbb{R}^3)$ such that, up to a subsequence,

$$u_n \rightharpoonup \bar{u} \quad \text{weakly in } H^1\big(\mathbb{R}^3\big), \tag{24}$$

$$u_n \to \bar{u} \quad \text{in } L^s(B), \text{ with } B \subset \mathbb{R}^3, \text{ bounded, and } 1 \leqslant s < 6. \tag{25}$$

To prove Theorem 1.4, we need some compactness on the sequence $(u_n)_n$.

We denote by ν_n the measure

$$\nu_n(\Omega) = \bigg(\frac{1}{2} - \frac{1}{p+1}\bigg)\int_{\Omega} |\nabla u_n|^2 + V(x)u_n^2 + \bigg(\frac{1}{4} - \frac{1}{p+1}\bigg)\int_{\Omega} \phi_{u_n} u_n^2.$$

Observe that, since there is no lower boundedness condition on the potential V, the measures ν_n may be not positive, and then we are not allowed to use the Lions' concentration arguments [19,20] on them. However, using a variant presented in [9], in the following theorem we are able to show that the functions u_k concentrate in the $H^1(\mathbb{R}^3)$-norms.

Theorem 2.13. *For any $\delta > 0$ there exists $\widetilde{R} > 0$ such that for any $n \geqslant \widetilde{R}$*

$$\int_{|x|>\widetilde{R}} \left(|\nabla u_n|^2 + |u_n|^2\right) < \delta.$$

Proof. By contradiction, suppose that there exist $\delta_0 > 0$ and a subsequence $(u_k)_k$ such that for any $k \geqslant 1$

$$\int_{|x|>k} \left(|\nabla u_k|^2 + |u_k|^2\right) \geqslant \delta_0. \tag{26}$$

We define

$$\rho_k(\Omega) = \int_{\Omega} |\nabla u_k|^2 + |u_k|^2 + \int_{\Omega} \phi_{u_k} u_k^2$$

and, for any $r > 0$, we set $A_r := \{x \in \mathbb{R}^3 \mid r \leqslant |x| \leqslant r+1\}$.

We claim that

$$\text{for any } \mu > 0 \text{ and } R > 0, \quad \text{there exists } r > R \text{ such that } \rho_k(A_r) < \mu \tag{27}$$

for infinitely many k. If not, then there should exist $\hat{\mu} > 0$ and $\widehat{R} \in \mathbb{N}$ such that, for any $m \geqslant \widehat{R}$, there exists $p(m)$ such that, for any $k \geqslant p(m)$,

$$\rho_k(A_m) \geqslant \hat{\mu}.$$

We are allowed to take $(p(m))_m$ not decreasing, so that for every $m \geqslant \widehat{R}$ we could get u_k such that, using (i) of Lemma 2.1,

$$C\|u_k\|^2\left(1 + \|u_k\|^2\right) \geqslant \|u_k\|^2 + \int_{\mathbb{R}^3} \phi_{u_k} u_k^2 \geqslant \rho_k(B_m \setminus B_{\widehat{R}}) \geqslant (m - \widehat{R})\hat{\mu}$$

contradicting the boundedness in $H^1(\mathbb{R}^3)$ of the sequence $(u_n)_n$.

So, we assume that (27) holds. Taking into account Lemmas 2.11 and 2.12, consider $\mu > 0$ such that

$$c < c(V_\infty - \mu) < c(V_\infty).$$

Using (V2), there exists $R_\mu \in \mathbb{N}$ such that for almost every $|x| \geqslant R_\mu$

$$V(x) \geqslant V_\infty - \mu > 0; \tag{28}$$

we take $r > R_\mu$ such that, up to a subsequence,

$$\rho_k(A_r) < \mu, \quad \text{for all } k \geqslant 1. \tag{29}$$

In particular, (28) and (29) imply

$$\int_{A_r} |\nabla u_k|^2 + V(x) u_k^2 = O(\mu), \quad \text{for all } k \geqslant 1, \tag{30}$$

$$\int_{A_r} \phi_{u_k} u_k^2 = O(\mu), \quad \text{for all } k \geqslant 1. \tag{31}$$

Let $\chi \in C^\infty$, such that $\chi = 1$ in B_r and $\chi = 0$ in $(B_{r+1})^c$, $0 \leqslant \chi \leqslant 1$ and $|\nabla \chi| \leqslant 2$. Set $v_k = \chi u_k$ and $w_k = (1-\chi)u_k$.

By simple computations, by (28) and (30) we infer

$$\int_{A_r} |\nabla v_k|^2 + V(x) v_k^2 = O(\mu), \qquad \int_{A_r} |v_k|^{p+1} = O(\mu),$$

$$\int_{A_r} |\nabla w_k|^2 + V(x) w_k^2 = O(\mu), \qquad \int_{A_r} |w_k|^{p+1} = O(\mu).$$

Hence, we deduce that

$$\int_{\mathbb{R}^3} |\nabla u_k|^2 + V(x) u_k^2 = \int_{\mathbb{R}^3} |\nabla v_k|^2 + V(x) v_k^2 + \int_{\mathbb{R}^3} |\nabla w_k|^2 + V(x) w_k^2 + O(\mu), \tag{32}$$

$$\int_{\mathbb{R}^3} |u_k|^{p+1} = \int_{\mathbb{R}^3} |v_k|^{p+1} + \int_{\mathbb{R}^3} |w_k|^{p+1} + O(\mu); \tag{33}$$

for large $k \geqslant 1$, by (26) and (28), we also deduce that there exists $\delta' > 0$ such that

$$\int_{\mathbb{R}^3} |\nabla w_k|^2 + V(x)|w_k|^2 \geqslant \delta' + O(\mu). \tag{34}$$

Moreover, arguing as in (13), we have

$$\int_{\mathbb{R}^3} \phi_{u_k} u_k^2 \geqslant \int_{\mathbb{R}^3} \phi_{v_k} v_k^2 + \int_{\mathbb{R}^3} \phi_{w_k} w_k^2 + O(\mu). \tag{35}$$

Hence, by (32) and (35), we get

$$J(u_k) \geqslant J(v_k) + J(w_k) + O(\mu),$$

and then, using (34) and (V3), we deduce

$$J(u_k) - C\delta' \geqslant J(v_k) + O(\mu), \tag{36}$$

$$J(u_k) \geqslant J(w_k) + O(\mu). \tag{37}$$

We recall the definition of the functional $\widetilde{G} : H^1(\mathbb{R}^3) \to \mathbb{R}$

$$\widetilde{G}(u) = \int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 + \phi_u u^2 - |u|^{p+1}$$

and that if $u \in \mathcal{N}$, then $\widetilde{G}(u) = 0$. By (32), (33) and (35), we have

$$0 = \widetilde{G}(u_k) \geqslant \widetilde{G}(v_k) + \widetilde{G}(w_k) + O(\mu). \tag{38}$$

We have to distinguish three cases.

Case 1: Up to a subsequence, $\widetilde{G}(v_k) \leqslant 0$.

By Lemma 2.7, for any $k \geqslant 1$, there exists $\theta_k > 0$ such that $\theta_k v_k \in \mathcal{N}$, and then

$$\int_{\mathbb{R}^3} |\nabla v_k|^2 + V(x)v_k^2 + \theta_k^2 \phi_{v_k} v_k^2 = \int_{\mathbb{R}^3} \theta_k^{p-1} |v_k|^{p+1}. \tag{39}$$

By (39) we have

$$\left(\theta_k^{p-1} - 1\right) \int_{\mathbb{R}^3} |\nabla v_k|^2 + V(x)v_k^2 + \left(\theta_k^{p-1} - \theta_k^2\right) \int_{\mathbb{R}^3} \phi_{v_k} v_k^2 \leqslant 0,$$

and, by (V3), we deduce that $\theta_k \leqslant 1$. Therefore, for all $k \geqslant 1$, by (V3) and (36),

$$c_V \leqslant I(\theta_k v_k) = J(\theta_k v_k) \leqslant J(v_k) \leqslant J(u_k) - C\delta' + O(\mu) = c_V - C\delta' + o_k(1) + O(\mu),$$

which is a contradiction.

Case 2: Up to a subsequence, $\widetilde{G}(w_k) \leqslant 0$.

Let $(\eta_k)_k$ be such that, for any $k \geqslant 1$, $\eta_k w_k \in \mathcal{N}$. Arguing as in the previous case, we deduce that $\eta_k \leqslant 1$. Define $\tilde{w}_k = \eta_k w_k$. Let $(t_k)_k$ be such that, for any $k \geqslant 1$, $t_k \tilde{w}_k \in \mathcal{N}_{V_\infty - \mu}$.

By (28),

$$\int_{\mathbb{R}^3} |\nabla \tilde{w}_k|^2 + (V_\infty - \mu)\tilde{w}_k^2 + \phi_{\tilde{w}_k} \tilde{w}_k^2 \leqslant \int_{\mathbb{R}^3} |\nabla \tilde{w}_k|^2 + V(x)\tilde{w}_k^2 + \phi_{\tilde{w}_k} \tilde{w}_k^2 = \int_{\mathbb{R}^3} |\tilde{w}_k|^{p+1},$$

and then $t_k \leqslant 1$. By (37) and (V3), we conclude that

$$\begin{aligned}
c(V_\infty - \mu) &\leqslant \frac{t_k^2}{2} \int_{\mathbb{R}^3} |\nabla \tilde{w}_k|^2 + (V_\infty - \mu)\tilde{w}_k^2 + \frac{t_k^4}{4} \int_{\mathbb{R}^3} \phi_{\tilde{w}_k} \tilde{w}_k^2 - \frac{t_k^{p+1}}{p+1} \int_{\mathbb{R}^3} |\tilde{w}_k|^{p+1} \\
&\leqslant \frac{t_k^2}{2} \int_{\mathbb{R}^3} |\nabla \tilde{w}_k|^2 + V(x)\tilde{w}_k^2 + \frac{t_k^4}{4} \int_{\mathbb{R}^3} \phi_{\tilde{w}_k} \tilde{w}_k^2 - \frac{t_k^{p+1}}{p+1} \int_{\mathbb{R}^3} |\tilde{w}_k|^{p+1} \\
&= \left(\frac{t_k^2}{2} - \frac{t_k^{p+1}}{p+1}\right) \int_{\mathbb{R}^3} |\nabla \tilde{w}_k|^2 + V(x)\tilde{w}_k^2 + \left(\frac{t_k^4}{4} - \frac{t_k^{p+1}}{p+1}\right) \int_{\mathbb{R}^3} \phi_{\tilde{w}_k} \tilde{w}_k^2 \\
&\leqslant J(\tilde{w}_k) = J(\eta_k w_k) \leqslant J(w_k) \leqslant J(u_k) + O(\mu) = c_V + o_k(1) + O(\mu),
\end{aligned}$$

but, letting μ go to zero and k go to ∞, by Lemma 2.11, this yields a contradiction with Lemma 2.12.

Case 3: Up to a subsequence, $\widetilde{G}(v_k) > 0$ and $\widetilde{G}(w_k) > 0$.

By (38), we infer that $\widetilde{G}(v_k) = O(\mu)$ and $\widetilde{G}(w_k) = O(\mu)$. Let $(\eta_k)_k$ be such that $\eta_k w_k \in \mathcal{N}$. If $\eta_k \leqslant 1 + O(\mu)$, we can repeat the arguments of Case 2. Suppose that

$$\lim_k \eta_k = \eta_0 > 1.$$

We have

$$O(\mu) = \widetilde{G}(w_k) = \int_{\mathbb{R}^3} |\nabla w_k|^2 + V(x)w_k^2 + \phi_{w_k} w_k^2 - |w_k|^{p+1} = \left(1 - \frac{1}{\eta_k^{p-1}}\right) \int_{\mathbb{R}^3} |\nabla w_k|^2 + V(x)w_k^2 + \left(1 - \frac{1}{\eta_k^{p-3}}\right) \int_{\mathbb{R}^3} \phi_{w_k} w_k^2$$

and so

$$\int_{\mathbb{R}^3} |\nabla w_k|^2 + V(x)w_k^2 = O(\mu),$$

which contradicts (34). □

Proof of Theorem 1.4. By Theorem 2.13, for any $\delta > 0$ there exists $r > 0$ such that

$$\|u_n\|_{H^1(B_r^c)} < \delta, \quad \text{uniformly for } n \geqslant 1. \tag{40}$$

Hence, arguing as in the constant potential case, we deduce that

$$u_n \to \bar{u} \quad \text{in } L^s(\mathbb{R}^3), \text{ for any } s \in [2, 6[. \tag{41}$$

Moreover

$$\begin{aligned} &\phi_{u_n} \to \phi_{\bar{u}} \quad \text{in } \mathcal{D}^{1,2}(\mathbb{R}^3), \text{ as } n \to \infty, \\ &\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \to \int_{\mathbb{R}^3} \phi_{\bar{u}} \bar{u}^2, \quad \text{as } n \to \infty, \end{aligned} \tag{42}$$

and for any $\psi \in C_0^\infty(\mathbb{R}^3)$

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n \psi \to \int_{\mathbb{R}^3} \phi_{\bar{u}} \bar{u} \psi. \tag{43}$$

By (23), we can suppose (see [28]) that $(u_n)_n$ is a Palais–Smale sequence for $I|_{\mathcal{N}}$ and, as a consequence, it is easy to see that $(u_n)_n$ is a Palais–Smale sequence for I. By (24), (41) and (43), we conclude that $I'(\bar{u}) = 0$.

Since $(u_n)_n$ is in $\mathcal{N}$, by 3 of Lemma 2.7 $(\|u_n\|_{p+1})_n$ is bounded below by a positive constant. As a consequence, (41) implies that $\bar{u} \neq 0$ and so $\bar{u} \in \mathcal{N}$.

Finally, by (23), (24), (41) and (42) and by (V2)–(V3) we get

$$c_V \leqslant I(\bar{u}) \leqslant \liminf I(u_n) = c_V,$$

so we can conclude that $(\bar{u}, \phi_{\bar{u}})$ is a ground state solution of (1). □

3. The critical case

This section is devoted to the study of the critical case and in particular we will give the proofs of Theorems 1.6, 1.7 and 1.8.

3.1. The nonexistence result

Proof of Theorem 1.6. Arguing as in [5,15], we can prove that if $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ is a solution of the problem (2), then (u, ϕ) satisfies the following Pohozaev identity:

$$\int_{\mathbb{R}^3} |\nabla u|^2 + 3 \int_{\mathbb{R}^3} V(x)u^2 + \int_{\mathbb{R}^3} \big(\nabla V(x) \mid x\big)u^2 + \frac{5}{2} \int_{\mathbb{R}^3} \phi u^2 = \int_{\mathbb{R}^3} u^6. \tag{44}$$

Multiplying the first equation of (2) by u and integrating, we have

$$\int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} V(x)u^2 + \int_{\mathbb{R}^3} \phi u^2 = \int_{\mathbb{R}^3} u^6; \tag{45}$$

on the other hand, multiplying the second equation of (2) by ϕ and integrating, we have

$$\int_{\mathbb{R}^3} |\nabla\phi|^2 = \int_{\mathbb{R}^3} \phi u^2. \tag{46}$$

By the combination of (44)–(46), we infer that

$$\int_{\mathbb{R}^3} \big[2V(x) + \big(\nabla V(x) \mid x\big)\big]u^2 + \frac{3}{2}\int_{\mathbb{R}^3} |\nabla\phi|^2 = 0,$$

which, together with (V5)–(V6), implies that $u = \phi = 0$. $\square$

3.2. The existence results

As in Section 2.1, for every $u \in L^{12/5}(\mathbb{R}^3)$ we denote by $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ the unique solution of

$$-\Delta\phi = u^2, \quad \text{in } \mathbb{R}^3.$$

It can be proved that $(u,\phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ is a solution of (3) if and only if $u \in H^1(\mathbb{R}^3)$ is a critical point of the functional $I^*: H^1(\mathbb{R}^3) \to \mathbb{R}$ defined as

$$I^*(u) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 - \frac{1}{q+1}\int_{\mathbb{R}^3} |u|^{q+1} - \frac{1}{6}\int_{\mathbb{R}^3} u^6,$$

and $\phi = \phi_u$.

The Nehari manifold of the functional I^*, defined as

$$\mathcal{N}^* := \left\{u \in H^1\big(\mathbb{R}^3\big) \setminus \{0\} \;\Big|\; \int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 + \phi_u u^2 - |u|^{q+1} - u^6 = 0\right\},$$

satisfies the equivalent of Lemma 2.7 and so it is a natural constraint for I^*. We are looking for critical points of I^* restricted to $\mathcal{N}^*$.

Set

$$c_1^* = \inf_{g\in\Gamma^*}\max_{t\in[0,1]} I^*\big(g(t)\big), \qquad c_2^* = \inf_{u\neq 0}\max_{t\geqslant 0} I^*(tu), \qquad c_3^* = \inf_{u\in\mathcal{N}^*} I^*(u),$$

where

$$\Gamma^* = \big\{g \in C\big([0,1], H^1\big(\mathbb{R}^3\big)\big) \;\big|\; g(0) = 0,\ I^*\big(g(1)\big) \leqslant 0,\ g(1) \neq 0\big\}.$$

It is standard to prove that

Lemma 3.1. *The following relations hold*

$$c_V^* := c_1^* = c_2^* = c_3^*.$$

We denote by S the best constant for the Sobolev embedding $\mathcal{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, namely

$$S = \inf_{u\in\mathcal{D}^{1,2}\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_6^2}.$$

3.2.1. The constant potential case

In this section we suppose that V is a positive constant. For simplicity we assume $V \equiv 1$ and we denote $c^* = c_V^*$.

Lemma 3.2. *The following inequality holds*

$$c^* < \frac{1}{3}S^{\frac{3}{2}}.$$

Proof. Consider the one parameter Talenti's functions $u_\varepsilon \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ defined by

$$u_\varepsilon := C_\varepsilon \frac{\varepsilon^{\frac{1}{4}}}{(\varepsilon + |x|^2)^{\frac{1}{2}}},$$

where $C_\varepsilon > 0$ is a normalizing constant (see [25]). Let φ be a smooth cut off function, namely $\varphi \in C_0^\infty(\mathbb{R}^3)$ and there exists $R > 0$ such that $\varphi|_{B_R} = 1$, $0 \leqslant \varphi \leqslant 1$ and $\operatorname{supp}\varphi \subset B_{2R}$. Set $w_\varepsilon := u_\varepsilon \varphi$ and $v_\varepsilon = w_\varepsilon / \|w_\varepsilon\|_6$. Using the estimates obtained in [6] we get

$$\|\nabla v_\varepsilon\|_2^2 = S + O\big(\varepsilon^{\frac{1}{2}}\big), \tag{47}$$

and, for any $s \in [2, 6[$,

$$\|v_\varepsilon\|_s^s = \begin{cases} O(\varepsilon^{\frac{s}{4}}), & \text{if } s \in [2,3[, \\ O(\varepsilon^{\frac{3}{4}}|\log(\varepsilon)|), & \text{if } s = 3, \\ O(\varepsilon^{\frac{6-s}{4}}), & \text{if } s \in]3,6[. \end{cases} \tag{48}$$

For every $\varepsilon > 0$ let $t_\varepsilon > 0$ such that $t_\varepsilon v_\varepsilon \in \mathcal{N}^*$. Obviously $(t_\varepsilon)_{\varepsilon>0}$ is bounded below by a positive constant; otherwise there should exist a sequence $(\varepsilon_n)_n$ such that $\lim_n t_{\varepsilon_n} = 0$ and then, by (47), Lemma 2.1 and (48),

$$0 < c^* \leqslant \lim_n I^*(t_{\varepsilon_n} v_{\varepsilon_n}) = 0.$$

Claim. *For any $\varepsilon > 0$ small enough $t_\varepsilon \leqslant (\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 + v_\varepsilon^2)^{1/4}$.*

Let $\gamma_\varepsilon(t) := I^*(t v_\varepsilon)$ and set $r_\varepsilon := (\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 + v_\varepsilon^2)^{1/4}$. By (47) and (48), $(r_\varepsilon)_{\varepsilon>0}$ is bounded below by a positive constant. Since $t_\varepsilon v_\varepsilon \in \mathcal{N}^*$, certainly $\gamma_\varepsilon'(t_\varepsilon) = 0$. On the other hand, by (i) of Lemma 2.1 and (48), for any ε small enough,

$$\gamma_\varepsilon'(t) = t r_\varepsilon^4 - t^5 + t^3 \int_{\mathbb{R}^3} \phi_{v_\varepsilon} v_\varepsilon^2 - t^q \|v_\varepsilon\|_{q+1}^{q+1} \leqslant t r_\varepsilon^4 - t^5 + C' t^3 \|v_\varepsilon\|_{\frac{12}{5}}^4 - t^q \|v_\varepsilon\|_{q+1}^{q+1} = t r_\varepsilon^4 - t^5 + t^3\big(C' O(\varepsilon) - t^{q-3} O\big(\varepsilon^{\frac{5-q}{4}}\big)\big),$$

where $O(\varepsilon)$ and $O(\varepsilon^{\frac{5-q}{4}})$ are nonnegative functions. We deduce that, for any $\varepsilon > 0$ small enough, $\gamma_\varepsilon'(t) < 0$ in $]r_\varepsilon, +\infty[$: the claim follows as a consequence.

Now, since the function

$$t \in \mathbb{R}_+ \mapsto \frac{1}{2} t^2 r_\varepsilon^4 - \frac{1}{6} t^6$$

is increasing in the interval $[0, r_\varepsilon[$, by (47) and (i) of Lemma 2.1 we have that

$$\begin{aligned} I^*(t_\varepsilon v_\varepsilon) &= \frac{t_\varepsilon^2}{2} \int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 + v_\varepsilon^2 + \frac{t_\varepsilon^4}{4} \int_{\mathbb{R}^3} \phi_{v_\varepsilon} v_\varepsilon^2 - \frac{t_\varepsilon^{q+1}}{q+1} \int_{\mathbb{R}^3} |v_\varepsilon|^{q+1} - \frac{t_\varepsilon^6}{6} \\ &\leqslant \frac{1}{3} \bigg(\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 + v_\varepsilon^2 \bigg)^{\frac{3}{2}} + C' \frac{t_\varepsilon^4}{4} \|v_\varepsilon\|_{\frac{12}{5}}^4 - \frac{t_\varepsilon^{q+1}}{q+1} \|v_\varepsilon\|_{q+1}^{q+1} \\ &= \frac{1}{3} \bigg(S + O\big(\varepsilon^{\frac{1}{2}}\big) + \int_{\mathbb{R}^3} v_\varepsilon^2 \bigg)^{\frac{3}{2}} + C' \frac{t_\varepsilon^4}{4} \|v_\varepsilon\|_{\frac{12}{5}}^4 - \frac{t_\varepsilon^{q+1}}{q+1} \|v_\varepsilon\|_{q+1}^{q+1}. \end{aligned}$$

Using the inequality $(a+b)^\delta \leqslant a^\delta + \delta(a+b)^{\delta-1} b$ which holds for any $\delta \geqslant 1$ and $a, b \geqslant 0$, by (48) and the previous chain of inequalities we get

$$I^*(t_\varepsilon v_\varepsilon) \leqslant \frac{1}{3} S^{\frac{3}{2}} + O\big(\varepsilon^{\frac{1}{2}}\big) + C_1(\varepsilon) O(\varepsilon) - C_2(\varepsilon) O\big(\varepsilon^{\frac{5-q}{4}}\big), \tag{49}$$

where $C_1(\varepsilon)$ and $C_2(\varepsilon)$ are in an interval $[\alpha, \beta]$ with $\alpha > 0$. Since $q > 3$, the conclusion follows from (49), for $\varepsilon > 0$ small enough. $\square$

Proof of Theorem 1.7. Let $(u_n)_n \subset \mathcal{N}^*$ such that

$$\lim_n I^*(u_n) = c^*. \tag{50}$$

We easily deduce that $(u_n)_n$ is bounded in $H^1(\mathbb{R}^3)$, so there exists $\bar{u} \in H^1(\mathbb{R}^3)$ such that, up to a subsequence,

$$\begin{aligned} &u_n \rightharpoonup \bar{u} \quad \text{weakly in } H^1\big(\mathbb{R}^3\big), \\ &u_n \to \bar{u} \quad \text{in } L^s(B), \text{ with } B \subset \mathbb{R}^3, \text{ bounded, and } 1 \leqslant s < 6. \end{aligned} \tag{51}$$

As in the first part of the paper, we use a concentration-compactness argument on the sequence of positive measures

$$\mu_n^*(\Omega)=\left(\frac{1}{2}-\frac{1}{q+1}\right)\int_\Omega |\nabla u_n|^2+u_n^2+\left(\frac{1}{4}-\frac{1}{q+1}\right)\int_\Omega \phi_{u_n}u_n^2+\left(\frac{1}{q+1}-\frac{1}{6}\right)\int_\Omega u_n^6.$$

We define the functional $J^*: H^1(\mathbb{R}^3)\to\mathbb{R}$ as:

$$J^*(u)=\left(\frac{1}{2}-\frac{1}{q+1}\right)\int_{\mathbb{R}^3} |\nabla u|^2+u^2+\left(\frac{1}{4}-\frac{1}{q+1}\right)\int_{\mathbb{R}^3} \phi_u u^2+\left(\frac{1}{q+1}-\frac{1}{6}\right)\int_{\mathbb{R}^3} u^6.$$

Vanishing does not occur.

Suppose by contradiction, that for all $r>0$

$$\lim_n \sup_{\xi\in\mathbb{R}^3}\int_{B_r(\xi)} d\mu_n^*=0.$$

By [20] we deduce that $u_n\to 0$ in $L^s(\mathbb{R}^3)$ for any $s\in\,]2,6[$.

By (i) of Lemma 2.1, since $(u_n)_n\subset\mathcal{N}^*$, it follows that

$$\lim_n\left[\int_{\mathbb{R}^3}|\nabla u_n|^2+u_n^2-\int_{\mathbb{R}^3}u_n^6\right]=0.$$

By the boundedness of $(u_n)_n$ in $H^1(\mathbb{R}^3)$, we infer that there exists $l>0$ such that, up to subsequence,

$$l:=\lim_n\int_{\mathbb{R}^3}|\nabla u_n|^2+u_n^2=\lim_n\int_{\mathbb{R}^3}u_n^6.$$

We have

$$c^*=\lim_n I^*(u_n)=\frac{1}{2}l-\frac{1}{6}l=\frac{1}{3}l \tag{52}$$

and

$$S\leqslant\frac{\int_{\mathbb{R}^3}|\nabla u_n|^2+u_n^2}{(\int_{\mathbb{R}^3}u_n^6)^{\frac{1}{3}}}\to l^{\frac{2}{3}}. \tag{53}$$

By (52) and (53) we get $c^*=\frac{1}{3}l\geqslant\frac{1}{3}S^{\frac{3}{2}}$, contradicting 2 of Lemma 3.2.

Dichotomy does not occur.

The proof uses similar argument as those in the proof of Theorem 1.1.

So the measures μ_n^* concentrate and, in particular, we have that there exists a sequence $(\xi_n)_n$ in $\mathbb{R}^N$ such that for any $\delta>0$ there exists $r=r(\delta)>0$ such that

$$\left(\frac{1}{2}-\frac{1}{q+1}\right)\int_{B_r^c(\xi_n)}|\nabla u_n|^2+u_n^2<\delta. \tag{54}$$

From now on, we only give a sketch of the remaining part of the proof, since it is similar to that of the subcritical case. We define $v_n:=u_n(\cdot-\xi_n)$. It is easy to see that $(v_n)_n\subset\mathcal{N}^*$. From (54) we have that for any $\delta>0$ there exists $r>0$ such that

$$\|v_n\|_{H^1(B_r^c)}<\delta,\quad \text{uniformly for } n\geqslant 1.$$

Hence we deduce

$$v_n\to\bar v\quad\text{in } L^s\big(\mathbb{R}^3\big),\ \text{for any } s\in[2,6[; \tag{55}$$

$$\phi_{v_n}\to\phi_{\bar v}\quad\text{in } \mathcal{D}^{1,2}\big(\mathbb{R}^3\big);$$

$$\int_{\mathbb{R}^3}\phi_{v_n}v_n^2\to\int_{\mathbb{R}^3}\phi_{\bar v}\bar v^2. \tag{56}$$

Moreover, for any $\psi\in C_0^\infty(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3}\phi_{v_n}v_n\psi\to\int_{\mathbb{R}^3}\phi_{\bar v}\bar v\psi,$$

and, by (55),

$$\int_{\mathbb{R}^3} v_n^5 \psi \to \int_{\mathbb{R}^3} \bar{v}^5 \psi.$$

By (50), we can suppose (see [28]) that $(v_n)_n$ is a Palais–Smale sequence for $I^*|_{|\mathcal{N}^*}$, and, consequently, it is a Palais–Smale sequence for I^*. By standard arguments, we infer that $\bar{v} \in \mathcal{N}^*$.

Finally, since $(v_n)_n$ and $\bar{v}$ are in $\mathcal{N}^*$, we have that

$$I^*(\bar{v}) = \frac{1}{3}\int_{\mathbb{R}^3} |\nabla \bar{v}|^2 + \bar{v}^2 + \frac{1}{12}\int_{\mathbb{R}^3} \phi_{\bar{v}} \bar{v}^2 + \left(\frac{1}{6} - \frac{1}{q+1}\right)\int_{\mathbb{R}^3} |\bar{v}|^{q+1},$$

$$I^*(v_n) = \frac{1}{3}\int_{\mathbb{R}^3} |\nabla v_n|^2 + v_n^2 + \frac{1}{12}\int_{\mathbb{R}^3} \phi_{v_n} v_n^2 + \left(\frac{1}{6} - \frac{1}{q+1}\right)\int_{\mathbb{R}^3} |v_n|^{q+1},$$

so, by (50), (51), (55) and (56),

$$c^* \leqslant I^*(\bar{v}) \leqslant \liminf I^*(v_n) = c^*.$$

We conclude that $(\bar{v}, \phi_{\bar{v}})$ is a ground state solution of (3). □

3.2.2. The non-constant potential case

In this section we suppose that V satisfies hypotheses (V1)–(V3).

We define the functional $I^*_\infty : H^1(\mathbb{R}^3) \to \mathbb{R}$ and the Nehari manifold $\mathcal{N}^*_\infty$ in the following way

$$I^*_\infty(u) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + V_\infty u^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 - \frac{1}{q+1}\int_{\mathbb{R}^3} |u|^{q+1} - \frac{1}{6}\int_{\mathbb{R}^3} u^6,$$

$$\mathcal{N}^*_\infty := \left\{ u \in H^1(\mathbb{R}^3) \setminus \{0\} \;\Big|\; \int_{\mathbb{R}^3} |\nabla u|^2 + V_\infty u^2 + \phi_u u^2 - |u|^{q+1} - u^6 = 0 \right\}.$$

We set

$$c^*_\infty = \inf_{u \in \mathcal{N}^*_\infty} I^*_\infty(u).$$

Lemma 3.3. *The following inequality holds*

$$c^*_V < \frac{1}{3} S^{\frac{3}{2}}.$$

Proof. By Theorem 1.7, there exists a ground state solution for (3) whenever $V \equiv V_\infty$; so, arguing as in Lemma 2.12, we can show that $c^*_V < c^*_\infty$. Therefore, the inequality follows by Lemma 3.2. □

Following [22], by Lemmas 3.2 and 3.3 and using a non-vanishing type argument as in the proof of Theorem 1.4, we can show that the corresponding versions of Lemmas 2.9, 2.10 and 2.11 hold for the functional I^*.

Proof of Theorem 1.8. Let $(u_n)_n \subset \mathcal{N}^*$ such that

$$\lim_n I^*(u_n) = c^*_V.$$

We easily deduce that $(u_n)_n$ is bounded in $H^1(\mathbb{R}^3)$, so there exists $\bar{u} \in H^1(\mathbb{R}^3)$ such that, up to a subsequence,

$$u_n \rightharpoonup \bar{u} \quad \text{weakly in } H^1(\mathbb{R}^3).$$

Arguing as in Theorem 2.13, we can prove that

$$\|u_n\|_{H^1(B_r^c)} < \delta, \quad \text{uniformly for } n \geqslant 1.$$

Hence we deduce

$$u_n \to \bar{u} \quad \text{in } L^s(\mathbb{R}^3), \text{ for any } s \in [2, 6[.$$

Then, arguing as in the proof of Theorem 1.7, we get the conclusion. □

Note added in proof

After this paper was completed, the authors became aware of a related work by L. Zhao, F. Zhao, Positive solutions for Schrödinger–Poisson equations with the critical exponent, Nonlinear Anal. (2008), doi:10.1016/j.na.2008.02.116, where the critical case is treated in a similar context.

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