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Asymptotic Analysis for the Dunkl Kernel

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This paper studies the asymptotic behavior of the integral kernel of the Dunkl transform, the so-called Dunkl kernel, when one of its arguments is fixed and the other tends to infinity either within a Weyl chamber of the associated reflection group, or within a suitable complex domain. The obtained results are based on the asymptotic analysis of an associated system of ordinary differential equations. They generalize the well-known asymptotics of the confluent hypergeometric function ${}_1F_1$ to the higher-dimensional setting and include a complete short-time asymptotics for the Dunkl-type heat kernel. As an application, it is shown that the representing measures of Dunkl's intertwining operator are generically continuous.

Key Words: Dunkl operators; Dunkl kernel; asymptotics.

1. INTRODUCTION AND RESULTS

In the theory of rational Dunkl operators as initiated by Dunkl [5], there is an analogue of the classical exponential function, commonly called the Dunkl kernel. It generalizes the usual exponential function in many respects, and can be characterized as the solution of a joint eigenvalue problem for the associated Dunkl operators. Generally speaking, Dunkl operators are parameterized differential-reflection operators attached to a finite reflection group. During the last decade, such operators have found considerable

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attention in various areas of mathematics and mathematical physics. They are, for example, useful in the study of integrable quantum many-body systems of Calogero–Moser–Sutherland type (for an up-to-date bibliography, we refer to [4]), and have led to a rapid development in the theory of special functions related with root systems; see for instance [10, 14]. Among the variants of Dunkl operators, especially the rational ones allow for a far-reaching harmonic analysis in close analogy to the classical Fourier analysis on $\mathbb{R}^N$. For example, there exists an analogue of the Fourier transform—the Dunkl transform—which establishes a natural correspondence between the action of Dunkl operators on one hand and multiplication operators on the other [7, 15]. The integral kernel of this transform is the Dunkl kernel. It was first introduced in [6] and has since then been studied in a variety of aspects among which we mention [7, 15, 17, 19]. The present paper contributes to a further study of its asymptotic and structural properties.

In order to introduce our setting and results, let R be a reduced (not necessarily crystallographic) root system in $\mathbb{R}^N$ with the standard Euclidean inner product $\langle .,. \rangle$. This means that $R \subset \mathbb{R}^N \setminus \{0\}$ is finite with $\sigma_\alpha R = R$ and $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ for all $\alpha \in R$, with σ_α denoting the reflection in the hyperplane H_α orthogonal to α. Let $G \subset O(\mathbb{R}^N)$ denote the finite reflection group generated by the σ_α, $\alpha \in R$, and put $\mathbb{R}^N_{\text{reg}} := \mathbb{R}^N \setminus \bigcup_{\alpha\in R} H_\alpha$. The connected components of $\mathbb{R}^N_{\text{reg}}$ are called the Weyl chambers of G. As customary in this context, we assume that the root system R is normalized by $|\alpha|^2 = 2$ for all α, and we denote the bilinear extension of $\langle .,. \rangle$ to $\mathbb{C}^N$ by $\langle .,. \rangle$ as well. Let further $k : R \to \mathbb{C}$ be a multiplicity function on R (i.e. k is invariant under the natural action of G on R). In the present paper, we shall assume throughout that k is non-negative, i.e. $k(\alpha) \geqslant 0$ for all $\alpha \in R$. The (rational) Dunkl operators associated with G and k are given by

$$T_\xi(k)f(x) := \partial_\xi f(x) + \sum_{\alpha\in R_+} k(\alpha)\langle \alpha, \xi\rangle \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle}, \qquad \xi \in \mathbb{R}^N.$$

Here ∂_ξ denotes the usual partial derivative in direction ξ and R_+ is an arbitrary but fixed positive subsystem of R. It is a remarkable property of the $T_\xi(k)$ that they mutually commute, see [5]. The Dunkl kernel $E_k(x, y)$ associated with G and k can be characterized as the unique solution of the joint eigenvalue problem for the corresponding Dunkl operators, more precisely: for each fixed $y \in \mathbb{C}^N$, the function $x \mapsto E_k(x, y)$ is the unique real-analytic solution of the system

$$T_\xi(k)f = \langle \xi, y\rangle f \qquad \text{for all } \xi \in \mathbb{R}^N \quad \text{and} \quad f(0) = 1, \tag{1.1}$$

cf. [17]. In case $k = 0$ we just have $E_k(x, y) = e^{\langle x,y\rangle}$. The generalized exponential kernel $E_k(x, y)$ is symmetric in its arguments and has a unique holomorphic extension to $\mathbb{C}^N \times \mathbb{C}^N$. It satisfies

$$E_k(\lambda z, w) = E_k(z, \lambda w) \qquad \text{and} \qquad E_k(gz, gw) = E_k(z, w) \tag{1.2}$$

for all $z, w \in \mathbb{C}^N, \lambda \in \mathbb{C}$ and $g \in G$.

Originally, E_k was defined in [6] by means of the so-called intertwining operator V_k. In fact, there exists a unique linear isomorphism V_k on polynomials, homogeneous of degree 0, and such that

$$T_\xi(k) V_k = V_k \partial_\xi \qquad \text{for all } \xi \in \mathbb{R}^N \quad \text{and} \quad V_k(1) = 1,$$

see [5, 9]. In [19] it was shown that V_k (k always being non-negative) admits a positive integral representation as follows: Let $M^1(\mathbb{R}^N)$ denote the space of probability measures on the Borel σ-algebra of $\mathbb{R}^N$. Then for every $x \in \mathbb{R}^N$, there exists a unique $\mu_x^k \in M^1(\mathbb{R}^N)$ such that

$$V_k p(x) = \int_{\mathbb{R}^N} p(\xi)\, d\mu_x^k(\xi) \tag{1.3}$$

for each polynomial function p on $\mathbb{R}^N$. The representing measures μ_x^k are compactly supported with $\operatorname{supp} \mu_x^k \subseteq \operatorname{co}\{gx, g \in G\}$, the convex hull of the orbit of x under G. By means of formula (1.3), V_k may be extended to various larger function spaces, e.g. the space $C(\mathbb{R}^N)$ of continuous functions on $\mathbb{R}^N$. We denote this extension by V_k again. Then for fixed $y \in \mathbb{C}^N$,

$$E_k(x, y) = V_k(e^{\langle\, \cdot\, , y\rangle})(x) = \int_{\mathbb{R}^N} e^{\langle \xi, y\rangle}\, d\mu_x^k(\xi) \qquad (x \in \mathbb{R}^N). \tag{1.4}$$

This, in particular, implies that

$$E_k(-ix, y) = \overline{E_k(ix, y)} \qquad \text{and} \qquad |E_k(ix, y)| \leqslant 1 \quad \text{for all } x, y \in \mathbb{R}^N. \tag{1.5}$$

As already indicated, the Dunkl kernel is especially of interest as it gives rise to a corresponding integral transform on $\mathbb{R}^N$. The Dunkl transform associated with G and k involves the weight function

$$w_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x\rangle|^{2k(\alpha)},$$

which is G-invariant and homogeneous of degree 2γ, with the index

$$\gamma := \gamma(k) = \sum_{\alpha \in R_+} k(\alpha) \geqslant 0.$$

It is defined by

$$\hat{f}^k(\xi) := c_k^{-1} \int_{\mathbb{R}^N} f(x) E_k(-i\xi, x) w_k(x)\, dx, \qquad f \in L^1(\mathbb{R}^N, w_k),$$

here c_k is the Mehta-type constant

$$c_k := \int_{\mathbb{R}^N} e^{-|x|^2/2} w_k(x)\, dx.$$

An explicit expression for c_k can be found in [17]. The Dunkl transform shares many properties of the classical Fourier transform. For example, there exist a Plancherel theorem and an inversion theorem for it. For details the reader is referred to [7, 15].

In this paper, we study the asymptotic behavior of $x \mapsto E_k(x, y)$ for large arguments x, with $y \in \mathbb{R}^N_{\text{reg}}$ considered as a fixed parameter. Let C denote the Weyl chamber attached with the positive subsystem R_+,

$$C = \{x \in \mathbb{R}^N : \langle \alpha, x \rangle > 0 \text{ for all } \alpha \in R_+\},$$

and for $\delta > 0$,

$$C_\delta := \{x \in C : \langle \alpha, x \rangle > \delta |x| \text{ for all } \alpha \in R_+\}.$$

Our main result is the following asymptotic behavior, uniform for the variable tending to infinity in cones C_δ:

THEOREM 1. *There exists a constant non-zero vector $v = (v_g)_{g \in G} \in \mathbb{C}^{|G|}$ such that for all $y \in C$, $g \in G$ and each $\delta > 0$,*

$$\lim_{|x| \to \infty,\, x \in C_\delta} \sqrt{w_k(x) w_k(y)}\, e^{-i\langle x, gy \rangle} E_k(ix, gy) = v_g.$$

The proof of this theorem, which will be given in Section 4, is based on the analysis of an associated system of first-order ordinary differential equations, which is derived from the eigenfunction characterization (1.1) of E_k. The idea for this approach goes back to [15], where it was used to obtain exponential estimates for the Dunkl kernel. An immediate consequence of Theorem 1 is the following ray asymptotic for the Dunkl kernel, making precise a conjectural remark in [7].

COROLLARY 1. *For all $x, y \in C$ and $g \in G$,*

$$\lim_{t\to\infty} t^{\gamma} e^{-it\,\langle x, gy\rangle} E_k(itx, gy) = \frac{v_g}{\sqrt{w_k(x)w_k(y)}},$$

the convergence being locally uniform with respect to the parameter x.

In the particular case $g = e$ (the unit of G), this result can be extended to a larger range of complex arguments by use of the Phragmén–Lindelöf principle. We consider the closed right half-plane $H = \{z \in \mathbb{C} : \operatorname{Re} z \geqslant 0\}$, and denote by $z \mapsto z^{\gamma}$ the holomorphic branch in $\mathbb{C}\setminus\{x \in \mathbb{R} : x \leqslant 0\}$ with $1^{\gamma} = 1$. We shall prove

THEOREM 2. *Let $x, y \in C$. Then*

$$\lim_{z\to\infty,\ z\in H} z^{\gamma} e^{-z\langle x,y\rangle} E_k(zx, y) = \frac{i^{\gamma} v_e}{\sqrt{w_k(x)w_k(y)}}.$$

An interesting consequence of this result concerns the short-time behavior of the Dunkl-type heat kernel

$$\Gamma_k(t, x, y) = \frac{1}{(2t)^{\gamma+N/2} c_k} e^{-(|x|^2+|y|^2)/4t} E_k\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right)$$

$(x, y \in \mathbb{R}^N,\ t > 0)$, which was first introduced in [18]. After suitable normalization, the kernel Γ_k behaves for short times like the free Gaussian heat kernel $\Gamma_0(t, x, y) = (4\pi t)^{-N/2} e^{-|x-y|^2/4t}$, as conjectured in [20]. More precisely,

COROLLARY 2. *For all $x, y \in C$,*

$$\lim_{t\downarrow 0} \frac{\sqrt{w_k(x)\, w_k(y)}\,\Gamma_k(t, x, y)}{\Gamma_0(t, x, y)} = 1.$$

Indeed, it is immediate from Theorem 2 that the limit above exists for all $x, y \in C$ and is equal to $i^{\gamma} v_e c_0 / c_k$. On the other hand, Theorem 3.3 of [20] shows—based on completely different methods—that the limit exists and equals 1 for a restricted range of arguments $x, y \in C$. This combination proves Corollary 2 and at the same time implies the value of v_e:

$$v_e = i^{-\gamma} \frac{c_k}{c_0}. \tag{1.6}$$

Remark 1. The explicit determination of the constants v_g with $g \neq e$ (or of v_e without falling back to [20]) is an open problem. The proof of

Theorem 1 yields the invariance property

$$v_g = v_{g^{-1}} \qquad \text{for all } g \in G,$$

but additional techniques seem to be necessary to obtain further information.

The asymptotic result of Theorem 1 also allows to deduce at least a certain amount of information about the structure of the intertwining operator V_k and its representing measures μ_x^k according to formula (1.3). These measures are explicitly known in very special cases only, namely essentially for the rank-one-case as well as the symmetric group S_3 [8], and very little is known about their general structure either. In particular, as to the authors' knowledge, no results towards the continuity properties of the μ_x^k have been obtained so far. We shall employ a well-known characterization of continuous measures due to Wiener by means of their (classical) Fourier–Stieltjes transform, which coincides with the kernel $E_k(-ix, .)$ in case of μ_x^k. (Recall that a measure $\mu \in M^1(\mathbb{R}^N)$ is called continuous, if $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}^N$.) Theorem 1 gives just sufficient information on the growth of the Dunkl kernel in order to apply Wiener's criterion. This yields

THEOREM 3. *If* $\gamma > 0$, *i.e. apart from the classical Fourier case*, *the measure* μ_x^k *is continuous for all* $x \in \mathbb{R}^N_{\text{reg}}$.

We conjecture that the measures μ_x^k are even absolutely continuous with respect to Lebesgue measure for all regular x, provided k is such that $\{\alpha \in R \mid k(\alpha) > 0\}$ spans $\mathbb{R}^N$. This is in fact true in the rank-one case, which provides a simple but illustrative example for our results. A short discussion of this example is given in Section 2. Section 3 contains the asymptotic analysis of the differential equation associated with the kernel E_k, as well as the implications concerning its asymptotic behavior. These results are the basis for the proofs of Theorems 1–3, which are completed in Section 4.

Remark 2. We mention that the group invariant counterpart of E_k, called "generalized Bessel function" in [17], can be considered as a natural generalization of the usual one-variable Bessel function, to which it reduces in the rank-one case (see below). For Weyl groups and certain half-integer multiplicity parameters k, generalized Bessel functions have an interpretation as spherical functions of a Cartan motion group. For the details concerning this identification we refer to [16, 17]. For such generalized Bessel functions corresponding to the group case, and with both the geometric and spectral variable in C, asymptotic results are derived in [2] which are more precise than can be obtained by averaging the results in Theorem 1.

Estimates on the generalized Bessel function in the group case with spectral variable in C, but with *arbitrary* geometric variable, can be found in [3]. The methods in [2, 3], however, use the presence of additional ambient structure for these special values of the multiplicity parameters, and therefore do not apply in our case of general non-negative k.

2. EXAMPLE: THE RANK-ONE CASE

Let $N = 1$. Then the only choice of R (being reduced) is $R = \{\pm\sqrt{2}\}$. Accordingly, $G = \{id, \sigma\} \cong \mathbb{Z}_2$ with $\sigma(x) = -x$. The Dunkl operator $T(k) = T_1(k)$ associated with the multiplicity parameter $k \geqslant 0$ is given by

$$T(k)f(x) = f'(x) + k\frac{f(x) - f(-x)}{x}.$$

The corresponding intertwining operator V_k and the kernel E_k were determined in [6, 7]. In particular,

$$E_k^{\mathbb{Z}_2}(z, w) = j_{k-1/2}(izw) + \frac{zw}{2k+1} j_{k+1/2}(izw).$$

Here j_α denotes the normalized spherical Bessel function

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha + 1)\frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha + 1)\sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n!\Gamma(n + \alpha + 1)}.$$

The integral representation (1.4) of $E_k^{\mathbb{Z}_2}$ is given by

$$\begin{aligned} E_k^{\mathbb{Z}_2}(z, w) &= \frac{\Gamma(k + 1/2)}{\Gamma(1/2)\,\Gamma(k)} \int_{-1}^{1} e^{tzw}(1 - t)^{k-1}(1 + t)^k \, dt \\ &= e^{zw}\, {}_1F_1(k, 2k + 1, -2zw). \end{aligned} \tag{2.1}$$

Thus for $x \neq 0$, the associated representing measure is

$$d\mu_x^k(u) = \frac{\Gamma(k + 1/2)}{\Gamma(1/2)\,\Gamma(k)} 1_{[-|x|,|x|]}(u)\frac{1}{|x|}\left(1 - \frac{u}{x}\right)^{k-1}\left(1 + \frac{u}{x}\right)^k du,$$

which is absolutely continuous with respect to Lebesgue measure. Further, recall the well-known asymptotic expansion of Kummer's function ${}_1F_1$ (see e.g. [1, (13.5.1)]):

$$\lim_{z\to\infty,\, z\in H} z^k\, {}_1F_1(k, 2k + 1, -2z) = \frac{\Gamma(2k + 1)}{2^k \Gamma(k + 1)}.$$

Thus the constants v_e and v_σ in Theorems 1 and 2 are given by

$$v_e = \frac{\Gamma(2k+1)}{2^k\Gamma(k+1)}\, i^{-k}, \qquad v_\sigma = \frac{\Gamma(2k+1)}{2^k\,\Gamma(k+1)}\, i^k.$$

3. ASYMPTOTICS OF E_k ALONG CURVES IN A WEYL CHAMBER

For $x, y \in \mathbb{R}^N$ define

$$\phi(x,y) = \sqrt{w_k(x)w_k(y)}\, e^{-i\langle x,y\rangle} E_k(ix,y).$$

Observe that ϕ is symmetric in its arguments. In this section, we shall study the asymptotic behavior of $x \mapsto \phi(x,y)$ along curves in a fixed Weyl chamber, with the second component $y \in \mathbb{R}^N_{\text{reg}}$ being fixed. In view of the invariance properties of w_k and E_k under the action of G, we may restrict ourselves to the case $x \in C$ (the chamber associated with R_+) and $y \in gC$ for some $g \in G$. Following an idea of [15], we introduce an auxiliary vector field $F = (F_g)_{g \in G}$ on $\mathbb{R}^N \times \mathbb{R}^N$ by

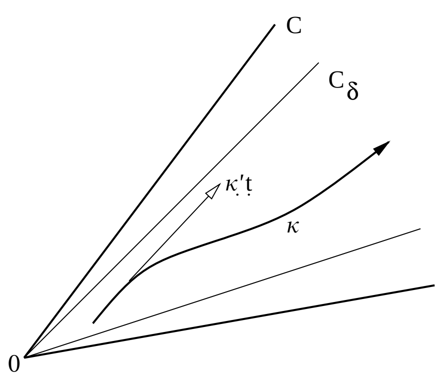
$$F_g(x,y) := \phi(x, gy).$$

For fixed y, we consider F along a differentiable curve $\kappa : (0,\infty) \to C$. The eigenfunction characterization (1.1) of E_k then translates into a first-order ordinary differential equation for $t \mapsto F(\kappa(t), y)$. We shall determine the asymptotic behavior of its solutions, provided κ is admissible in the following sense:

DEFINITION 1. A C^1-curve $\kappa : (0,\infty) \to C$ is called *admissible*, if it satisfies the subsequent conditions:

(1) There exists a constant $\delta > 0$ such that $\kappa(t) \in C_\delta$ for all $t > 0$.

(2) $\lim_{t\to\infty} |\kappa(t)| = \infty$ and $\kappa'(t) \in C$ for all $t > 0$.



Note that conditions (1) and (2) imply that $\lim_{t\to\infty}\langle\alpha,\kappa(t)\rangle=\infty$ for all $\alpha\in R_+$. An important class of admissible curves are the rays $\kappa(t)=tx$ with some fixed $x\in C$. Corollary 1 describes the asymptotic behavior of $x\mapsto F(x,y)$ along such rays. In this section, we prove that $t\mapsto F(\kappa(t),y)$ is asymptotically constant as $t\to\infty$ for arbitrary admissible curves, not just for rays. In the next section it will become clear that the limit value is actually independent of y and κ.

THEOREM 4. *If $\kappa:(0,\infty)\to C$ is admissible, then for every $y\in C$, the limit*

$$\lim_{t\to\infty} F(\kappa(t),y)$$

exists in $\mathbb{C}^{|G|}$, and is different from 0.

The subsequent proof of Theorem 4 is based on the following variant of formula (3.1) in [15]:

LEMMA 1. *For fixed $\xi,y\in\mathbb{R}^N$,*

$$\partial_\xi F_g(x,y)=\sum_{\alpha\in R_+}k(\alpha)\frac{\langle\alpha,\xi\rangle}{\langle\alpha,x\rangle}e^{-i\langle\alpha,x\rangle\langle\alpha,gy\rangle}F_{\sigma_\alpha g}(x,y)\qquad(x\in\mathbb{R}^N_{\mathrm{reg}}).$$

Proof. The eigenfunction characterization (1.1) of the kernel E_k, together with the invariance property $E_k(gx,gy)=E_k(x,y)$, implies that

$$\partial_\xi E_k(x,y)=\langle\xi,y\rangle E_k(x,y)-\sum_{\alpha\in R_+}k(\alpha)\frac{\langle\alpha,\xi\rangle}{\langle\alpha,x\rangle}(E_k(x,y)-E_k(x,\sigma_\alpha y)).$$

Moreover, if $x\in\mathbb{R}^N_{\mathrm{reg}}$, then

$$\partial_\xi\sqrt{w_k(x)}=\left(\sum_{\alpha\in R_+}k(\alpha)\frac{\langle\alpha,\xi\rangle}{\langle\alpha,x\rangle}\right)\sqrt{w_k(x)}.$$

It follows that

$$\partial_\xi F_g(x,y)=\sqrt{w_k(x)w_k(y)}\,e^{-i\langle x,gy\rangle}\sum_{\alpha\in R_+}k(\alpha)\frac{\langle\alpha,\xi\rangle}{\langle\alpha,x\rangle}E_k(ix,\sigma_\alpha gy)$$

$$=\sum_{\alpha\in R_+}k(\alpha)\frac{\langle\alpha,\xi\rangle}{\langle\alpha,x\rangle}e^{-i\langle x,gy\rangle}e^{i\langle x,\sigma_\alpha gy\rangle}F_{\sigma_\alpha g}(x,y).$$

As $\sigma_\alpha(x)-x=-\langle\alpha,x\rangle\alpha$, this implies the assertion. ∎

We are interested in the derivative of $x \mapsto F(x, y)$ along differentiable curves in C, with the second component y being fixed. The following is immediate from the previous lemma:

COROLLARY 3. *For a C^1-curve $\kappa : (0, \infty) \to \mathbb{R}^N_{\text{reg}}$ and fixed $y \in \mathbb{R}^N$ define $F^\kappa := (F^\kappa_g)_{g \in G}$ by $F^\kappa_g(t) := F_g(\kappa(t), y)$. Then F^κ satisfies the differential equation*

$$(F^\kappa)'(t) = A^\kappa(t) F^\kappa(t), \tag{3.1}$$

where $A^\kappa : (0, \infty) \to \mathbb{C}^{|G| \times |G|}$ is given by $A^\kappa = \sum_{\alpha \in R_+} k(\alpha) B^\kappa_\alpha$, and

$$(B^\kappa_\alpha(t))_{g,h} = \begin{cases} \dfrac{\langle \alpha, \kappa'(t) \rangle}{\langle \alpha, \kappa(t) \rangle} \, e^{-i\langle \alpha, \kappa(t) \rangle \langle \alpha, gy \rangle} & \text{if } h = \sigma_\alpha g, \\ 0 & \text{otherwise.} \end{cases}$$

If κ is a ray, i.e. $\kappa(t) = tx$ with some fixed $x \in C$, then

$$(B^\kappa_\alpha(t))_{g, \sigma_\alpha g} = \frac{1}{t} e^{-it\langle \alpha, x \rangle \langle \alpha, gy \rangle}.$$

Note that in this typical case, $t = \infty$ is an essential singularity of A^κ. However, $A^\kappa(t) = O(\frac{1}{t})$ as $t \to \infty$, i.e. the system is asymptotically constant (in fact asymptotically zero) in the sense of [11]. This suggests that the solution F^κ should be asymptotically constant as $t \to \infty$, which just means that $\lim_{t \to \infty} F^\kappa(t)$ exists as asserted in Theorem 4.

The decisive criterion for the proof of Theorem 4 is the following result on the asymptotic integration of ordinary linear differential equations, which is a special case of the Levinson-type Theorem 1.11.1 in [11], and is originally due to [22]:

PROPOSITION 1(Eastham [11], Wintner [22]). *Consider the linear differential equation*

$$x'(t) = A(t) x(t), \tag{3.2}$$

where $A : [t_0, \infty) \to \mathbb{C}^{n \times n}$ is a continuous matrix-valued function satisfying the following integrability conditions:

(1) *The matrix-valued improper Riemann integral $\int_{t_0}^\infty A(t) dt$ converges. In particular, $\tilde{A}(t) := \int_t^\infty A(s) ds$ is well defined on $[t_0, \infty)$.*

(2) $A\tilde{A} \in L^1([t_0, \infty), \mathbb{C}^{n \times n})$.

Then (3.2) *has a basis of solutions $\{x_k(t), 1 \leqslant k \leqslant n\}$ of the asymptotic form $x_k(t) = e_k + o(1)$ as $t \to \infty$, where e_k is the kth unit vector in $\mathbb{R}^n$. In* *particular, for each solution* x *of* (3.2), *the limit* $\lim_{t\to\infty} x(t)$ *exists. Moreover, if* $x \neq 0$, *then* $\lim_{t\to\infty} x(t) \neq 0$.

Proof of Theorem 4. We shall verify that the matrix A^κ satisfies the conditions of Proposition 1 with arbitrary $t_0 > 0$. For (1), let $g \in G$ and $\alpha \in R_+$. Then if $T > t \geqslant t_0$,

$$\int_t^T (B_\alpha^\kappa(s))_{g,\sigma_\alpha g}\, ds = \int_t^T \frac{\langle \alpha, \kappa'(s)\rangle}{\langle \alpha, \kappa(s)\rangle} e^{-i\langle \alpha,\kappa(s)\rangle\langle \alpha, gy\rangle}\, ds$$

$$= \int_{\langle \alpha,gy\rangle\langle \alpha,\kappa(t)\rangle}^{\langle \alpha,gy\rangle\langle \alpha,\kappa(T)\rangle} \frac{1}{u} e^{-iu}\, du.$$

This integral exists, because the admissible curve κ remains in the Weyl chamber C. For abbreviation, put $\varphi_{\alpha,g}(t) := \langle \alpha, \kappa(t)\rangle |\langle \alpha, gy\rangle|$, which is strictly positive for all t. Also note that $\lim_{t\to\infty} \varphi_{\alpha,g}(t) = +\infty$, by admissibility of κ. We thus obtain

$$\lim_{T\to\infty} \int_t^T (B_\alpha^\kappa(s))_{g,\sigma_\alpha g}\, ds = i\,\mathrm{sign}(\langle \alpha, gy\rangle)\mathrm{si}(\varphi_{\alpha,g}(t)) - \mathrm{Ci}(\varphi_{\alpha,g}(t)), \tag{3.3}$$

where for $\tau > 0$, $\mathrm{si}(\tau) = -\int_\tau^\infty \frac{\sin u}{u}\, du$ and $\mathrm{Ci}(\tau) = -\int_\tau^\infty \frac{\cos u}{u}\, du$ are the integral sine and cosine, respectively. Thus in particular, condition (1) is satisfied. To verify condition (2), notice first that the matrix entries of $A^\kappa(t)\tilde{A}^\kappa(t)$ are linear combinations with constant coefficients, of terms of the following kind:

$$I_{\alpha,\beta,g}(t) := \frac{\langle \alpha, \kappa'(t)\rangle}{\langle \alpha, \kappa(t)\rangle} e^{-i\langle \alpha,\kappa(t)\rangle\langle \alpha, gy\rangle} \Big(\pm\, i\,\mathrm{si}(\varphi_{\beta,\sigma_\alpha g}(t)) - \mathrm{Ci}(\varphi_{\beta,\sigma_\alpha g}(t)) \Big)$$

with $g \in G, \alpha, \beta \in R_+$. In order to estimate the integral sine and cosine terms, we use that for $\tau > 0$,

$$|\mathrm{si}(\tau)| \leqslant 2/\tau, \quad |\mathrm{Ci}(\tau)| \leqslant 2/\tau. \tag{3.4}$$

In fact, integration by parts yields

$$\mathrm{si}(\tau) = -\frac{\cos\tau}{\tau} + \int_\tau^\infty \frac{\cos u}{u^2}\, du,$$

which readily implies the first part of (3.4), and the second one is seen in a similar way. Moreover, as κ is admissible, we have $\langle \alpha, \kappa(t)\rangle > 0$ for all $t > 0$, and there exists some constant $\delta > 0$ such that

$$\langle \beta, \kappa(t)\rangle \geqslant \delta|\kappa(t)| \geqslant \delta\langle \alpha, \kappa(t)\rangle/\sqrt{2} \qquad \text{for all } \alpha, \beta \in R_+.$$

Together with (3.4), this yields the estimation

$$|I_{\alpha,\beta,g}(t)| \leqslant C_1 \frac{\langle \alpha, \kappa'(t)\rangle}{\langle \alpha, \kappa(t)\rangle\langle \beta, \kappa(t)\rangle} \leqslant C_2 \frac{\langle \alpha, \kappa'(t)\rangle}{\langle \alpha, \kappa(t)\rangle^2}$$

with constants C_1, C_2 depending on g, α, β only. As

$$\int_{t_0}^{\infty} \frac{\langle \alpha, \kappa'(t)\rangle}{\langle \alpha, \kappa(t)\rangle^2}\, dt = \int_{\langle \alpha, \kappa(t_0)\rangle}^{\infty} \frac{1}{u^2}\, du < \infty,$$

it follows that A^κ fulfills condition (2). ∎

4. PROOFS OF THE MAIN THEOREMS

For the proof of Theorem 1 we consider $x \mapsto F(x, y)$ along arbitrary admissible curves, and infer by an interpolation technique that the limit in Theorem 4 is independent of the special choice of the admissible curve κ and also of $y \in C$. We start with a supplementary notation.

DEFINITION 2. A sequence $(x_n)_{n\in\mathbb{N}} \subset C$ with $\lim_{n\to\infty} x_n = \infty$ is called admissible, if there exists an interpolating admissible curve for it, i.e. an admissible $\kappa : (0, \infty) \to C$ such that $x_n = \kappa(t_n)$ for suitable parameters t_n with $\lim_{n\to\infty} t_n = \infty$.

Remark 3. The following special situation will be of importance in the sequel: Suppose that $(x_n)_{n\in\mathbb{N}}$ is contained in C_δ for some $\delta > 0$ and satisfies $\lim_{n\to\infty} |x_n| = \infty$ as well as $x_{n+1} - x_n \in C$ for all $n \in \mathbb{N}$. Then $(x_n)_{n\in\mathbb{N}}$ is admissible. An admissible interpolating curve is obtained by slightly smoothening the piecewise linear connection of the successive points x_n.

Proof of Theorem 1. In a first step, we show that there exists a non-zero vector $v(y) = (v_g(y))_{g\in G} \in \mathbb{C}^{|G|}$ such that for each admissible curve κ in C,

$$\lim_{t\to\infty} F(\kappa(t), y) = v(y). \tag{4.1}$$

For this, fix $y \in C$ and let κ_1 , κ_2 be any two admissible curves, both contained in some C_δ. With the above remark in mind, we can inductively construct an admissible sequence $(x_n)_{n\in\mathbb{N}} \subset C_\delta$ with $x_{2n-1} \in \kappa_1$ and $x_{2n} \in \kappa_2$ for all $n \in \mathbb{N}$. In fact, suppose that $x_1, \ldots, x_n$ are already constructed, and consider the set $S_n = \{x \in C_\delta : x - x_n \notin C\}$, which is bounded. The curves κ_i being admissible, we can therefore choose x_{n+1} from the part of the relevant curve κ_i which is contained in the complement of S_n, and we can do this in such a way that $\lim_{n\to\infty} |x_n| = \infty$. Now join the successive points x_n

by an interpolating admissible curve κ. Then according to Theorem 4, all three limits

$$\lim_{t\to\infty} F(\kappa_1(t), y), \qquad \lim_{t\to\infty} F(\kappa_2(t), y), \qquad \lim_{t\to\infty} F(\kappa(t), y)$$

exist, are different from zero, and must in fact be equal by our choice of the interpolating curve κ. This proves (4.1). Next, we focus on admissible rays. Observe that $F_g(tx, y) = F_{g^{-1}}(ty, x)$ for all $g \in G$ and $x, y \in C$. Together with (4.1), this implies that $v_g(y) = v_{g^{-1}}(x)$, and therefore $v_g(x) = v_{g^{-1}}(x) = v_g(y) =: v_g$. Put $v = (v_g)_{g\in G}$. Then

$$\lim_{t\to\infty} F(\kappa(t), y) = v \tag{4.2}$$

for every admissible κ and every $y \in C$. Now assume that the statement of Theorem 1 is false. Then there exist $\varepsilon > 0$ and a sequence $(x_n)_{n\in\mathbb{N}} \subset C_\delta$ with $\lim_{n\to\infty} |x_n| = \infty$ and such that

$$\max_{g\in G} |F_g(x_n, y) - v_g| > \varepsilon \qquad \text{for all } n \in \mathbb{N}.$$

We may also assume without restriction that $(x_n)_{n\in\mathbb{N}}$ is admissible—again because for each $x \in C_\delta$ the set $\{z \in C_\delta : z - x \notin C\}$ is bounded. Hence relation (4.2) entails $\lim_{n\to\infty} F_g(x_n, y) = v_g$, a contradiction. ∎

The proof of Theorem 2 is based on the Phragmén–Lindelöf Theorems (see [21, Sect. 5.6]) for the right half-plane $H = \{z \in \mathbb{C} : \operatorname{Re} z \geqslant 0\}$.

Proof of Theorem 2. We may assume that $\gamma > 0$. For fixed $x, y \in C$ define

$$G(z) := z^\gamma \sqrt{w_k(x) w_k(y)}\, e^{-z\langle x,y\rangle} E_k(zx, y),$$

which is regular in $H\backslash\{0\}$ and continuous in H with $G(0) = 0$. The integral representation (1.4) easily implies that

$$|E_k(zx, y)| \leqslant \max_{g\in G} e^{\operatorname{Re} z\langle gx,y\rangle} \qquad \text{for all } z \in \mathbb{C},$$

cf. [19, Corollary 5.4] or, alternatively, [15]. As x and y are contained in the same Weyl chamber, the inequality $\langle gx, y\rangle \leqslant \langle x, y\rangle$ holds for all $g \in G$ ([13, Theorem 3.1.2]). This shows that

$$|G(z)| \leqslant |z|^\gamma \sqrt{w_k(x) w_k(y)} e^{\operatorname{Re} z(\langle gx,y\rangle - \langle x,y\rangle)} \leqslant |z|^\gamma \sqrt{w_k(x) w_k(y)}$$

as long as $\operatorname{Re} z \geqslant 0$. Hence G is of subexponential growth when restricted to H. More precisely, for every $\delta > 0$,

$$G(z) = O(e^{\delta|z|}) \qquad \text{as } z \to \infty \quad \text{within } H.$$

Next, consider G along the boundary lines of H, $\kappa_{\pm i}(t) = \pm it,\ t > 0$. According to Theorem 1, $\lim_{t\to\infty} G(it) = i^{\gamma} v_e$. Moreover, $G(-it) = \overline{G(it)}$ for $t > 0$ (cf. (1.5)); hence $\lim_{t\to\infty} G(-it)$ exists as well. Employing the Phragmén–Lindelöf Theorems 5.62 and 5.64 of [21], we deduce that G is in fact bounded in H and that $\lim_{z\to\infty,\, z\in H} G(z) = i^{\gamma} v_e$. ∎

We finally come to the proof of Theorem 3. The key for our approach is the following simple observation: according to formula (1.4), one may write

$$E_k(x, -i\xi) = \int_{\mathbb{R}^N} e^{-i\langle \xi, y\rangle}\, d\mu_x^k(y) = \widehat{\mu_x^k}(\xi) \qquad (x, \xi \in \mathbb{R}^N), \tag{4.3}$$

where $\hat{\mu}$ stands for the classical Fourier–Stieltjes transform of $\mu \in M^1(\mathbb{R}^N)$,

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^N} e^{-i\langle \xi, y\rangle}\, d\mu(y).$$

Equation (4.3) suggests to employ Wiener's theorem, which characterizes Fourier–Stieltjes transforms of continuous measures on locally compact abelian groups (here $(\mathbb{R}^N, +)$), see for instance [12, Lemma 8.3.7]:

LEMMA 2 (Wiener). *For $\mu \in M^1(\mathbb{R}^N)$ the following properties are equivalent*:

(1) *μ is continuous.*
(2) $\lim_{n\to\infty} \frac{1}{n^N} \int_{\{\xi\in\mathbb{R}^N : |\xi| \leqslant n\}} |\hat{\mu}(\xi)|^2\, d\xi = 0.$

Apart from this, our argumentation relies on the following growth estimate for E_k, which is an easy consequence of Theorem 1 and of some interest in its own:

PROPOSITION 2. *Let $y \in C$. Then for each $\delta > 0$ there exists a constant $M_\delta(y) > 0$ such that*

$$w_k(x)|E_k(ix, gy)|^2 \leqslant M_\delta(y) \qquad \textit{for all } x \in C_\delta, \quad g \in G.$$

Remark 4. It is important at this point to note that the asymptotics of Theorem 4 implies boundedness of $x \mapsto w_k(x)|E_k(ix, gy)|^2$ only within suitable subsets of C. We do not know at present whether this function remains bounded when the range of x is all of C.

Proof of Theorem 3. For $n \in \mathbb{N}$ put $K_n := \{\xi \in \mathbb{R}^N : 1 \leqslant |\xi| \leqslant n\}$. In view of the properties of E_k ((1.2) and (1.5)), the assertion of the theorem is equivalent to

$$\lim_{n\to\infty} \frac{1}{n^N} \int_{K_n \cap C} |E_k(ix, \xi)|^2 \, d\xi = 0 \qquad \text{for all} \quad x \in \mathbb{R}^N_{\text{reg}}. \tag{4.4}$$

For fixed $x \in \mathbb{R}^N_{\text{reg}}$ and $\delta > 0$, define

$$I_1^\delta(n) := \frac{1}{n^N} \int_{K_n \cap C_\delta} |E_k(ix, \xi)|^2 \, d\xi,$$

$$I_2^\delta(n) := \frac{1}{n^N} \int_{K_n \cap (C \setminus C_\delta)} |E_k(ix, \xi)|^2 \, d\xi.$$

Let further ω denote the Lebesgue surface measure on $S^{N-1} = \{\xi \in \mathbb{R}^N : |\xi| = 1\}$. By use of (1.5), one obtains

$$\begin{aligned} I_2^\delta(n) &= \frac{1}{n^N} \int_1^n \int_{S^{N-1} \cap (C \setminus C_\delta)} |E_k(ix, t\xi)|^2 \, d\omega(\xi) \, t^{N-1} \, dt \\ &\leqslant \frac{\omega(S^{N-1} \cap (C \setminus C_\delta))}{n^N} \int_1^n t^{N-1} \, dt \leqslant \frac{1}{N} \omega(S^{N-1} \cap (C \setminus C_\delta)), \end{aligned}$$

which tends to 0 as $\delta \to 0$. Thus for given $\varepsilon > 0$, we can find $\delta > 0$ such that $I_2^\delta(n) \leqslant \varepsilon$ for all $n \in \mathbb{N}$. With this δ fixed, the upper estimate on E_k according to Proposition 2 yields

$$I_1^\delta(n) \leqslant \frac{M_\delta(x)}{n^N} \int_{K_n \cap C_\delta} \frac{d\xi}{w_k(\xi)}.$$

The weight w_k being homogeneous of degree 2γ, we further have

$$\int_{K_n \cap C_\delta} \frac{d\xi}{w_k(\xi)} = A_\delta \int_1^n t^{N-2\gamma-1} \, dt$$

with

$$A_\delta = \int_{S^{N-1} \cap C_\delta} \frac{d\omega(\xi)}{w_k(\xi)} < \infty.$$

As γ is strictly positive, this implies that $\lim_{n\to\infty} I_1^\delta(n) = 0$ and finishes the proof. ∎

Remark 5. As already mentioned, we conjecture that for $x \in \mathbb{R}^N_{\text{reg}}$ the measures μ_x^k are even absolutely continuous with respect to Lebesgue measure, provided $R' := \{\alpha \in R \mid k(\alpha) > 0\}$ spans $\mathbb{R}^N$. We comment briefly on the hypotheses for this conjecture. First note that some regularity condition on x is necessary: In fact, for $x = 0$ the representing measure is always given by the unit mass at the origin. As to the condition on R', let V' be the span of R', where $V' = \{0\}$ by convention if $R' = \emptyset$. Suppose $V' \neq \mathbb{R}^n$. Let $V'' := (V')^{\perp}$ be the orthoplement with corresponding decomposition $\mathbb{R}^N = V' \oplus V''$. For $x \in \mathbb{R}^N$, write $x = x' + x''$ with $x' \in V', x'' \in V''$. Then it is easily seen from characterization (1.1) of E_k that

$$E_k(x, y) = E_k(x', y') \cdot e^{\langle x'', y'' \rangle} \qquad \text{for all} \quad x, y \in \mathbb{R}^N$$

and accordingly,

$$\mu_x^k = \mu_{x'}^k \otimes \delta_{x''},$$

where on the right side, k is understood as a multiplicity function on R'. Thus for all $x \in \mathbb{R}^N$, μ_x^k is supported in a translate of V' and is therefore not absolutely continuous.

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